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EXISTENCE OF SOLUTIONS FOR A $p(x)$-LAPLACIAN NON-HOMOGENEOUS EQUATIONS

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Abstract. We study the boundary value problem

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. Our attention is focused on the cases when

$$f(x,u) = \pm(-\lambda|u|^{p(x)-2}u + |u|^{q(x)-2}u),$$

where $p(x) < q(x) < N \cdot p(x)/(N - p(x))$ for x in Ω.

1. Introduction and preliminary results

In the recent years increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics or calculus of variations. For more information on modelling physical phenomena by equations involving $p(x)$-growth condition we refer to [1, 5, 11, 22, 26, 30]. The appearance of such physical models was facilitated by the development of variable Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real-valued function. Variable exponent Lebesgue spaces appeared for the first time in literature as early as 1931 in an article by Orlicz [21]. The spaces $L^{p(x)}$ are special cases of Orlicz spaces L^{φ} originated by Nakano [20] and developed by Musielak and Orlicz [18, 19], where $f \in L^{\varphi}$ if and only if $\int \varphi(x, |f(x)|)dx < \infty$ for a suitable φ. Variable exponent Lebesque spaces on the real line have been independently developed by Russian researchers. In that context we refer to the studies of Tsenov [29], Sharapudinov [27] and Zhikov [32, 33].

This paper is motivated by the phenomena that can be modelled by the equations

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary and $1 < p(x)$, $p(x) \in C(\overline{\Omega})$. Our goal will be to obtain nontrivial weak solutions for (1.1) in the

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generalized Sobolev space $W^{1,p(x)}(\Omega)$ for some particular nonlinearities of the type $f(x,u)$. Problems of type (1.1) have been intensively studied in the past decades. We refer to [2, 8, 9, 10, 13, 15, 16, 17, 24, 25, 31], for some interesting results. We point out the presence in (1.1) of the $p(x)$-Laplace operator. This is a natural extension of the p-Laplace operator, with p a positive constant. However, such generalizations are not trivial since the $p(x)$-Laplace operator possesses a more complicated structure than p-Laplace operator, for example it is inhomogeneous.

We recall some definitions and properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. Roughly speaking, anisotropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For $p \in C_+(\overline{\Omega})$, we introduce *the variable exponent Lebesgue space*

$$\begin{gathered} L^{p(\cdot)}(\Omega) = \big\{u : u \text{ is a measurable real-valued function} \\ \text{such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\big\}, \end{gathered}$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\big\{\mu > 0;\ \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\,dx \le 1\big\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [12]. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous, [12, Theorem 2.8].

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise that is, $1/p(x) + 1/p'(x) = 1$, [12, Corollary 2.7]. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{1.2}$$

is valid.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold

$$|u|_{p(\cdot)} < 1\ (= 1; > 1) \ \Leftrightarrow\ \rho_{p(\cdot)}(u) < 1\ (= 1; > 1) \tag{1.3}$$

$$|u|_{p(\cdot)} > 1 \ \Rightarrow\ |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+} \tag{1.4}$$

$$|u|_{p(\cdot)} < 1 \ \Rightarrow\ |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-} \tag{1.5}$$

$$|u_n - u|_{p(\cdot)} \to 0 \ \Leftrightarrow\ \rho_{p(\cdot)}(u_n - u) \to 0, \tag{1.6}$$

since $p^+ < \infty$. For a proof of these facts see [12]. Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [6].

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = Np(x)/(N - p(x))$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$ [12, Theorem 3.9 and 3.3] (see also [7, Theorem 1.3 and 1.1]).

2. Main results

In this paper we study (1.1) in the particular cases when

$$f(x,t) = \pm(-\lambda|t|^{p(x)-2}t + |t|^{q(x)-2}t)$$

where $p(x)$, $q(x) \in C_+(\Omega)$ with $p(x) < q(x) < N \cdot p(x)/(N - p(x))$ for any $x \in \overline{\Omega}$ and $\lambda > 0$.

First, we consider the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = -\lambda|u|^{p(x)-2}u + |u|^{q(x)-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.1}$$

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (2.1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \lambda \int_\Omega |u|^{p(x)-2}uv\, dx - \int_\Omega |u|^{q(x)-2}uv\, dx = 0$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

We will prove the following result.

