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COMPARISON RESULTS FOR SEMILINEAR ELLIPTIC EQUATIONS VIA PICONE-TYPE IDENTITIES

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Abstract. By means of a Picone's type identity, we prove uniqueness and oscillation of solutions to an elliptic semilinear equation with Dirichlet boundary conditions.

1. Introduction

The aim of this work is to provide some comparison and uniqueness results for semilinear Dirichlet problems in a smooth, open and bounded domain $G \subset \mathbb{R}^n$, $n \geq 3$. The problems are related to the elliptic operators

$$\ell u := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + f(x,u) + c(x)u\,. \tag{1.1}$$

The notation in this article is as follows:

$$D_i\{.\} := \frac{\partial}{\partial x_i}\{.\} := \{.\}_{,i}\,;$$

$$\forall Y, W \in \mathbb{R}^n \text{ and } a \in M_{n\times n}, \quad a(Y,W) := \sum_{i,j=1}^{n} a_{ij} Y^i W^j,$$

where $M_{n\times n}$ denotes the space of $n\times n$-matrices. The hypotheses on the coefficients are:

(H1) The functions $a_{ij} \in C^1(\overline{G};\mathbb{R}_+)$ are symmetric and continuous with

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \quad \forall (x,\xi) \in G\times\mathbb{R}^n \quad (> 0 \text{ if } \xi \neq 0).$$

(H2) The function $c \in C(\overline{G};\mathbb{R})$; $f \in C(\mathbb{R}^n\times\mathbb{R};\mathbb{R})$ is non constant; $\mathbb{R}_+ := (0,\infty)$ and $\bar{\mathbb{R}}_+ := [0,\infty)$. The (classical) solutions for (1.1) belong to the space $C^1(\overline{G}) \cap C^2(G)$.

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Dedicated to my late son Nkayum Tadie Abissi (+ 11/03/07) and to my cousin Tagne David Pierre (+ 01/11/08); requiescate in pacem.

2. Preliminaries

For the (smooth) functions u, w, as in [1], from the expressions $D_i\{ua_{ij}D_ju - (u^2/w)a_{ij}D_jw\}$ and $u\ell u$ we have that if $w \neq 0$,

$$\begin{aligned} &\sum_{i,j=1}^{n} D_i\{ua_{ij}(x)D_ju - \frac{u^2}{w}\ a_{ij}D_jw\} \\ &= w^2 a\Big(\nabla[\frac{u}{w}], \nabla[\frac{u}{w}]\Big) + u\ell u - \frac{u^2}{w}\ell w + u^2\{\frac{f(x,w)}{w} - \frac{f(x,u)}{u}\} \end{aligned} \tag{2.1}$$

and if $u \neq 0$, then

$$\begin{aligned} &\sum_{i,j=1}^{n} D_i\Big\{wa_{ij}(x)D_jw - \frac{w^2}{u}\ a_{ij}D_ju\Big\} \\ &= u^2 a\Big(\nabla[\frac{w}{u}], \nabla[\frac{w}{u}]\Big) + w\ell w - \frac{w^2}{u}\ell u + w^2\{\frac{f(x,u)}{u} - \frac{f(x,w)}{w}\}\,; \end{aligned} \tag{2.2}$$

also for $\lambda \neq 0$ if $\ell u = 0$, then

$$\ell(\lambda u) = f(x, \lambda u) - \lambda f(x,u)\,. \tag{2.3}$$

Remark 2.1. Most of the results will be established by the means of integrating over G (which is a regular domain) allowing the integration by parts along its boundary ∂G; this in cases like the left side of say, (2.1) , (2.2) and many other cases makes the left side of the integral to be zero when $u|_{\partial G} = 0$.

Lemma 2.2. *If u_1 and w_1 are classical solutions of*

$$\ell v = \sum_{ij=1}^{n} D_i\big(a_{ij}(x)D_j\big)v + c(x)v = 0 \quad \text{in } G\,; \quad v\big|_{\partial G} = 0, \tag{2.4}$$

then

$$\begin{aligned} \sum_{i.j=1}^{n} D_i\{u_1a_{ij}D_ju_1 - \frac{u_1^2}{w_1}a_{ij}D_jw_1\} &= w_1^2\sum_{i.j=1}^{n} a_{ij}D_i[\frac{u_1}{w_1}]D_j[\frac{u_1}{w_1}] \\ &= w_1^2 a(\nabla[\frac{u_1}{w_1}], \nabla[\frac{u_1}{w_1}])\,. \end{aligned} \tag{2.5}$$

The proof of the above lemma follows from the identities (2.1)-(2.2) where $f \equiv 0$.

