A CLASS OF GENERALIZED INTEGRAL OPERATORS

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Abstract. In this paper, we introduce a class of generalized integral operators that includes Fourier integral operators. We establish some conditions on these operators such that they do not have bounded extension on $L^2(\mathbb{R}^n)$. This permits us in particular to construct a class of Fourier integral operators with bounded symbols in $S_{0,1}^0(\mathbb{R}^n \times \mathbb{R}^n)$ and in $\bigcap_{0 < \rho < 1} S_{\rho,1}^0(\mathbb{R}^n \times \mathbb{R}^n)$ which cannot be extended to bounded operators in $L^2(\mathbb{R}^n)$.

1. Introduction

The integral operators of type

$$A\phi(x) = \int e^{iS(x,\theta)}a(x, \theta)\mathcal{F}\phi(\theta)d\theta$$

appear naturally for solving the hyperbolic partial differential equations and expressing the $C^\infty$-solution of the associate Cauchy problem’s (see e.g. [10, 11]).

If we write formally the expression of the Fourier transform $\mathcal{F}\phi(\theta)$ in (1.1), we obtain the following Fourier integral operators, so-called canonical transformations,

$$A\phi(x) = \int \int e^{i(S(x,\theta)-y\theta)}a(x, y, \theta)\phi(y)dyd\theta$$

in which appear two $C^\infty$-functions, the phase function $\phi(x, y, \theta) = S(x, \theta) - y\theta$ and the amplitude $a$ called the symbol of the operator $A$. In the particular case where $S(x, \theta) = x\theta$, one recovers the notion of pseudodifferential operators (see e.g. [6, 15]).

Since 1970, many of Mathematicians have been interested to these type of operators: Duistermaat [3], Hörmander [6, 7] Kumano-Go [8], and Fujiwara [2]. We mention also the works of Hasanov [4], and the recent results of Messirdi Senoussaoui [12] and Aiboudi-Messirdi-Senoussaoui [11].

In this paper we study a general class of integral operators including the class of Fourier integral operators, specially we are interested in their continuity on $L^2(\mathbb{R}^n)$.

The continuity of the operator $A$ on $L^2(\mathbb{R}^n)$ is guaranteed if the weight of the symbol $a$ is bounded, if this weight tends to zero then $A$ is compact on $L^2(\mathbb{R}^n)$ (see e.g. [12]).

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If the symbol \( a \) is only bounded the associated Fourier integral operator \( A \) is not necessary bounded on \( L^2(\mathbb{R}^n) \). Indeed, in 1998 Hasanov [4] constructed an example of unbounded Fourier integral operators on \( L^2(\mathbb{R}) \).

Aiboudi-Messirdi-Senoussaoui [1] constructed recently in a class of Fourier integral operators with bounded symbols in the Hörmander class \( \bigcap_{0 < \rho < 1} S^{0,1}_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n) \) that cannot be extended to be a bounded operator in \( L^2(\mathbb{R}^n), n \geq 1 \).

These results of unboundedness was obtained by using the properties of the operators \( B\varphi(x) = \int_{\mathbb{R}^n} k(z)\varphi((b(x)z + a(x))dz \) (1.3) on \( L^2(\mathbb{R}^n), n \geq 1 \), where \( k(z) \in S(\mathbb{R}^n) \) (the space of \( C^\infty \)-functions on \( \mathbb{R}^n \), whose derivatives decrease faster than any power of \( |x| \) as \( |x| \to +\infty \)), \( a(x) \) and \( b(x) \) are real-valued, measurable functions on \( \mathbb{R}^n \). Operators of type (1.3) was considered by Hasanov [4] and a slightly different way by Aiboudi Messirdi Senoussaoui [1].

We also give in this paper a generalization of these results since we consider a class of integral operators which is general than thus of type (1.3):

\[ C\varphi(x) = \int_{\mathbb{R}^n} K(x,z)\varphi(F(x,z))dz \]

(1.4)

where \( K(x,z) \) and \( F(x,z) \) are real-valued, measurable functions on \( \mathbb{R}^{2n} \). The generalized integral operator \( C \) includes Hilbert, Mellin and the Fourier-Bros-Iagolnitzer transforms which they has been used by many authors and for many purposes, in particular respectively by Hörmander [5] for the analysis of linear partial differential operators, Robert [13] about the functional calculus of pseudodifferential operators, Sjöstrand [14] in the area of microlocal and semiclassical analysis and Stein [15] for the study of singular integral operators.