Theorem 2.1. *For every $\lambda > 0$, problem (2.1) has infinitely many weak solutions provided $2 \leq p^-$, $p^+ < q^-$ and $q^+ < N \cdot p^-/(N - p^-)$.*

Next, we study the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{p(x)-2}u - |u|^{q(x)-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.2}$$

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (2.2) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx - \lambda \int_\Omega |u|^{p(x)-2}uv dx + \int_\Omega |u|^{q(x)-2}uv\, dx = 0$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

Next, we prove the following result.

Theorem 2.2. *There exists $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$ problem (2.2) has a nontrivial weak solution provided $p^+ < q$ and $q^+ < N \cdot p^-/(N - p^-)$.*

We remark that in the particular case corresponding to $p(x) = 2$ and $q(x) = q$, q being a constant, (2.1) becomes

$$\begin{gathered} -\Delta u = -\lambda u + |u|^{q-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.3}$$

This problem has been studied by Ambrosetti and Rabinowitz [3] provided $2 < q < 2* = 2N/(N-2)$. Using the Mountain Pass Theorem combined with the observation that the operator $-\Delta + \lambda I$ ($\lambda > 0$) is coercive in $H_0^1(\Omega)$, Ambrosetti and Rabinowitz showed that problem (2.3) has a positive solution for any $\lambda > 0$.

3. Proof of Theorem 1

The key argument in the proof is the following version of the Mountain Pass Theorem (see [23, Theorem 9.12]):

Mountain Pass Theorem. Let X be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition (i.e., any sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $I'(x_n) \to 0$ in X^* has a convergent subsequence) and $I(0) = 0$. Suppose that

- (I1) there exists two constants ρ, $a > 0$ such that $I(x) \geq a$ if $\|x\| = \rho$,
- (I2) for each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1; I(x) \geq 0\}$ is bounded.

Then I has an unbounded sequence of critical values.

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$ and let $\lambda > 0$ be arbitrary but fixed.

The energy functional corresponding to problem (2.1) is defined as $J_\lambda : E \to \mathbb{R}$,

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \lambda \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx.$$

A simple calculation based on relations (1.4) and (1.5) and the compact embedding of E into $L^{r(x)}(\Omega)$ for all $r \in C_+(\overline{\Omega})$ with $r(x) < p^*(x)$ on $\overline{\Omega}$ shows that J_λ is well-defined on E and $J_\lambda \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle J'_\lambda(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \lambda \int_\Omega |u|^{p(x)-2} uv\, dx - \int_\Omega |u|^{q(x)-2} uv\, dx$$

for any u, $v \in E$. Thus the weak solutions of (2.1) are exactly the critical points of J_λ.

We show now that the Mountain Pass Theorem can be applied in this case.

Lemma 3.1. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in E$ with $\|u\|_{p(x)} = \eta$*

Proof. We first point out that we have

$$|u(x)|^{q^-} + |u(x)|^{q^+} \geq |u(x)|^{q(x)}, \quad \forall x \in \overline{\Omega} \tag{3.1}$$

Using (3.1) we deduce that

$$J_\lambda(u) \geq \frac{1}{p^+} \cdot \int_\Omega |\nabla u|^{p(x)} dx - \frac{1}{q^-} \cdot \Big(\int_\Omega |u|^{q^-} dx + \int_\Omega |u|^{q^+} dx \Big) \tag{3.2}$$

Since $p^+ < q^- \leq q^+ < p^*(x)$ for any $x \in \overline{\Omega}$ and E is continuously embedded in $L^{q^-}(\Omega)$ and in $L^{q^+}(\Omega)$, it follows that there exist two positive constant C_1 and C_2 such that

$$\|u\|_{p(x)} \geq C_1 \cdot |u|_{q^+}, \quad \|u\|_{p(x)} \geq C_2 \cdot |u|_{q^-}, \quad \forall u \in E. \tag{3.3}$$

