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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR THIRD-ORDER BOUNDARY-VALUE PROBLEM ON THE HALF-LINE WITH DEPENDENCE ON THE FIRST ORDER DERIVATIVE

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Abstract. By using fixed-point theorem for operators on a cone, sufficient conditions for the existence of multiple positive solutions for a third-order boundary-value problem on the half-line are established. In the case of the p-Laplace operator our results for $p > 1$ generalize previous known results. The interesting point lies in the fact that the nonlinear term is allowed to depend on the first order derivative u'.

1. Introduction

Boundary value problems on the half-line arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations, see [3], and various physical phenomena [2, 5], such as an unsteady flow of gas through a semi-infinite porous media, the theory of drain flows, plasma physics, in determining the electrical potential in an isolated neutral atom. In recently years, the boundary-value problems on the half-line have received a great deal of attention in literature (see [1, 7, 10, 11, 15, 16] and references therein). However, in [6, 13, 14] the authors only studied multi-point boundary-value problems on the finite interval. They showed that there exist multiple positive solutions by using fixed-point theorems for operators on a cone. But so far, very few results are obtained for a third-order multi-point boundary-value problems on the half-line. To the author's knowledge, there are no results about third order multi-point boundary-value problems, whose non- linear term does not depend on the first derivative u'. The goal of this paper is to fill the gap in this area. In this paper, by using fixed-point theorem for operators on a cone, some sufficient conditions for the existence of multiple positive solutions for third-order boundary-value problem on the half-line are established, which are the complement of previously known results. In the case of the p-Laplace operator our results for some $p > 1$ generalize previous known results.

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In this paper, we study the existence of multiple positive solutions for the following third-order boundary-value problem with dependence on the first order derivative on the half-line

$$\begin{gathered}(\varphi(-u'')(t))' = a(t)f(t,u,u'), \quad 0 < t < +\infty,\\ u(0) - \beta u'(0) = 0,\\ u'(\infty) = 0, \quad u''(0) = 0,\end{gathered} \tag{1.1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\varphi(0) = 0$ and $f \in C([0,+\infty)^3, [0,+\infty))$ and $\beta \in (0,+\infty)$. $a(t)$ is a nonnegative measurable function defined in $(0,+\infty)$ and $a(t)$ does not identically vanish on any subinterval of $(0,+\infty)$.

A projection $\varphi : \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and positive homomorphism (see [8]), if the following conditions are satisfied:

- (i) if $x \le y$, then $\varphi(x) \le \varphi(y)$, for all $x, y \in \mathbb{R}$;
- (ii) φ is a continuous bijection and its inverse mapping is also continuous;
- (iii) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in [0,+\infty)$.

In above definition, we can replace the condition (iii) by the following stronger condition:

- (iv) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in \mathbb{R}$, where $\mathbb{R} = (-\infty,+\infty)$.

Remark 1.1. (1) If conditions (i), (ii) and (iv) hold, then it implies that φ is homogenous generating a p-Laplace operator; i.e., $\varphi(x) = |x|^{p-2}x$, for some $p > 1$.

(2) It is well known that a p-Laplacian operator is odd. However, the operator which we defined above is not necessary odd, see (5.2). We emphasize that the results of the papers [10, 12, 13, 14] cannot be applied if φ is defined as above.

(3) The nonlinear term is allowed to depend on the first order derivative u' which is the complement of previously known results [6, 7, 13, 14].

In this article, the following hypotheses are needed:

- (C1) $f \in C([0,+\infty)^3, [0,+\infty))$, $f(t,0,0) \not\equiv 0$ on any subinterval of $(0,+\infty)$ and when u is bounded $f(t,(1+t)u,u')$ is bounded on $[0,+\infty)$;
- (C2) $a(t)$ satisfies the following relations:

$$\int_0^{+\infty} \varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds < +\infty, \qquad \int_0^{+\infty} s\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds < +\infty.$$

The plan of the article is as follows. In Section 2 for the convenience of the reader we give some background and definitions. In Section 3 we present some lemmas in order to prove our main results. Section 4 is devoted to presenting and proving our main results. Some examples are presented in Section 5 to demonstrate the application of our main results.

2. Some definitions and fixed point theorems

In this section, we provide background definitions from the cone theory in Banach spaces.