The operators \( C \) appears also in the study of the width of the quantum resonances (see e.g. [9]).

We shall discuss in the second section bounded extension problems for the class of operators type \( C \). We give some technical conditions on the functions \( K(x,z) \) and \( F(x,z) \) such that \( C \) do not admit a bounded extension on \( L^2(\mathbb{R}^n) \). We also indicate a connection between transformations \( C \) and Fourier integral operators.

In the third section, we construct an example of Fourier integral with bounded symbols belongs respectively to \( S^{0,1}_{1,1}(\mathbb{R}^n) \), (the case \( n = 1 \) is given in [4] and generalized for \( n \geq 2 \) in [1]), and \( \bigcap_{0 < \rho < 1} S^{0,1}_{\rho} \) that cannot be extended as a bounded operator on \( L^2(\mathbb{R}^n), n \geq 2 \). In the case of the Hörmander symbolic class \( S^{0,1}_{1,1}(\mathbb{R}^n) \) our constructions are direct and technical.

2. Unboundedness of the generalized integral operators

In this section we construct a class of operators \( C \) that cannot be extended to be a bounded operator in \( L^2(\mathbb{R}^n), n \geq 1 \). We have first an easy boundedness criterion of the operator \( C \).
Proposition 2.1. Let $F(x,.) \in C^1(\mathbb{R}^n)$, and $K(x,.) \in L^2(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$. Suppose that there exists a function $g(x)$ such that
\[ g(x) > 0, \quad \forall x \in \mathbb{R}^n \]
\[ |\det \left( \frac{\partial F(x, z)}{\partial z} \right) | \geq g(x), \quad \forall x, z \in \mathbb{R}^n \]
\[ \|K(x,.)\|_{L^2(\mathbb{R}^n)}/\sqrt{g(x)} \in L^2(\mathbb{R}^n) \]
then $C$ is a bounded operator on $L^2(\mathbb{R}^n)$.

Proof. Using Hölder inequality and the change of variable $y = F(x, z)$, it’s inverse is denoted $z = G(x, y)$, we obtain for all $\varphi \in L^2(\mathbb{R}^n)$,
\[
\|C\varphi\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} K(x, z)\varphi(F(x, z))dz \right|^2 dx \\
\leq \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |K(x, z)\varphi(F(x, z))|dz \right]^2 dx \\
\leq \int_{\mathbb{R}^n} \left[ \|K(x,.)\|_{L^2(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n} |\varphi(F(x, z))|^2 dz \right] dx \\
= \int_{\mathbb{R}^n} \left[ \|K(x,.)\|_{L^2(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n} |\varphi(y)|^2 \left| \det(\frac{\partial F(x, z)}{\partial z})(z=G(x,y)) \right|^{-1} dy \right] dx \\
\leq \|\varphi\|_{L^2(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n} \frac{\|K(x,.)\|_{L^2(\mathbb{R}^n)}}{\sqrt{g(x)}} dx \tag{2.1}
\]

hence $C$ is bounded operator on $L^2(\mathbb{R}^n)$ with $\|C\| \leq \frac{\|K(x,.)\|_{L^2(\mathbb{R}^n)}}{\sqrt{g(x)}}$.

Now we give the main result of this paper. We proof that under some conditions the operator $C$ do not admit a bounded extension on $L^2(\mathbb{R}^n)$.

Theorem 2.2. Let $\delta \in ]0,1[\,$ and the operator $C$ defined by (1.4) on $L^2(\mathbb{R}^n)$ for $x = (x_1, \ldots, x_n) \in ]0, \delta[^n$ such that:

(H1) For $\varepsilon > 0$ and for all $x \in \mathbb{R}^n$
\[ \{ z \in \mathbb{R}^n : |F(x, z)| \leq \varepsilon \} = \prod_{i=1}^{n} [a_i^-(x, \varepsilon), a_i^+(x, \varepsilon)] \]
where $a_i^\pm(x,t)$ are real-measurable functions on $\mathbb{R}^n \times ]0, +\infty[$ satisfying 1- for any $p \in \mathbb{N}^*$ and $i \in \{1, \ldots, n\}$,
\[ \lim_{x_i \to 0^+} a_i^+(px, x_i) = +\infty \]
2- for any $\lambda \in ]0,1[, i \in \{1, \ldots, n\}$ and $p \in \mathbb{N}^*$, the functions $a_i^+(px, \lambda)$ and $a_i^-(px, \lambda)$ are respectively decreasing and increasing with respect to $x$ in $]0, \delta[^n$.