Next, we focus our attention on the case when $u \in E$ with $\|u\|_{p(x)} < 1$. For such a u by relation (1.5) we obtain

$$\int_\Omega |\nabla u|^{p(x)} dx \geq \|u\|_{p(x)}^{p^+}. \tag{3.4}$$

Relations (3.2), (3.3) and (3.4) imply

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+} \cdot \|u\|_{p(x)}^{p^+} - \frac{1}{q^-} \cdot \Big[\Big(\frac{1}{C_1} \cdot \|u\|_{p(x)} \Big)^{q^+} + \Big(\frac{1}{C_2} \cdot \|u\|_{p(x)} \Big)^{q^-} \Big] \\ &= (\beta - \gamma \cdot \|u\|_{p(x)}^{q^+ - p^+} - \delta \cdot \|u\|_{p(x)}^{q^- - p^+}) \cdot \|u\|_{p(x)}^{p^+} \end{aligned}$$

for any $u \in E$ with $\|u\|_{p(x)} < 1$, where β, γ and δ are positive constants.

We remark that the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(t) = \beta - \gamma \cdot t^{q^+ - p^+} - \delta \cdot t^{q^- - p^+}$$

is positive in a neighborhood of the origin. We conclude that Lemma 3.1 holds. $\square$

Lemma 3.2. *If $E_1 \subset E$ is a finite dimensional subspace, the set $S = \{u \in E_1; J_\lambda \geq 0\}$ is bounded in E.*

Proof. To prove this lemma, we first show that

$$\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \leq K_1 \cdot \Big(\|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+} \Big), \quad \forall u \in E \tag{3.5}$$

where K_1 is a positive constant. Indeed, using relations (1.4) and (1.5) we obtain

$$\int_\Omega |\nabla u|^{p(x)} dx \leq |\nabla u|_{p(x)}^{p^-} + |\nabla u|_{p(x)}^{p^+} = \|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}, \quad \forall u \in E. \tag{3.6}$$

On the other hand

$$\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \leq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)} dx$$

and thus (3.5) holds. Also, for each $\lambda > 0$ there exists a positive constant $K_2(\lambda)$ such that

$$\lambda \cdot \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \leq K_2(\lambda) \cdot \Big(\|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+} \Big), \quad \forall u \in E. \tag{3.7}$$

By inequalities (3.5) and (3.7), we get

$$J_\lambda(u) \leq K_1 \cdot \Big(\|u\|_{p(x)}^{p-} + \|u\|_{p(x)}^{p^+} \Big) + K_2(\lambda) \cdot \Big(\|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+} \Big) - \frac{1}{q^+} \int_\Omega |u|^{q(x)} dx$$

for all $u \in E$.

Let $u \in E$ be arbitrary but fixed. We define

$$\Omega_1 = \{x \in \Omega; |u(x)| < 1\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

Then we have

$$\begin{aligned}
J_\lambda(u) &\le K_1\cdot\Big(\|u\|_{p(x)}^{p-}+\|u\|_{p(x)}^{p^+}\Big)+K_2(\lambda)\cdot\Big(\|u\|_{p(x)}^{p^-}+\|u\|_{p(x)}^{p^+}\Big)-\frac{1}{q^+}\int_\Omega |u|^{q(x)}dx\\
&\le K_1\cdot\Big(\|u\|_{p(x)}^{p-}+\|u\|_{p(x)}^{p^+}\Big)+K_2(\lambda)\cdot\Big(\|u\|_{p(x)}^{p^-}+\|u\|_{p(x)}^{p^+}\Big)-\frac{1}{q^+}\int_{\Omega_2} |u|^{q(x)}dx\\
&\le K_1\cdot\Big(\|u\|_{p(x)}^{p-}+\|u\|_{p(x)}^{p^+}\Big)+K_2(\lambda)\cdot\Big(\|u\|_{p(x)}^{p^-}+\|u\|_{p(x)}^{p^+}\Big)-\frac{1}{q^+}\int_{\Omega_2} |u|^{q^-}dx\\
&\le K_1\cdot\Big(\|u\|_{p(x)}^{p-}+\|u\|_{p(x)}^{p^+}\Big)+K_2(\lambda)\cdot\Big(\|u\|_{p(x)}^{p^-}+\|u\|_{p(x)}^{p^+}\Big)\\
&\quad -\frac{1}{q^+}\int_\Omega |u|^{q^-}dx+\frac{1}{q^+}\int_{\Omega_1}|u|^{q^-}dx.
\end{aligned}$$