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:

- (a) if $y \in P$ and $\lambda \ge 0$, then $\lambda y \in P$;
- (b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative, continuous, concave functional on a cone P of a real Banach space E, if $\alpha : P \to [0, \infty)$ is continuous, and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all x, $y \in P$ and $t \in [0,1]$.

Definition 2.3. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

The following fixed point theorems are fundamental and important to the proofs of our main results.

Theorem 2.4. [4] *Let E be a Banach space and $P \subset E$ be a cone in E. Let $r > 0$ define $\Omega_r = \{x \in P : \|x\| < r\}$. Assume that $T : P \bigcap \overline{\Omega}_r \to P$ is completely continuous operator such that $Tx \neq x$ for $x \in \partial\Omega_r$.*

(i) *If $\|Tx\| \leq \|x\|$ for $x \in \partial\Omega_r$, then $i(T, \Omega_r,\ P) = 1$.*
(ii) *If $\|Tx\| \geq \|x\|$ for $x \in \partial\Omega_r$, then $i(T, \Omega_r,\ P) = 0$.*

Theorem 2.5. [6] *Let K be a cone in a Banach space X. Let D be an open bounded set with $D_k = D \cap K \neq \emptyset$ and $\overline{D}_k \neq K$. Let $T : \overline{D}_k \to K$ be a compact map such that $x \neq Tx$ for $x \in \partial D_k$. Then the following results hold:*

(1) *If $\|Tx\| \leq \|x\|$ for $x \in \partial D_k$, then $i_k(T, D_k) = 1$.*
(2) *Suppose there is $e \in K$, $e \neq 0$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(T, D_k) = 0$.*
(3) *Let D^1 be open in X such that $\overline{D^1} \subset D_k$. If $i_k(T, D_k) = 1$ and $i_k(T, D^1_k) = 0$, then T has a fixed point in $D_k \backslash \overline{D^1_k}$. Then same result holds if $i_k(T, D_k) = 0$ and $i_k(T, D^1_k) = 1$.*

3. Preliminaries and Lemmas

Let E be the set defined as

$$E = \big\{u \in C^1[0, +\infty) : \sup_{0 \leq t < +\infty} \frac{|u(t)|}{1+t} < +\infty,\ \lim_{t \to +\infty} u'(t) = 0.\big\}$$

Then E is a Banach space, equipped with the norm $\|u\| = \|u\|_1 + \|u\|_2$, where $\|u\|_1 = \sup_{0 \leq t < +\infty} \frac{|u(t)|}{1+t} < +\infty$, $\|u\|_2 = \sup_{0 \leq t < +\infty} |u'(t)|$.

Also, define the cone $K \subset E$ by

$$\begin{aligned} K = \big\{u \in E : u(t) \geq 0, t \in [0, +\infty),\ u(0) - \beta u'(0) = 0,\\ u(t) \text{ is concave on } [0, +\infty).\big\} \end{aligned}$$

To prove the main results in this paper, we will employ several lemmas.

Lemma 3.1. *For any $p \in C[0, +\infty)$, the problem*

$$(\varphi(-u''(t)))' = p(t), \quad 0 < t < +\infty, \tag{3.1}$$

$$u(0) - \beta u'(0) = 0, \quad u'(\infty) = 0, \quad u''(0) = 0 \tag{3.2}$$

has a unique solution

$$
\begin{aligned}
u(t)=&\int_t^{+\infty}(t-s)\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds+\int_0^{+\infty}s\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds\\
&+\beta\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds.
\end{aligned}
\tag{3.3}
$$

Proof. Necessity. By taking the integral of the equation (3.1) on $[0,t]$, we have

$$
\varphi(-u''(t))-\varphi(-u''(0))=\int_0^t p(\tau)d\tau. \tag{3.4}
$$

By the boundary condition and $\varphi(0)=0$, we have

$$
u''(t)=-\varphi^{-1}\left(\int_0^t p(\tau)d\tau\right). \tag{3.5}
$$

By taking the integral of (3.5) on $[t,+\infty)$, we obtain

$$
u'(\infty)-u'(t)=-\int_t^{+\infty}\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds. \tag{3.6}
$$

By the boundary condition $u'(\infty)=0$, we obtain

$$
u'(t)=\int_t^{+\infty}\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds. \tag{3.7}
$$