Lemma 2.3. *If $u, v \in C^2$ with $v \neq 0$ then*

$$\begin{aligned} &v^2 a(\nabla[\frac{u}{v}], \nabla[\frac{u}{v}]) + \sum_{i,j=1}^{n} D_i\Big(\frac{u^2}{v}a_{ij}D_jv\Big) \\ &= a(\nabla u, \nabla u) + u^2\frac{\ell v}{v} - c(x)u^2 - \frac{u^2 f(x,v)}{v}\,. \end{aligned} \tag{2.6}$$

Proof. As in [6], for all $u, v \in C^2$ with $v \neq 0$,

$$D_i\{a_{ij}\ \frac{u^2}{v}\ D_jv\} = \frac{2u}{v}a_{ij}D_iuD_jv - \big(\frac{u}{v}\big)^2 a_{ij}D_ivD_iv + \frac{u^2}{v}D_i(a_{ij}v_j)$$

and

$$\begin{aligned} &v^2 a_{ij}D_i\Big(\frac{u}{v}\Big)D_j\Big(\frac{u}{v}\Big) \\ &= a_{ij}D_iuD_ju - \frac{u}{v}a_{ij}(D_iuD_jv + D_juD_iv\ ) + \Big(\frac{u}{v}\Big)^2 a_{ij}D_ivD_iv\,; \end{aligned} \tag{2.7}$$

thus

$$\begin{aligned}
&\sum_{i,j=1}^{n}\Big\{v^2 a_{ij}D_i\big(\frac{u}{v}\big)D_j\big(\frac{u}{v}\big)+D_i\Big(\frac{u^2}{v}a_{ij}D_jv\Big)\Big\}\\
&=v^2a(\nabla[\frac{u}{v}],\nabla[\frac{u}{v}])+\sum_{i,j=1}^{n}D_i\Big(\frac{u^2}{v}a_{ij}D_jv\Big)\\
&=\sum_{i,j=1}^{n}a_{ij}D_iuD_ju+\frac{u^2}{v}\sum_{i,j=1}^{n}D_i(a_{ij}D_jv)\\
&:=a(\nabla u,\nabla u)+u^2\frac{\ell v}{v}-c(x)u^2-\frac{u^2f(x,v)}{v}\,.
\end{aligned}$$

Then (2.6) follows. □

To ensure that solutions can be extended in the whole $\mathbb{R}^n$ we set the hypothesis

(H3) for all $x\in\mathbb{R}^n$ and all $t\in\mathbb{R}\setminus\{0\}$, it holds $tf(x,t)>0$.

Lemma 2.4. *Assume* (H1)–(H3) *hold. Let u and v be respectively solutions of*

$$\ell v:=\sum_{ij=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)v+c(x)v+f(x,v)=0\quad \textit{in } G; \tag{2.8}$$

$$Lu:=\sum_{ij=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u+c(x)u=0\quad \textit{in } G; \tag{2.9}$$

$$u\Big|_{\partial G}=0\,;\quad u>0 \textit{ in } G \textit{ and } v>0 \textit{ somewhere in } G. \tag{2.10}$$

Then v has a zero inside G. The same conclusion holds in the case where the inequalities are reverse in (2.10). *Consequently any component of the support of u or that of $-u$ contains a zero of and vise versa.*

Proof. Assume that $v>0$ in G. The integration over G of (2.1) where v replaces w , gives

$$0=\int_G\Big[v^2a\Big(\nabla[\frac{u}{v}],\nabla[\frac{u}{v}]\Big)+u^2\frac{f(x,v)}{v}\Big]dx \tag{2.11}$$

which cannot hold as the second member is strictly positive. If the inequalities in (2.10) are reverse we get the same conclusion by applying the result to $-u$ and $-v$. □

2.1. **Oscillatory solutions.**

Definition. A function u is said to be oscillatory in $\mathbb{R}^n$ if for all $R>0$, u has a simple zero in $\Omega_R:=\{x\in\mathbb{R}^n:|x|>R\}$. Equation (1.1) is said to be oscillatory if it has oscillatory solutions.