But there exists a positive constant K_3 such that

$$\frac{1}{q^+}\int_{\Omega_1}|u|^{q^-}dx\le K_3,\quad \forall u\in E.$$

Thus we deduce that

$$J_\lambda(u)\le K_1\cdot\big(\|u\|_{p(x)}^{p-}+\|u\|_{p(x)}^{p^+}\big)+K_2(\lambda)\cdot\big(\|u\|_{p(x)}^{p^-}+\|u\|_{p(x)}^{p^+}\big)-\frac{1}{q^+}\int_\Omega |u|^{q^-}dx+K_3,$$

for all $u\in E$. The functional $|\cdot|_{q^-}:E\to\mathbb{R}$ defined by

$$|u|_{q^-}=\Big(\int_\Omega |u|^{q^-}dx\Big)^{1/q^-}$$

is a norm in E. In the finite dimensional subspace E_1 the norms $|\cdot|_{q^-}$ and $\|\cdot\|_{p(x)}$ are equivalent, so there exists a positive constant $K=K(E_1)$ such that

$$\|u\|_{p(x)}\le K\cdot |u|_{q^-},\quad \forall u\in E_1.$$

As a consequence we have that there exists a positive constant K_4 such that

$$J_\lambda(u)\le K_1\cdot\Big(\|u\|_{p(x)}^{p-}+\|u\|_{p(x)}^{p^+}\Big)+K_2(\lambda)\cdot\Big(\|u\|_{p(x)}^{p^-}+\|u\|_{p(x)}^{p^+}\Big)-K_4\cdot\|u\|^{q^-}+K_3,$$

for all $u\in E_1$. Hence

$$K_1\cdot\Big(\|u\|_{p(x)}^{p-}+\|u\|_{p(x)}^{p^+}\Big)+K_2(\lambda)\cdot\Big(\|u\|_{p(x)}^{p^-}+\|u\|_{p(x)}^{p^+}\Big)-K_4\cdot\|u\|_{p(x)}^{q^-}+K_3\ge 0,$$

for all $u\in S$. and since $q^->p^+$ we conclude that S is bounded in E. The proof is complete. □

Lemma 3.3. *If $\{u_n\}\subset E$ is a sequence which satisfies the conditions*

$$|J_\lambda(u_n)|<M, \tag{3.8}$$

$$J'_\lambda(u_n)\to 0\quad as\quad n\to\infty \tag{3.9}$$

where M is a positive constant, then $\{u_n\}$ possesses a convergent subsequence.

Proof. First, we show that $\{u_n\}$ is bounded in E. Assume the contrary. Then, passing if necessary to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\|_{p(x)}\to\infty$ as $n\to\infty$. Thus, we may assume that $\|u_n\|_{p(x)}>1$ for any integer n.

By (3.9) we deduce that there exists $N_1>0$ such that for any $n>N_1$, we have

$$\|J'_\lambda(u_n)\|\le 1.$$

On the other hand, for any $n > N_1$ fixed, the application

$$E \ni v \to \langle J'_\lambda(u_n), v\rangle$$

is linear and continuous. The above information implies

$$|\langle J'_\lambda(u_n), v\rangle| \le \|J'_\lambda(u_n)\| \cdot \|v\|_{p(x)} \le \|v\|_{p(x)}, \quad \forall v \in E, n > N_1.$$

Setting $v = u_n$ we have

$$-\|u_n\|_{p(x)} \le \int_\Omega |\nabla u_n|^{p(x)} dx + \lambda \int_\Omega |u_n|^{p(x)} dx - \int_\Omega |u_n|^{q(x)} dx \le \|u_n\|_{p(x)}$$

for all $n > N_1$. We obtain

$$-\|u_n\|_{p(x)} - \int_\Omega |\nabla u_n|^{p(x)} dx - \lambda \int_\Omega |u_n|^{p(x)} dx \le - \int_\Omega |u_n|^{q(x)} dx \tag{3.10}$$

for any $n > N_1$.

