Electronic Journal of Differential Equations, Vol. 2009(2009), No. 56, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

A UNIQUENESS RESULT FOR ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULAR COEFFICIENTS

YIFEI PAN, MEI WANG

Abstract. We consider the uniqueness of solutions of ordinary differential equations where the coefficients may have singularities. We derive upper bounds on the order of singularities of the coefficients and provide examples to illustrate the results.

1. Results and examples

Classical results on the existence and uniqueness of ordinary differential equations are mostly concerned with continuous coefficients [2]. Here we consider the uniqueness of ordinary differential equation solutions of coefficients with singularities. We study upper bounds on the order of singularities of the coefficients that guarantee the uniqueness of the solution.

Main theorems are stated below. Two examples are given to illustrate and to address the sharpness aspect of the results. Proofs are provided in the subsequent section.

Theorem 1.1. *Let $f(x) \in \mathcal{C}^\infty(-a,a)$ be a solution (real or complex) of*

$$y^{(n)} + a_{n-1}(x,y)y^{(n-1)} + \dots + a_0(x,y)y = 0, \quad x \in (-a,a),\ a > 0 \tag{1.1}$$

with initial conditions

$$f(0) = f'(0) = \dots = f^{(n-1)}(0) = 0.$$

If

$$\lim_{x\to 0} |x|^{n-k}|a_k(x,y)| \le \frac{1}{e}, \quad k = 0,1,\dots,n-1, \tag{1.2}$$

where e is the Euler's number, then there exists $\delta > 0$ such that $f \equiv 0$ on $[-\delta,\delta]$.

Remarks:

- For fixed n, the inequality (1.2) can be relaxed to

$$\lim_{x\to 0} |x|^{n-k}|a_k(x,y)| < \frac{1}{B_n}, \quad k = 0,1,\dots,n-1, \quad B_n = \sum_{k=0}^{n-1} \frac{1}{k!}. \tag{1.3}$$

2000 *Mathematics Subject Classification.* 34A12, 65L05.
Key words and phrases. Uniqueness; unique continuation.

Submitted December 24, 2008. Published April 21, 2009.

- Notice that the coefficients $a_k(x,y)$ in (1.1) can be functions of $y^{(k)}$ for any k, even for $k>n$, as evidently shown in the proofs in the next section.

Corollary 1.2. *Let $f(x)\in\mathcal{C}^\infty(-a,a)$ be a solution of (1.1) with initial conditions*
$$f(0)=f'(0)=\dots=f^{(n-1)}(0)=0.$$
If $|a_k(x,y)|=o(\frac{1}{|x|^{n-k}})$ as $x\to 0$, $k=0,1,\dots,n-1$, then there exists $\delta>0$ such that $f\equiv 0$ on $[-\delta,\delta]$.

Corollary 1.3. *Let $f(x)\in\mathcal{C}^\infty(-a,a)$ be a solution of (1.1) with initial conditions*
$$f(0)=f'(0)=\dots=f^{(n-1)}(0)=0.$$
If $|a_k(x,y)|\le M$ as $x\to 0$, $k=0,1,\dots,n-1$ for some $M>0$, then exists $\delta>0$ such that $f\equiv 0$ on $[-\delta,\delta]$.

Example 1.4. The uniqueness in Theorem 1.1 may not be true for solutions not sufficiently smooth. For $\alpha\in(0,1)$, the function
$$y=\begin{cases} x^\alpha\sin(x), & x\in[0,\infty)\\ (-x)^\alpha\sin(-x), & x\in(-\infty,0)\end{cases}$$
satisfies the differential equation
$$y''-\frac{2\alpha}{x}y'+\big(1+\frac{\alpha^2+\alpha}{x^2}\big)y=0 \quad\text{with}\quad y(0)=y'(0)=0. \tag{1.4}$$
Let $\alpha=1/2e$. Then condition (1.2) in Theorem 1.1 is satisfied (for $n=2$):
$$\lim_{x\to 0}|x||a_1(x,y)|=\frac{1}{e},\quad \lim_{x\to 0}|x|^2|a_0(x,y)|=\frac{1}{2e}\big(1+\frac{1}{2e}\big)<\frac{1}{e}.$$
But $y\not\equiv 0$. Thus solutions to equation (1.4) are not unique. Notice that $y\in\mathcal{C}^{1,\alpha}$ (first derivative of Hölder continuity of order α), $y\notin\mathcal{C}^\infty$. The example also shows that the non-uniqueness cannot be remedied by using a smaller bound in (1.2), because for any given $\varepsilon>0$, we may choose $\alpha<\varepsilon/2$ such that
$$\lim_{x\to 0}|x|^{2-k}|a_k(x,y)|\le\max_{\alpha}\{2\alpha,\alpha^2+\alpha\}<\varepsilon,\quad k=0,1.$$