For the equation

$$Lu:=\sum_{ij=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u+c(x)u=0\quad\text{in }\mathbb{R}^n \tag{2.12}$$

and for $r>0$ and $I_n:=\{(i,j):i,j\in 1,2,\dots n.\}$, define

$$A(r):=\max_{\{I_n:|x|=r\}}\{a_{ij}(x)\}\,,\quad C(r):=\min_{|x|=r}c(x)\,,$$
$$p(r):=r^{n-1}A(r)\,,\quad q(r):=r^{n-1}C(r)$$

and the associated equation

$$\big(p(r)y'\big)' + q(r)y = 0 \quad \text{in } \mathbb{R}_+ \,. \tag{2.13}$$

For some $r_0 > 0$, define

$$P(t) := \int_{r_0}^{t} \frac{dr}{p(r)} \quad \text{if } \lim_{\infty} p(t) = \infty$$

and

$$\Pi(t) := \int_{r_0}^{t} \frac{dr}{p(r)} \quad \text{if } \lim_{\infty} p(t) < \infty.$$

From [3, Lemma 3.1 and Theorem 3.1], we have the following result.

Lemma 2.5. *Let $r_0 > 0$,*

(i) $\int_{r_0}^{\infty} q(r)dr = \infty$ *or*

$$\int_{r_0}^{\infty} q(r)dr < \infty \quad \textit{and} \quad \lim_{r\nearrow\infty} \inf \big\{P(r)\int_{r}^{\infty} q(s)ds\big\} > \frac{1}{4}$$

(ii) Π *is bounded and* $\int_{r_0}^{\infty} \Pi(r)^2 q(r)dr = \infty$, *or*

$$\int_{r_0}^{\infty} \Pi(r)^2 q(r)dr < \infty \quad \textit{and} \quad \lim_{r\nearrow\infty} \inf \big\{\frac{1}{\Pi(r)}\int_{r}^{\infty} \Pi(s)^2 q(s)ds\big\} > \frac{1}{4}$$

If either (i) or (ii) holds, then (2.13) *is oscillatory, and so is* (2.12).

From [3, Remark 3.3], Lemma 2.4 also holds when $A(r)$ and $C(r)$ are replaced, respectively, by

$$\overline{a}(r) := \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \max_{I_n}\{a_{ij}(x)\}ds,$$
$$\overline{C}(r) := \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} c(x)ds$$

where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$.

3. Main results

Theorem 3.1. *Consider the problem*

$$Lu := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + c(x)u = 0 \quad \text{in } G \tag{3.1}$$

with either

$$u\big|_{\partial G} = 0\,; \quad u > 0 \quad \text{in } G \tag{3.2}$$

or

$$\nabla u|_{\partial G} = 0\,; \quad u > 0 \quad \text{in } G. \tag{3.3}$$

Under the hypotheses (H1)-(H2), *any two solutions u and v of the problem* (3.1), (3.2) *or the problem* (3.1), (3.3) *must satisfy $u = kv$ for some constant $k \in \mathbb{R}$.*

Proof. If u and v are two such solutions then after integrating both sides of (2.5), we get the right side strictly positive while the left one is zero (see Remark 2.1. This is absurd unless $\nabla[\frac{u}{v}] \equiv 0$ in G. □

Theorem 3.2. *Assume that* (H1)-(H2) *hold. For the problem*

$$\ell u := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + f(x,u) + c(x)u = 0 \quad \text{in } G \tag{3.4}$$

with either

$$u\big|_{\partial G} = 0\,; \quad u > 0 \quad \text{in } G \tag{3.5}$$

or

$$\nabla u|_{\partial G} = 0\,; \quad u > 0 \ \text{in } G. \tag{3.6}$$

(1) If $f(x,t)$ or $\frac{f(x,t)}{t}$ is decreasing in $t > 0$ for any $x \in G$ then any of the problems (3.1), (3.2)*; or* (3.1), (3.3) *of* (1.1) *has at most one positive classical solution.*
(2) Moreover if $t \mapsto \frac{f(x,t)}{t}$ is monotone in $t > 0$ uniformly for $x \in G$ then any two solutions u and v of (1.1) *must intersect in the sense that each of the sets $G_u := \{x \in G : u(x) > v(x)\}$ and $G_v := \{x \in G : u(x) < v(x)\}$ has a non zero measure.*