By taking the integral of (3.7) on $[t,+\infty)$, we have

$$
u(t)=\int_t^{+\infty}\int_s^{+\infty}\varphi^{-1}\Big(\int_0^\tau p(\eta)d\eta\Big)d\tau ds+u(0). \tag{3.8}
$$

Substituting $u(0)=\beta u'(0)$ and integrating by parts, we obtain

$$
\begin{aligned}
u(t)=&\int_t^{+\infty}(t-s)\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds+\int_0^{+\infty}s\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds\\
&+\beta\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds.
\end{aligned}
$$

Sufficiency: Let u be as in (3.3). Taking the derivative of (3.3), it implies that

$$
u'(t)=\int_t^{+\infty}\varphi^{-1}\Big(\int_0^s p(\tau)d\tau\Big)ds.
$$

Furthermore, we obtain

$$
u''(t)=-\varphi^{-1}\Big(\int_0^t p(\tau)d\tau\Big),
$$

$$
\varphi(-u''(t))=\int_0^t p(\tau)d\tau,
$$

taking the derivative of this expression yields $(\varphi(-u''(t)))'=p(t)$. Routine calculation verifies that u satisfies the boundary value conditions, so that u given in (3.3) is a solution of (3.1)-(3.2).

It is easy to see that the problem

$$
(\varphi(-u''(t)))'=0,\quad u(0)-\beta u'(0)=0,\quad u'(\infty)=0,\quad u''(0)=0,
$$

has only the trivial solution. Thus u in (3.3) is the unique solution of (3.1)-(3.2). The proof is complete. $\square$

Lemma 3.2. *For any $p(t) \in C[0,+\infty)$ and $p(t) \geq 0$, the unique solution u of (3.1)-(3.2) satisfies*
$$u(t) \geq 0, \quad \text{for } t \in [0,+\infty).$$

Lemma 3.3. *For any $u \in K$, it holds that $\beta\|u\|_2 \leq \|u\|_1 \leq \mu\|u\|_2$, where $\mu = \max\{\beta, 1\}$.*

Proof. Since $u(t)$ is concave and nondecreasing, together with $u'(\infty) = 0$, we have $\|u\|_2 = u'(0)$ and
$$\frac{u(t)}{1+t} \leq \frac{1}{1+t}\Big(\int_0^t u'(s)ds + \beta u'(0)\Big) \leq \frac{t+\beta}{1+t}u'(0) \leq \mu\|u\|_2.$$
On the other hand,
$$\|u\|_1 = \sup_{0\leq t<+\infty} \frac{|u(t)|}{1+t} \geq \frac{u(0)}{1+0} = \beta u'(0) = \beta\|u\|_2.$$
So we can obtain the desired result. □

Lemma 3.4. *Let $u \in K$. Then $\min_{t\in[1/a,a]} u(t) \geq \delta(t)\|u\|$, where $a > 1$, $\delta(t) = \frac{1}{2}\min\{\lambda(t), \beta/a\}$,*
$$\lambda(t) = \begin{cases} \sigma, & t \geq \sigma, \\ t, & t \leq \sigma, \end{cases}$$
and $\sigma = \inf\{\xi \in [0,+\infty) : \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t} = \frac{u(\xi)}{1+\xi}\}$.

Proof. From [7, Lemma 3.2], we know that
$$\min_{t\in[\frac{1}{a},a]} u(t) \geq \lambda(t)\|u\|_1. \tag{3.9}$$
On the other hand, since $u \in K$, we have
$$\min_{t\in[\frac{1}{a},a]} u(t) = u(\frac{1}{a}) \geq \frac{\beta}{a}u'(0) = \frac{\beta}{a}\|u\|_2. \tag{3.10}$$
So (3.9) and (3.10) imply that the result of Lemma 3.4 holds. □

Remark 3.5. From the definition of $\lambda(t)$, we know that $0 < \delta(t) < 1$, for $t \in (0,1)$.