Provided that $\|u_n\|_{p(x)} > 1$ relations (3.8), (3.10) and (1.4) imply

$$\begin{aligned}
M &> J_\lambda(u_n) \\
&\ge \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big) \cdot \int_\Omega (|\nabla u_n|^{p(x)}) dx \\
&\quad + \lambda \cdot \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big) \cdot \int_\Omega |u_n|^{p(x)} dx - \frac{1}{q^-} \cdot \|u_n\|_{p(x)} \\
&\ge \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big) \cdot \int_\Omega |\nabla u_n|^{p(x)} dx - \frac{1}{q^-} \cdot \|u_n\|_{p(x)} \\
&\ge \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big) \cdot \|u_n\|_{p(x)}^{p^-} - \frac{1}{q^-} \cdot \|u_n\|_{p(x)}.
\end{aligned}$$

Letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E.

Since $\{u_n\}$ is bounded in E we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u_0 \in E$ such that $\{u_n\}$ converges weakly to u_0 in E. Using Theorem 1.3 in [7], E is compactly embedded in $L^{p(x)}(\Omega)$ and in $L^{q(x)}(\Omega)$ it follows that $\{u_n\}$ converges strongly to u_0 in $L^{p(x)}(\Omega)$ and $L^{q(x)}(\Omega)$. The above information and relation (3.9) imply

$$\langle J'_\lambda(u_n) - J'_\lambda(u_0), u_n - u_0\rangle \to 0 \quad \text{as } n \to \infty.$$

On the other hand, we have

$$\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u_0|^{p(x)-2} \nabla u_0) \cdot (\nabla u_n - \nabla u_0) dx \\
&= \langle J'_\lambda(u_n) - J'_\lambda(u_0), u_n - u_0\rangle - \lambda \cdot \int_\Omega (|u_n|^{p(x)-2} u_n - |u_0|^{p(x)-2} u_0)(u_n - u_0) dx \\
&\quad + \int_\Omega (|u_n|^{q(x)-2} u_n - |u_0|^{q(x)-2} u_0)(u_n - u_0) dx.
\end{aligned}$$

Using the fact that $\{u_n\}$ converges strongly to u_0 in $L^{q(x)}(\Omega)$ and inequality (1.2), we have

$$\begin{aligned}
&\Big|\int_\Omega (|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0)(u_n-u_0)dx\Big|\\
&\le \Big|\int_\Omega |u_n|^{q(x)-2}u_n(u_n-u_0)dx\Big| + \Big|\int_\Omega |u_0|^{q(x)-2}u_0(u_n-u_0)dx\Big|\\
&\le C_3\cdot \||u_n|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\cdot |u_n-u_0|_{q(x)} + C_4\cdot \||u_0|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\cdot |u_n-u_0|_{q(x)}
\end{aligned}$$

where C_3 and C_4 are positive constants. Since $|u_n-u_0|_{q(x)}\to 0$ as $n\to\infty$ we deduce that

$$\lim_{n\to\infty}\int_\Omega (|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0)(u_n-u_0)dx = 0, \tag{3.11}$$

$$\lim_{n\to\infty}\int_\Omega (|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0)(u_n-u_0)dx = 0. \tag{3.12}$$

By (3.11) and (3.12), we obtain

$$\lim_{n\to\infty}\int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_0|^{p(x)-2}\nabla u_0)\cdot(\nabla u_n-\nabla u_0)dx = 0. \tag{3.13}$$

It is known that

$$(|z|^{r-2}z-|t|^{r-2}t)\cdot(z-t)\ge \Big(\frac{1}{2}\Big)^r|z-t|^r, \quad \forall r\ge 2,\ z,t\in\mathbb{R}^N. \tag{3.14}$$

Relations (3.13) and (3.14) yield

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n-\nabla u_0|^{p(x)}dx = 0\,.$$

This fact and relation (1.6) imply $\|u_n-u_0\|_{p(x)}\to\infty$ as $n\to\infty$. The proof is complete. □

Completed proof of Theorem 2.1. It is clear that the functional J_λ is even and verifies $J_\lambda(0)=0$. Lemma 3.3 implies that J_λ satisfies the Palais-Smale condition. On the other hand, Lemmas 3.1 and 3.2 show that conditions (I1) and (I2) are satisfied. The Mountain Pass Theorem can be applied to the functional J_λ. We conclude that equation (2.1) has infinitely many weak solutions in E. The proof is complete. □

4. Proof of Theorem 2.2

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$ and let $\lambda>0$ be arbitrary but fixed.