Example 1.5. This example shows that a bound in condition (1.2) in Theorem 1.1 is necessary. Consider the Bessel differential equation (ref. [3])
$$y''+\frac{1}{x}y'+\big(1-\frac{\nu^2}{x^2}\big)y=0. \tag{1.5}$$
A real solution can be of the form
$$y_\nu(x)=\sum_{k=0}^\infty\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\big(\frac{x}{2}\big)^{2k+\nu}=x^\nu g(x)$$
where $g(x)$ is real analytic, $g(0)\ne 0$. Let $\nu=m\ge 2$ be an integer. Then
$$y_m(x)=x^m g(x)\in\mathcal{C}^\infty$$
is a solution to (1.5) with $y_m'(x)=mx^{m-1}g(x)+x^m g'(x)$ and $y_m(0)=y_m'(0)=0$. But $y_m(x)\not\equiv 0$. Thus solutions to equation (1.5) are not unique. Notice that the only assumption not satisfied in Theorem 1.1 is Condition (1.2):
$$\lim_{x\to 0}|x|^n|a_0(x,y)|=\lim_{x\to 0}|x|^2\big|1-\frac{m^2}{x^2}\big|=m^2>\frac{1}{e}.$$

Example 1.6. In the case of Cauchy-Euler or equi-dimensional equations,

$$x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \dots + a_0 xy = 0, \quad x \in (-a, a) \tag{1.6}$$

where a_k's are constants, Condition (1.2) is simplified to

$$|a_k| < \frac{1}{e}, \quad k = 0, \dots, n-1.$$

For $n = 2$, the solutions for (1.6) have the forms $y = c_1 x^\alpha + c_2 x^\beta, y = c_1 x^\alpha \ln(x) + c_2 x^\beta$ or $y = c_1 x^\alpha \cos(\beta \ln(x)) + c_2 x^\alpha \sin(\beta \ln(x))$. These solutions do not fall into the categories described in Example 1.4 or Example 1.5.

2. Proofs

We need our previous result ([1], Theorem 5) which is stated here as a lemma.

Lemma 2.1. *Assume f (real or complex) is in $\mathcal{C}^\infty(a,b), 0 \in (a,b)$, and for $n \geq 2$ and some constant C,*

$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k}}, \quad x \in (a, b). \tag{2.1}$$

Then $f^{(k)}(0) = 0$, for all $k \geq 0$ implies $f \equiv 0$.

First we prove a lemma that provides an upper bound on the vanishing order of f near 0 when $f \not\equiv 0$.

Lemma 2.2. *Assume $f(x) \in \mathcal{C}^\infty(a,b)$, $0 \in (a,b)$, and (2.1) holds for $n \geq 2$ and some constant C. If $f \not\equiv 0$ on (a,b), then at $x = 0$, f is of finite vanishing order N,*

$$N \leq B_n C + n - 1, \quad B_n = \sum_{k=0}^{n-1} \frac{1}{k!},$$

i.e., there exists $N > 0$ such that for x near 0,

$$f(x) = a_N x^N + O\left(x^{N+1}\right).$$

Proof. When $f \not\equiv 0$, by Lemma 2.1, there must exist $N > 0$ and a_N such that

$$f^{(j)}(0) = 0, \quad \forall j < N, j \geq 0, \quad \text{and} \quad f^{(N)}(0) = N! a_N \neq 0.$$

Since $f(x) \in \mathcal{C}^\infty(a,b)$, Taylor's theorem yields

$$f(x) = a_N x^N + O\left(x^{N+1}\right).$$

If $N \geq n-1$, then

$$\begin{aligned} \frac{|f^{(k)}(x)|}{|x|^{n-k}} &= \frac{|N(N-1)\dots(N-k+1)a_N x^{N-k} + O(x^{N-k+1})|}{|x|^{n-k}} \\ &= N(N-1)\dots(N-k+1)|a_N x^{N-n}| + O\big(|x|^{N-n+1}\big) \end{aligned}$$

for $k = 1, 2, \dots, n-1$. By (2.1), for $x \in (a,b)$, as x approach 0,

$$\begin{aligned} |f^{(n)}(x)| &= N(N-1)\dots(N-n+1)|a_N x^{N-n}| + O(|x|^{N-n+1}) \\ &\le C\sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k}} \\ &= C\Big(1 + \sum_{k=1}^{n-1} N(N-1)\dots(N-k+1)\Big)|a_N x^{N-n}| + O(|x|^{N-n+1}). \end{aligned}$$

If $N \ge n-1$, dividing both sides by $N(N-1)\dots(N-n+2)|a_N x^{N-n}|$ we obtain

$$N - n + 1 + O(|x|) \le C\frac{1 + \sum_{k=1}^{n-1} N(N-1)\dots(N-k+1)}{N(N-1)\dots(N-n+2)} + O(|x|).$$

Letting $x \to 0$,

$$\begin{aligned} N - n + 1 &\le C\frac{1 + \sum_{k=1}^{n-1} N(N-1)\dots(N-k+1)}{N(N-1)\dots(N-n+2)} \\ &= C\frac{1 + N + N(N-1) + \dots + N(N-1)\dots(N-n+2)}{N(N-1)\dots(N-n+2)} \\ &= C\Big(\frac{1}{N(N-1)\dots(N-n+2)} + \frac{1}{(N-1)\dots(N-n+2)} + \dots \\ &\quad + \frac{1}{N-n+2} + 1\Big) \\ &\le C\Big(\frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \dots + \frac{1}{2!} + \frac{1}{1!} + 1\Big) = CB_n. \end{aligned}$$

Notice that the last inequality achieves equality when $N = n-1$. Thus when $N \ge n-1$, the order of $f(x) = a_N x^N + O(x^{N+1})$ satisfies $n-1 \le N \le B_n C + n - 1$. Combining with the case of $N < n-1$, we obtain

$$N \le B_n C + n - 1.$$

This completes the proof of Lemma (2.2). $\square$

Next, we consider a proposition slightly more general than Corollary 1.2.