Define $T : K \to E$ by
$$\begin{aligned}(Tu)(t) = &\int_t^{+\infty} (t-s)\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds \\ &+ \int_0^{+\infty} s\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds \\ &+ \beta\int_0^{+\infty} \varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds.\end{aligned} \tag{3.11}$$
Obviously, we have $(Tu)(t) \geq 0$, for $t \in (0,+\infty)$, and
$$(Tu)'(t) = \int_t^{+\infty} \varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds \geq 0.$$
Furthermore,
$$(Tu)''(t) = -\varphi^{-1}\Big(\int_0^t a(s)f(s,u(s),u'(s))ds\Big) \leq 0,$$

and $(Tu)(0)-\beta(Tu)'(0)=0$. This shows $(TK)\subset K$.

To obtain the complete continuity of T, the following lemma is needed.

Lemma 3.6 ([9]). *Let W be a bounded subset of K. Then W is relatively compact in E if $\{\frac{v(t)}{1+t}, v\in W\}$ and $\{v'(t), v\in W\}$ are equicontinuous on any finite subinterval of $[0,+\infty)$ and for any $\varepsilon>0$, there exists $T=T(\varepsilon)>0$ such that*

$$\Big|\frac{v(t_1)}{1+t_1}-\frac{v(t_2)}{1+t_2}\Big|<\varepsilon,\quad |v'(t_1)-v'(t_2)|<\varepsilon$$

uniformly with respect to $v\in W$ as $t_1,t_2\geq T$.

Lemma 3.7. *Let* (C1), (C2) *hold. Then $T:K\to K$ is completely continuous.*

Proof. Firstly, it is easy to check that $T:K\to K$ is well defined. Let $u_n\to u$ in K, then from the definition of E, we can choose r_0 such that $\sup_{n\in N\setminus\{0\}}\|u_n\|<r_0$. Let $A_{r_0}=\sup\{f(t,(1+t)u,v),(t,u,v)\in[0,+\infty)\times[0,r_0]^2\}$ and we have

$$\begin{aligned}&\int_t^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)|f(\tau,u_n,u_n')-f(\tau,u,u')|d\tau\Big)ds\\&\leq 2\varphi^{-1}(A_{r_0})\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds.\end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}|(Tu_n)'(t)-(Tu)'(t)|&=\Big|\int_t^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)(f(\tau,u_n,u_n')-f(\tau,u,u'))d\tau\Big)ds\Big|\\&\leq\int_t^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)|f(\tau,u_n,u_n')-f(\tau,u,u')|d\tau\Big)ds\\&\to 0,\quad\text{as } n\to+\infty.\end{aligned}$$

From Lemma 3.3, we have

$$\|Tu_n-Tu\|\leq(1+\mu)\|Tu_n-Tu\|_2\to 0,\quad\text{as } n\to+\infty.$$

Thus T is continuous.

Let Ω be any bounded subset of K. Then there exists $r>0$ such that $\|u\|\leq r$ for all $u\in\Omega$ and we have

$$\begin{aligned}\|Tu\|_2&=\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u,u')d\tau\Big)ds\\&\leq\varphi^{-1}(A_r)\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds<+\infty.\end{aligned}$$

From Lemma 3.3, we have $\|Tu\|\leq(1+\mu)\|Tu\|_2<+\infty$. So $T\Omega$ is bounded. Moreover for any $S\in(0,+\infty)$ and $t_1,t_2\in[0,S]$. Without loss of generality, let $t_1\geq t_2$. Then we have

$$\begin{aligned}\Big|\frac{(Tu)(t_1)}{1+t_1}-\frac{(Tu)(t_2)}{1+t_2}\Big|&\leq\Big[\int_0^{+\infty}s\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u,u')d\tau\Big)ds\\&\quad+\beta\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u,u')d\tau\Big)ds\Big]\Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big|\\&\quad+\int_{t_2}^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u,u')d\tau\Big)ds\Big|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\Big|\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{+\infty} s\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u,u')d\tau\Big)ds\Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big| \\
&+\frac{1}{1+t_2}\int_{t_2}^{t_1} s\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u,u')d\tau\Big)ds \\
&\to 0,\quad \text{uniformly as } t_1\to t_2
\end{aligned}
$$

and

$$
|(Tu)'(t_2)-(Tu)'(t_1)| = \int_{t_1}^{t_2}\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds \to 0,
$$

uniformly as $t_1\to t_2$. We obtain that $T\Omega$ is equicontinuous on any finite subinterval of $[0,+\infty)$.