Proof. Let u and v be two such solutions.
(1) From (2.1)-(2.2)

$$0 = \int_G v^2 a(\nabla[\frac{u}{v}], \nabla[\frac{u}{v}]) + u^2\big\{\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\big\}dx$$
$$0 = \int_G u^2 a(\nabla[\frac{v}{u}], \nabla[\frac{v}{u}]) - v^2\big\{\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\big\}dx$$

whence

$$0 = \int_G \Big[v^2 a(\nabla[\frac{u}{v}], \nabla[\frac{u}{v}]) + u^2 a(\nabla[\frac{v}{u}], \nabla[\frac{v}{u}]) + \{u^2 - v^2\}\big\{\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\big\}\Big]dx \tag{3.7}$$

and the conclusion follows from the fact that in any of the cases, the left hand side of (3.7) is zero and the right strictly positive.
(2) From (2.1)-(2.2), with $X(x) := \frac{f(x,v)}{v} - \frac{f(x,u)}{u}$

$$\begin{aligned} 0 &= \int_G \Big\{v^2 a(\nabla[\frac{u}{v}], \nabla[\frac{u}{v}]) + u^2 X(x)\Big\}dx \\ &= \int_G \Big\{u^2 a(\nabla[\frac{v}{u}], \nabla[\frac{v}{u}]) - v^2 X(x)\Big\}dx \end{aligned}$$

whence

$$0 = \int_G \Big[v^2 a(\nabla[\frac{u}{v}], \nabla[\frac{u}{v}]) + u^2 a(\nabla[\frac{v}{u}], \nabla[\frac{v}{u}]) + \{u^2 - v^2\}X(x)\Big]dx\,. \tag{3.8}$$

If $t \mapsto \frac{f(x,t)}{t}$ is increasing and $u - v$ does not change sign in G then (3.8) cannot hold as its second member would be strictly positive. Thus to have two distinct solutions in this case none of G_u and G_v must have zero measure. □

Theorem 3.3. *Assume that there is $\lambda_0 > 1$ such that for all $(\lambda, x, t) \in (\lambda_0, \infty) \times G \times (0, \infty)$,*

$$\lambda f(x,t) - f(x, \lambda t) > 0\,. \tag{3.9}$$

Then if for all $x \in G$, $t \mapsto \frac{f(x,t)}{t}$ is strictly increasing in $t > 0$, (1.1) *has at most one positive solution.*

Proof. Let u and v be two distinct solutions; for $G_u := \{x \in G : u(x) > v(x)\}$, we have $\nabla\{u - v\}|_{\partial G_u} \not\equiv 0$; otherwise from (2.2),

$$0 = \int_{G_u} \Big[u^2 a(\nabla[\frac{v}{u}], \nabla[\frac{v}{u}]) + v^2\{\frac{f(x,u)}{u} - \frac{f(x,v)}{v}\}\Big]dx$$

which would not hold as the second member would be strictly positive.

Let $W \in C(G)$ be defined by $W(x) := (u \vee v)(x) := \max\{u(x),\ v(x)\}$. Then W is a weak subsolution of (1.1). We chose $\lambda_0 > 1$ such that for all $(x, \lambda) \in G \times (\lambda_0, \infty)$ $W(x) < \lambda u(x) := V(x)$. By (3.9), V is a supersolution for (1.1) and there is a solution w, say, such that $W \le w \le V$ in G , by the super-sub-solutions method. This conflicts with the fact that any two solutions of (1.1) must intersect by Theorem 3.2. In fact such w would not intersect u nor v in the sense of Theorem 3.2. □

Theorem 3.4. *Assume that* (H1)–(H3) *hold in the whole* $\mathbb{R}^n$*. If in addition (i) and (ii) of the Lemma 2.4 hold, then*

$$\ell u := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + f(x,u) + c(x)u = 0$$

is oscillatory in $\mathbb{R}^n$*.*

The proof of the above theorem is a mere application of Lemmas 2.4 and 2.5.

Theorem 3.5 (Wirtinger-type inequalities)**.** *Assume that* (H1)–(H2) *hold. Let* v *be a classical solution of* (1.1) *and* u *be a function in* $C^1(\overline{G})$ *such that* $u\big|_{\partial G} = 0$*. Then*

$$\int_G v^2 a(\nabla[\frac{u}{v}], \nabla[\frac{u}{v}])\,dx \le \int_G a(\nabla u, \nabla u)dx$$

and

$$\int_G \big\{c(x)u^2 + \frac{u^2}{v} f(x,v)\big\}dx \le \int_G a(\nabla u, \nabla u)\,dx\,.$$

The proof of the above theorem follows from the integration over G of both sides of (2.6).

Concluding remarks. Some of these results can be extended to more general quasilinear equations including the p-Laplacian equations; see [8].

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