Proposition 2.3. *Let $f \in \mathcal{C}^\infty(-a,a)$ be a solution of (1.1) such that*

$$|a_k(x,y)| = O\big(\frac{1}{|x|^{n-k}}\big) \quad \text{as } x \to 0,\ k = 0, 1, \dots, n-1. \tag{2.2}$$

If

$$f^{(k)}(0) = 0, \quad \forall k \le B_n C_n + n - 1, \tag{2.3}$$

where

$$C_n = \max_{0\le k\le n-1} \limsup_{x\to 0}\{|a_k(x,y)||x|^{n-k}\}, \quad B_n = \sum_{k=0}^{n-1} \frac{1}{k!},$$

then there exists $\delta > 0$ such that $f \equiv 0$ on $[-\delta, \delta]$.

Proof. It follows from the differential equation (1.1) that

$$|f^{(n)}(x)| \le \sum_{k=0}^{n-1} |a_k(x,y)||f^{(k)}(x)|, \quad \forall x \in (-a,a).$$

Then

$$C_n = \max_{0 \le k \le n-1} c_k, \quad \text{with } c_k = \limsup_{x \to 0} \{|x|^{n-k} |a_k(x,y)|\},$$

and c_k's are finite by Assumption (2.2). Therefore, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f^{(n)}(x)| \le (C_n + \varepsilon) \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k}}, \quad \forall x \in [-\delta, \delta].$$

If $f \not\equiv 0$ on $[-\delta, \delta]$, we would have $f^{(N)}(0) \ne 0$ for some $N \le B_n(C_n + \varepsilon) + n - 1$ by Lemma 2.2, and the arbitrariness of ε would imply $f^{(N)}(0) \ne 0$ for some $N \le M$, where $M = \lfloor B_n C_n + n - 1 \rfloor$ is the largest integer $\le B_n C_n + n - 1$. However Condition (2.3) implies $f^{(k)}(0) = 0, \forall k \le M$. Hence we must have $f \equiv 0$ on $[-\delta, \delta]$ for some $\delta > 0$. This completes the proof of Proposition 2.3. $\square$

Remarks: Notice that Example 1.5 satisfies Condition (2.2) in Proposition 2.3:

$$|a_0(x,y)| = |1 - \frac{m^2}{x^2}| = O\big(\frac{1}{|x|^{n-0}}\big), \quad |a_1(x,y)| = |\frac{1}{x}| = O\big(\frac{1}{|x|^{n-1}}\big) \tag{2.4}$$

as $x \to 0$ for $k = 0, 1$ $(n = 2)$. However the uniqueness does not hold because Condition (2.3) is not satisfied: $y_m^{(m)} \ne 0$, where $m < M = B_n C_n + 1, C_n = m^2$.

The proof of Theorem 1.1 follows from Proposition 2.3, as stated below.

Proof of Theorem 1.1. By the assumption in this Theorem, $C_n = 1/e$. Since

$$B_n C_n + n - 1 = B_n \frac{1}{e} + n - 1 < e\frac{1}{e} + n - 1 = n,$$

the initial conditions $f^{(k)}(0) = 0$, for all $k < n$ imply

$$f^{(k)}(0) = 0, \quad \forall k \le B_n C_n + n - 1.$$

Therefore $f \equiv 0$ on $|x| \le \delta$ for some $\delta > 0$ by the result in Proposition 2.3. This completes the proof of Theorem 1.1. $\square$

Similarly, Corollary 1.2 follows immediately.

Proof of Corollary 1.2. By the assumption, $C_n = 0$, $B_n C_n + n - 1 = n - 1$. Since $f^{(k)}(0) = 0$ for all $k \le n - 1$, the result of Corollary 1.2 follows from Proposition 2.3. $\square$

Acknowledgements. We thank the referee for the careful reading of the original manuscript.

References

[1] Y. Pan and M. Wang; *When is a function not flat?* J. Math. Anal. App. **340**(2008) (1), 536-542.

[2] M. Tenenbaum and H. Pollard; *Ordinary Differential Equations*, Harper & Row. New York (1963).

[3] G. N. Watson; *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University press (1944).

Yifei Pan
Department of Mathematical Sciences, Indiana University - Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA
School of Mathematics and Informatics, Jiangxi Normal University, Nanchang, China
E-mail address: pan@ipfw.edu

Mei Wang
Department of Statistics, University of Chicago, Chicago, IL 60637, USA
E-mail address: meiwang@galton.uchicago.edu