For any $u\in\Omega$, we have

$$
\begin{aligned}
\lim_{t\to+\infty}\Big|\frac{(Tu)(t)}{1+t}\Big| &= \lim_{t\to+\infty}\frac{1}{1+t}\Big\{\int_t^{+\infty}(t-s)\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds \\
&\quad +\int_0^{+\infty} s\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds \\
&\quad +\beta\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds\Big\} \\
&\le \lim_{t\to+\infty}\varphi^{-1}(A_r)\int_t^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds \\
&= 0
\end{aligned}
$$

and

$$
\lim_{t\to+\infty}|(Tu)'(t)| = \lim_{t\to+\infty}\varphi^{-1}(A_r)\int_t^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds = 0.
$$

So $T\Omega$ is equiconvergent at infinity. By Lemma 3.6, $T\Omega$ is relatively compact. Therefore we know that T is compact. So $T:K\to K$ is completely continuous. The proof is complete. $\square$

Let $k>1$ be a fixed constant and choose $a=k$. Then define

$$
\gamma = \delta(\frac{1}{k})\frac{\delta\big(\frac{1}{k}\big)\beta\int_{1/k}^{k}\varphi^{-1}\Big(\int_{1/k}^{s}a(\tau)d\tau\Big)ds}{(1+\mu)\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds},
$$

$$
\gamma_1 = \frac{\delta\big(\frac{1}{k}\big)\beta\int_{1/k}^{k}\varphi^{-1}\Big(\int_{1/k}^{s}a(\tau)d\tau\Big)ds}{(1+\mu)\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds},
$$

$$
K_\rho = \{u\in K: \|u\|_1\le\rho\},
$$

$$
\Omega_\rho = \{u\in K: \min_{t\in[\frac{1}{k},k]}u(t)<\gamma\rho\} = \{u\in K: \gamma\|u\|_1\le \min_{t\in[\frac{1}{k},k]}u(t)<\gamma\rho\}.
$$

Lemma 3.8 ([6]). *The set Ω_ρ has the following properties:*

(a) *Ω_ρ is open relative to K.*
(b) *$K_{\gamma\rho}\subset\Omega_\rho\subset K_\rho$.*
(c) *$u\in\partial\Omega_\rho$ if and only if $\min_{t\in[\frac{1}{k},k]}u(t)=\gamma\rho$.*
(d) *$u\in\partial\Omega_\rho$, then $\gamma\rho\le u(t)\le\rho$ for $t\in[\frac{1}{k},k]$.*

Now, we introduce the following notation. Let

$$f^{\rho}_{\gamma\rho} = \min\big\{\frac{f(t,(1+t)u,v)}{\varphi(\rho)} : t \in [1/k,k],\ u \in [\gamma\rho,\rho], v \in [0,\rho/\beta]\big\},$$
$$f^{\rho}_{0} = \sup\big\{\frac{f(t,(1+t)u,v)}{\varphi(\rho)} : t \in [0,+\infty),\ u \in [0,\rho], v \in [0,\rho/\beta]\big\},$$
$$f^{\alpha} = \limsup_{u\to\alpha}\big\{\frac{f(t,(1+t)u,v)}{\varphi(u)} : t \in [0,+\infty)\big\},$$
$$f_{\alpha} = \lim_{u\to\alpha}\min\big\{\frac{f(t,(1+t)u,v)}{\varphi(u)} : t \in [1/k,k]\big\} \quad (\alpha := \infty \text{ or } 0^{+}),$$
$$\frac{1}{m} = (1+\mu)\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds,$$
$$\frac{1}{M} = \delta\big(\frac{1}{k}\big)\beta\int_{1/k}^{k}\varphi^{-1}\Big(\int_{1/k}^s a(\tau)d\tau\Big)ds.$$

Remark 3.9. It is easy to see that $0 < m$, $M < \infty$ and $M\gamma = M\gamma_1\delta(\frac{1}{k}) = \delta(\frac{1}{k})m < m$.

Lemma 3.10. *If f satisfies the condition*

$$f^{\rho}_0 \le \varphi(m) \quad \textit{and} \quad u \ne Tu \quad \textit{for } u \in \partial K_\rho, \tag{3.12}$$

then $i_k(T,K_\rho) = 1$.