(H2) There exists a constant $R > 0$ such that for any $r \geq R$ and for all $x \in ]0, \delta[^n$
\[ |\int_{[-r,r]^n} K(x, z)dz| \geq \delta \]
Then the operator $C$ cannot be extended to a bounded operator on $L^2(\mathbb{R}^n)$. 
Proof. Let us define the generalized sequence of functions
\[
\varphi_\varepsilon(x) = \begin{cases} 
1, & \text{if } x \in [-\varepsilon, \varepsilon]^n \\
0, & \text{otherwise}
\end{cases} 
\] (2.2)
then \( \varphi_\varepsilon \in L^2(\mathbb{R}^n) \) for all \( \varepsilon > 0 \) and we have
\[
C\varphi_\varepsilon(x) = \int_{\prod_{i=1}^n [a_i^-(x,\varepsilon), a_i^+(x,\varepsilon)]]} K(x, z) dz
\]
Consequently,
\[
C\varphi_\varepsilon(x) = \int_{\prod_{i=1}^n [a_i^-(x,\varepsilon_j), a_i^+(x,\varepsilon_j)]]} K(x, z) dz 
\] (2.3)
where \( \varepsilon_j \geq 0 \) and \( \lim_{j \to +\infty} \varepsilon_j = 0 \).

By condition 1 of the the assumption \((H1)\), for any \( p \in \mathbb{N}^* \) there exists a number \( \varepsilon_p \geq 0 \) such that
\[
a_i^+(p\Lambda_p, \varepsilon_p) \geq R 
\] (2.4)
and
\[
a_i^-(p\Lambda_p, \varepsilon_p) \leq -R 
\] (2.5)
for \( \Lambda_p = (\varepsilon_p, \varepsilon_p, \ldots, \varepsilon_p) \), \( p\varepsilon_p \leq \delta < 1 \) and \( i \in \{1, \ldots, n\} \).

It follows from (2.4), (2.5) and condition 2 of the assumption \((H1)\) that for \( x \in [0, p\varepsilon_p]^n \) and \( i \in \{1, \ldots, n\} \) we have
\[
a_i^+(x, \varepsilon_p) \geq a_i^+(p\Lambda_p, \varepsilon_p) \geq R; 
\] (2.6)
\[
a_i^-(x, \varepsilon_p) \leq a_i^-(p\Lambda_p, \varepsilon_p) \leq -R. 
\] (2.7)
Finally using (H2), (2.3), (2.6) and (2.7) we deduce
\[
\|C\varphi_\varepsilon_p\|_{L^2(\mathbb{R}^n)} \geq \int_{[0, p\varepsilon_p]^n} |C\varphi_\varepsilon_p(x)|^2 dx \geq \delta^2 p^n \varepsilon_p^n 
\] (2.8)
If we consider that \( C \) has a bounded extension to \( L^2(\mathbb{R}^n) \), then by virtue of (2.1) we obtain for \( \varphi = \varphi_\varepsilon_p \in L^2(\mathbb{R}^n) \)
\[
\delta^2 p^n \varepsilon_p^n \leq \|C\varphi_\varepsilon_p\|_{L^2(\mathbb{R}^n)} \leq M^2 \varepsilon_p^n 
\]
and for any \( p \in \mathbb{N}^* \)
\[
p^n \leq \frac{M^2}{\delta^2} 
\]
This is a contradiction. Consequently \( A \) cannot be a bounded operator in \( L^2(\mathbb{R}^n) \). □

Remark 2.3. (1) If in particular \( K(x, z) = K(z) \) is independent on \( x \) and \( F(x, z) = b(x) \circ z + a(x) \), where \( K(z) \) is a real-valued measurable function, \( b(x), a(x) \in \mathbb{R}^n \) are measurable functions on \( \mathbb{R}^n \), we obtain the so-called generalized Hilbert transforms introduced in [4]

(2) The operator \( C \) is an Fourier integral operator for an appropriate choice of the functions \( K(x, z) \) and \( F(x, z) \).
\[
C\varphi(x) = \int_{\mathbb{R}^n} K(x, z) \varphi(F(x, z)) dz = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{iz\cdot \xi} F K(x, \xi) \varphi(F(x, z)) d\xi dz,
\]
where $\mathcal{F}K(x, \xi)$ is the Fourier transform of the partial function $z \rightarrow K(x, z)$. Setting $y = F(x, z)$ and $z = G(x, y)$, we have

$$C\varphi(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{iG(x,y)\cdot \xi} \mathcal{F}K(x, \xi)\varphi(y)|\det(\frac{\partial G}{\partial y})|d\xi dy$$

which is a Fourier integral operator with the phase function $\phi(x, y, \xi) = G(x, y)\xi$ and the symbol $p(x, y, \xi) = \mathcal{F}K(x, \xi)|\det(\frac{\partial G}{\partial y})|$ if $K$ and $G$ are infinitely regular with respect to $x, y$ and $\xi$.