We start by introducing the energy functional corresponding to problem (2.1) as $I_\lambda: E\to\mathbb{R}$,

$$I_\lambda(u)=\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx-\lambda\int_\Omega\frac{1}{p(x)}|u|^{p(x)}dx+\int_\Omega\frac{1}{q(x)}|u|^{q(x)}dx.$$

The same arguments as those used in the case of the functional J_λ show that I_λ is well-defined on E and $I_\lambda\in C^1(E,\mathbb{R})$ with the derivative given by

$$\langle I_\lambda'(u),v\rangle=\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v dx-\lambda\int_\Omega|u|^{p(x)-2}uv\,dx+\int_\Omega|u|^{q(x)-2}uvdx$$

for any u, $v \in E$. We obtain that the weak solutions of (2.1) are the critical points of I_λ.

This time our idea is to show that I_λ possesses a nontrivial global minimum point in E. With this end in view we start by proving two auxiliary results.

Lemma 4.1. *The functional I_λ is coercive on E.*

Proof. To prove this lemma, we first show that for any a, $b > 0$ and $0 < k < l$ the following inequality holds:

$$a \cdot t^k - b \cdot t^l \le a \cdot \big(\frac{a}{b}\big)^{k/(l-k)}, \quad \forall t \ge 0. \tag{4.1}$$

Indeed, since the function $[0, +\infty) \ni t \to t^\theta$ is increasing for any $\theta > 0$ it follows that

$$a - b \cdot t^{l-k} < 0, \quad \forall t > \big(\frac{a}{b}\big)^{1/(l-k)}$$

and

$$t^k \cdot (a - b \cdot t^{l-k}) \le a \cdot t^k < a \cdot \big(\frac{a}{b}\big)^{k/(l-k)}, \forall t \in [0, (\frac{a}{b})^{1/(l-k)}].$$

The above two inequalities show that (4.1) holds. Using (4.1) we deduce that for any $x \in \Omega$ and $u \in E$, we have

$$\begin{aligned}
\frac{\lambda}{p^-}|u(x)|^{p(x)} - \frac{1}{q^+}|u(x)|^{q(x)} &\le \frac{\lambda}{p^-}\big[\frac{\lambda \cdot q^+}{p^-}\big]^{p(x)/(q(x)-p(x))} \\
&\le \frac{\lambda}{p^-}\big[\big(\frac{\lambda \cdot q^+}{p^-}\big)^{p^+/(q^- - p^+)} + \big(\frac{\lambda \cdot q^+}{p^-}\big)^{p^-/(q^+ - p^-)}\big] \\
&= C
\end{aligned}$$

where C is a positive constant independent of u and x. Integrating the above inequality over Ω we obtain

$$\frac{\lambda}{p^-}\int_\Omega |u|^{p(x)}dx - \frac{1}{q^+}\int_\Omega |u|^{q(x)}dx \le D \tag{4.2}$$

where D is a positive constant independent of u.

Using inequalities (3.1) and (4.2) we obtain that for any $u \in E$ with $\|u\|_{p(x)} > 1$,

$$\begin{aligned}
I_\lambda(u) &\ge \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{\lambda}{p^-}\int_\Omega |u|^{p(x)}dx + \frac{1}{q^+}\int_\Omega |u|^{q(x)}dx \\
&\ge \frac{1}{p^+}\|u\|_{p(x)}^{p^-} - \Big(\frac{\lambda}{p^-}\int_\Omega |u|^{p(x)}dx - \frac{1}{q^+}\int_\Omega |u|^{q(x)}dx\Big) \\
&\ge \frac{1}{p^+}\|u\|_{p(x)}^{p^-} - D.
\end{aligned}$$