Proof. By (3.11) and (3.12), for $u(t) \in \partial K_\rho$, we have $\|u\|_1 = \sup_{0\le t<+\infty}\frac{|u(t)|}{1+t} = \rho$. Moreover, Lemma 3.3 implies that $\|u\|_2 \le \frac{1}{\beta}\|u\|_1 = \frac{1}{\beta}\rho$, Therefore, from definition of f^{ρ}_0 we have

$$f(t,u,u') \le \varphi(\rho)\varphi(m) = \varphi(\rho m).$$

Therefore,

$$\begin{aligned}
\|Tu\| &= \|Tu\|_1 + \|Tu\|_2 \le (1+\mu)\|Tu\|_2\\
&= (1+\mu)\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds\\
&\le \rho m(1+\mu)\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds\\
&\le \rho = \|u\|_1 \ \le \|u\|.
\end{aligned}$$

This implies that $\|Tu\| \le \|u\|$ for $u(t) \in \partial K_\rho$. By Theorem 2.4 (1) we have $i_k(T,K_\rho) = 1$. □

Lemma 3.11. *If f satisfies the conditions*

$$f^{\rho}_{\gamma\rho} \ge \varphi(M\gamma) \quad \textit{and} \quad u \ne Tu \quad \textit{for } u \in \partial\Omega_\rho, \tag{3.13}$$

then $i_k(T,\Omega_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0,+\infty)$. Then $e \in \partial K_1$, and we claim that

$$u \ne Tu + \lambda e, \quad u \in \partial\Omega_\rho, \quad \lambda > 0.$$

If not, there exist $u_0 \in \partial\Omega_\rho$ and $\lambda_0 > 0$ such that $u_0 = Tu_0 + \lambda_0 e$. By (3.11) and (3.13) we have

$$u_0 = Tu_0(t) + \lambda_0 e$$

$$
\begin{aligned}
&\geq \delta\big(\frac{1}{k}\big)\|Tu_0\|_1 + \lambda_0 \\
&= \delta\big(\frac{1}{k}\big) \sup_{t\in[0,+\infty)} \frac{|Tu_0|}{1+t} + \lambda_0 \\
&\geq \delta\big(\frac{1}{k}\big) \frac{(Tu)(0)}{1+0} + \lambda_0 \\
&= \delta\big(\frac{1}{k}\big)\beta \int_0^{+\infty} \varphi^{-1}\Big(\int_0^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds + \lambda_0 \\
&\geq \delta\big(\frac{1}{k}\big)\beta \int_{1/k}^{k} \varphi^{-1}\Big(\int_{1/k}^s a(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds + \lambda_0 \\
&\geq \delta\big(\frac{1}{k}\big)M\gamma\rho\beta \int_{1/k}^{k} \varphi^{-1}\Big(\int_{1/k}^s a(\tau)d\tau\Big)ds + \lambda_0 \\
&\geq \gamma\rho + \lambda_0.
\end{aligned}
$$

This implies that $\gamma\rho \geq \gamma\rho + \lambda_0$ which is a contradiction. Hence by Theorem 2.4 (2), we have $i_k(T, \Omega_\rho) = 0$. □

4. Main results

The main results in this articles are the following.

Theorem 4.1. *Assume that one of the following conditions holds:*

(C3) *There exist $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ with $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$ such that*

$$f_0^{\rho_1} \leq \varphi(m), \quad f_{\gamma\rho_2}^{\rho_2} \geq \varphi(M\gamma), \quad u \neq Tu \text{ for } u \in \partial\Omega_{\rho_2} \text{ and } f_0^{\rho_3} \leq \varphi(m).$$

(C4) *There exist $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3$ such that*

$$f_{\gamma\rho_1}^{\rho_1} \geq \varphi(M\gamma), \quad f_0^{\rho_2} \leq \varphi(m), \quad u \neq Tu \text{ for } u \in \partial K_{\rho_2} \text{ and } f_{\gamma\rho_3}^{\rho_3} \geq \varphi(M\gamma).$$

Then (1.1) *has two positive solutions in K. Moreover if in (C3) $f_0^{\rho_1} \leq \varphi(m)$ is replaced by $f_0^{\rho_1} < \varphi(m)$, then* (1.1) *has a third positive solution $u_3 \in K_{\rho_1}$.*

Proof. The proof is similar to the one in [6, Theorem 2.10]. We omit it here. □

As a special case of Theorem 4.1 we obtain the following result.