3. A class of unbounded Fourier integral operators on $L^2(\mathbb{R}^n)$

It follows from theorem 2.2 that with an appropriate choice of $K(x, z)$ and $F(x, z)$ we can construct a class of Fourier integral operators which cannot be extended as bounded operators on $L^2(\mathbb{R}^n)$.

An example of unbounded Fourier integral operator with a symbol in $S^0_{\rho, 1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\bigcap_{0 < \rho < 1} S^0_{\rho, 1}(\mathbb{R}^n \times \mathbb{R}^n)$ was given respectively in [4] and [1], where if $\rho \in \mathbb{R}$,

$$S^0_{\rho, 1}(\mathbb{R}^n \times \mathbb{R}^n) = \{ p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \forall (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n \exists C_{\alpha, \beta} > 0 ; \quad |\partial^\alpha_x \partial^\beta_\xi p(x, \theta)| \leq C_{\alpha, \beta} \lambda^{-\rho|\beta| + |\alpha|}(\theta) \} \quad (3.1)$$

3.1. A class with symbols in $S^0_{1, 1}(\mathbb{R}^n \times \mathbb{R}^n)$. Here, we generalize the example given by Hasanov on $\mathbb{R}$ to high dimensions. Namely, in the same spirit of [8], we have easily if we get $K(z) \in S(\mathbb{R}^n)$ and $b \in C^\infty(\mathbb{R}^n, \mathbb{R})$.

**Proposition 3.1.** If $K(z) \in S(\mathbb{R}^n)$ and $b \in C^\infty(\mathbb{R}^n, \mathbb{R})$, then for all $\alpha, \beta \in \mathbb{N}^n$ there exists $C_{\alpha, \beta} > 0$ such that

$$|\partial^\alpha_x \partial^\beta_\xi K(b(x)\xi)| \leq C_{\alpha, \beta}(1 + |\xi|)|\alpha| - |\beta| \quad (3.2)$$

for all $(x, \xi) \in [-1, 1]^n \times \mathbb{R}^n$.

**Proof.** It suffices to use the fact that $K \in S(\mathbb{R}^n)$ and $\beta$ is bounded on $[-1, 1]^n$. □

Let also $a = (a_1, a_2, \ldots, a_n) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $a, b, K$ satisfy (H1) and (H2), with

$$b(x) > 0$$

$$a_i^\pm(x, t) = \frac{\pm t + a_i(x)}{b(x)}, \quad t > 0, \quad x \in \mathbb{R}^n \quad (3.3)$$

Then, for $q(x, \xi) = K(b(x)\xi)$ defined on $[-1, 1]^n \times \mathbb{R}^n$, we have

$$|\partial^\alpha_x \partial^\beta_\xi q(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)|\alpha - |\beta|$$

on $[-1, 1]^n \times \mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n$, $C_{\alpha, \beta}$ being constants.

Thus, $q \in S^0_{1, 1}([-1, 1]^n \times \mathbb{R}^n)$, in particular $q(x, \xi)$ is a well bounded symbol. Take a function $\eta \in C^\infty(\mathbb{R}^n)$ with supp $\eta \subset [-\delta, \delta]^n$ and $\eta(x) = 1$ for $x \in [-\delta, \delta]^n$, $\delta < 1$.

It is now obvious to see that the function $p(x, \xi) = \eta(x)q(x, \xi) \in S^0_{1, 1}(\mathbb{R}^n \times \mathbb{R}^n)$.