Thus I_λ is coercive and the proof of is complete. $\square$

Lemma 4.2. *The functional I_λ is weakly lower semicontinuous.*

Proof. First we prove that the functional $A : E \to \mathbb{R}$,

$$A(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx,$$

is convex. Indeed, since the function $[0, \infty) \ni t \to t^s$ is convex for any $s > 1$, we deduce that for each $x \in \Omega$ fixed it the inequality

$$\big|\frac{z+t}{2}\big|^{p(x)} \le \big|\frac{|z|+|t|}{2}\big|^{p(x)} \le \frac{1}{2}|z|^{p(x)} + \frac{1}{2}|t|^{p(x)}, \quad \forall z,\ t \in \mathbb{R}^N$$

holds. Using the above inequality we deduce that

$$\Big|\frac{\nabla u+\nabla v}{2}\Big|^{p(x)}\leq \frac{1}{2}|\nabla u|^{p(x)}+\frac{1}{2}|\nabla v|^{p(x)},\quad \forall u,v\in E,\ x\in\Omega.$$

Multiplying with $1/p(x)$ and integrating over Ω we obtain

$$A\Big(\frac{u+v}{2}\Big)\leq \frac{1}{2}A(u)+\frac{1}{2}A(v),\quad \forall u,v\in E.$$

Thus A are convex.

Next, we show that the functional A is weakly lower semicontinuous on E. Taking into account that A is convex, by [4, Corollary III.8] it is sufficient to show that A is strongly lower semicontinuous on E. We fix $u\in E$ and $\varepsilon>0$. Let $v\in E$ be arbitrary. Since A is convex and inequality (1.2) holds; we have

$$\begin{aligned}
A(u) &\geq A(u)+\langle A'(u),v-u\rangle\\
&\geq A(u)-\int_\Omega |\nabla u|^{p(x)-1}|\nabla(v-u)|dx\\
&\geq A(u)-D_1\cdot \||\nabla u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}\cdot|\nabla(u-v)|_{p(x)}\\
&\geq A(u)-D_2\cdot\|u-v\|_{p(x)}\\
&\geq A(u)-\varepsilon
\end{aligned}$$

for all $v\in E$ with $\|u-v\|_{p(x)}<\varepsilon/[\||\nabla u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}]$. We have denoted by D_1 and D_2 two positive constants. It follows that A is strongly lower semicontinuous and since it is convex we obtain that A is weakly lower semicontinuous.

Finally, we remark that if $\{u_n\}\subset E$ is a sequence which converges weakly to u in E then $\{u_n\}$ converges strongly to u in $L^{p(x)}(\Omega)$ and $L^{q(x)}(\Omega)$. Thus, I_λ is weakly lower semicontinuous. The proof is complete. $\square$

Proof of Theorem 2.2. By Lemmas 4.1 and 4.2, we deduce that I_λ is coercive and weakly lower semicontinuous on E. Then [28, Theorem 1.2] implies that there exist a global minimizer $u_\lambda\in E$ of I_λ and thus a weak solution of problem (2.2).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0>1$ be a fixed real and Ω_1 an open subset of Ω with $|\Omega_1|>0$ we deduce that there exists $u_0\in C_0^\infty(\Omega)\subset E$ such that $u_0(x)=t_0$ for any $x\in\overline{\Omega}_1$ and $0\leq u_0(x)\leq t_0$ in $\Omega\setminus\Omega_1$. We have

$$\begin{aligned}
I_\lambda(u_0) &= \int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}dx-\lambda\int_\Omega\frac{1}{p(x)}|u_0|^{p(x)}dx+\int_\Omega\frac{1}{q(x)}|u_0|^{q(x)}dx\\
&\leq L-\frac{\lambda}{p^+}\int_{\Omega_1}|u_0|^{p(x)}dx\\
&\leq L-\frac{\lambda}{p^+}\cdot t_0^{p^-}\cdot|\Omega_1|
\end{aligned}$$

where L is a positive constant. Thus, there exists $\lambda^*>0$ such that $I_\lambda(u_0)<0$ for any $\lambda\in[\lambda^*,\infty)$. It follows that $I_\lambda(u_\lambda)<0$ for any $\lambda\geq\lambda^*$ and thus u_λ is a nontrivial weak solution of problem (2.2) for λ large enough. The proof of is complete. $\square$

Remark. After this article was accepted, the author learned that the results here are a particular case of the results in [14].

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