Now the Fourier integral operator defined by

$$C\varphi(x) = \int_{\mathbb{R}^n} e^{-i(a(x), \xi + p(x, \xi))} p(x, \xi)\varphi(\xi)dyd\xi$$

$$= \int_{\mathbb{R}^n} e^{-i(a(x), \xi + p(x, \xi))} \eta(x)K(b(x)\xi)\varphi(\xi)dyd\xi$$
is of the type \([1.4]\). Indeed, for \(s = b(x)\xi\) and \(x \in [0, \delta]^n\)
\[
C\varphi(x) = \int_{\mathbb{R}^{2n}} e^{-i\frac{a(x)+y}{b(x)}} K(s) \frac{1}{b^n(x)} \varphi(y) dy ds
\]
Finally, if we pose \(\frac{a(x)+y}{b(x)} = z\), we have
\[
C\varphi(x) = \int \mathcal{F}K(z)\varphi(b(x)z-a(x))dz
\]
By theorem \([2.2]\) we conclude that the operator \(C\) cannot be extended as a bounded operator on \(L^2(\mathbb{R}^n)\).

3.2. **A class with symbols in** \(\bigcap_{0 < \rho < 1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)\). We describe in this section the results of Aiboudi-Messirdi-Senoussaoui \([1]\), they constructed a class of unbounded Fourier integral operators with a separated variables phase function and a symbol in the Hörmander class \(\bigcap_{0 < \rho < 1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)\).

Precisely, let \(K \in S(\mathbb{R})\) with \(K(t) = 1\) on \([-\delta, \delta]\) and \(b(t)\) is continuous function on \([0, 1]\) such that
\[
b(t) \in C^\infty([0, 1]), \quad b(0) = 0, \quad b'(t) > 0 \text{ in } [0, 1]
\]
\[
|b^{(k)}(t)| \leq \frac{C_k}{t^k} \text{ in } [0, 1], \quad k \in \mathbb{N}^*, \quad C_k > 0
\]
\(\chi(x), \psi(\xi) \in C^\infty(\mathbb{R}^n, \mathbb{R})\) homogeneous of degree 1. Thus the function
\[
q(x, \xi) = e^{-i x \cdot (x) \psi(\xi)} \prod_{j=1}^n K(b(|x|)|x|\xi_j), \quad \xi = (\xi_1, \ldots, \xi_n)
\]
belongs to \(C^\infty([-1, 1]^n \times \mathbb{R}^n)\) and satisfies, as in the proposition \([3.1]\), the following estimates

**Proposition 3.2.** For all \(\alpha, \beta \in \mathbb{N}^n\),
\[
|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C_{\alpha \beta} \frac{(1 + |\xi|)^{\alpha - |\beta|}}{b((1 + |\xi|)^{-1})^{|\beta|}}
\]
on \([-1, 1]^n \times \mathbb{R}^n\) where \(C_{\alpha \beta} > 0\).

Now if \(\phi(x)\) is a \(C_0^\infty(\mathbb{R})\)-function such that
\[
\phi(s) = 1 \quad \text{on } [-\delta, \delta], \quad \delta < 1
\]
\(\text{supp } \phi \subset [-1, 1]\)
define the global \(C^\infty\) symbol on \(\mathbb{R}^n \times \mathbb{R}^n\) by
\[
p(x, \xi) = e^{-i x \cdot \psi(\xi)} \prod_{j=1}^n \phi(x_j) K(b(|x|)|x|\xi_j)
\]
\(\quad x = (x_1, \ldots, x_n), \quad \xi = (\xi_1, \ldots, \xi_n)\).
Then \(p(x, \xi) \in \bigcap_{0 < \rho < 1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)\) and the corresponding Fourier integral operator is
\[
C\varphi(x) = \int_{\mathbb{R}^n} e^{i x \cdot \psi(\xi)} p(x, \xi) \mathcal{F}\varphi(\xi) d\xi
\]
\[
= \prod_{j=1}^n \phi(x_j) \int_{\mathbb{R}^n} K(b(|x|)|x|\xi_j) \mathcal{F}\varphi(\xi) d\xi
\]
By using an adequate change of variable in the integral \([3.8]\), we have
\[
C\varphi(x) = \int_{\mathbb{R}^n} \varphi(b(|x|)|x| z) \prod_{j=1}^{n} F K(z_j) d\xi, \quad z = (z_1, \ldots, z_n) \tag{3.9}
\]
which is of the form \(C\) in theorem \(2.2\) where the functions \(F(x, z) = b(|x|)|x| z\) and \(K(x, z) = \prod_{j=1}^{n} F K(z_j)\) satisfy (H1) and (H2). Consequently, the operator \(C\) cannot be continuously extended on \(L^2(\mathbb{R}^n)\).

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