Corollary 4.2. *If there exists $\rho > 0$ such that one of the following conditions holds:*

(C5) $0 \leq f^0 < \varphi(m)$, $f_{\gamma\rho}^{\rho} \geq \varphi(M\gamma)$, *$u \neq Tu$ for $u \in \partial\Omega_\rho$ and* $0 \leq f^\infty < \varphi(m)$,

(C6) $\varphi(M) < f_0 \leq \infty$, $f_0^\rho \geq \varphi(m)$, *$u \neq Tu$ for $u \in \partial K_\rho$ and* $\varphi(M) < f_\infty \leq \infty$.

Then (1.1) *has two positive solutions in K.*

Proof. We show that (C5) implies (C3). It is easy to verify that $0 \leq f^0 < \varphi(m)$ implies that there exist $\rho_1 \in (0, \gamma\rho)$ such that $f_0^{\rho_1} < \varphi(m)$. Let $a \in (f^\infty, \varphi(m))$. Then there exists $r > \rho$ such that $\sup_{t\in[0,+\infty)} f(t,(1+t)u,v) \leq a\varphi(u)$ for $u \in [r,\infty)$ since $0 \leq f^0 < \varphi(m)$. Let

$$\beta = \max\big\{\sup_{t\in[0,+\infty)} f(t,(1+t)u,v) : \ 0 \leq u \leq r, \ 0 \leq v \leq \frac{r}{\beta}\big\}$$

and

$$\rho_3 > \varphi^{-1}\big(\frac{\beta}{\varphi(m)-a}\big).$$

Then

$$\sup_{t\in[0,+\infty)} f(t,(1+t)u,v) \le a\varphi(u)+\beta \le a\varphi(\rho_3)+\beta < \varphi(m)\varphi(\rho_3)$$

for $u \in [0,\rho_3]$. This implies that $f_0^{\rho_3} < \varphi(m)$ and (C3) holds. Similarly, (C6) implies (C4). □

By an argument similar to that of Theorem 4.1 we obtain the following result.

Theorem 4.3. *Assume that one of the following conditions holds:*

- (C7) *There exist $\rho_1,\rho_2 \in (0,\infty)$ with $\rho_1 < \gamma\rho_2$ such that $f_0^{\rho_1} \le \varphi(m)$ and $f_{\gamma\rho_2}^{\rho_2} \ge \varphi(M\gamma)$.*
- (C8) *There exist $\rho_1,\rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that $f_{\gamma\rho_1}^{\rho_1} \ge \varphi(M\gamma)$ and $f_0^{\rho_2} \le \varphi(m)$.*

Then (1.1) *has a positive solution in K.*

As a special case of Theorem 4.3 we obtain the following result.

Corollary 4.4. *If there exists $\rho > 0$ such that one of the following conditions holds:*

- (C5) *$0 \le f^0 < \varphi(m)$, $f_{\gamma\rho}^{\rho} \ge \varphi(M\gamma)$, $u \ne Tu$ for $u \in \partial\Omega_\rho$ and $0 \le f^\infty < \varphi(m)$,*
- (C6) *$\varphi(M) < f_0 \le \infty$, $f_0^{\rho} \ge \varphi(m)$, $u \ne Tu$ for $u \in \partial K_\rho$ and $\varphi(M) < f_\infty \le \infty$.*

Then (1.1) *has two positive solutions in K.*

5. Examples

As an example we mention the boundary-value problem

$$\begin{gathered}(\varphi(-u'')(t))' = a(t)f(t,u,u'), \quad 0<t<+\infty,\\ u(0)-u'(0)=0,\\ u'(\infty)=0, \quad u''(0)=0,\end{gathered}\tag{5.1}$$

where

$$\varphi(u) = \begin{cases} \frac{u^3}{1+u^2}, & u \le 0,\\ u^2 & u>0,\end{cases}\tag{5.2}$$

and

$$f(t,u,v) = \begin{cases} 10^{-5}|\sin t| + \big(\frac{u}{1+t}\big)^9 + \frac{1}{100}\big(\frac{v}{1000}\big), & u\le 2,\\ 10^{-5}|\sin t| + \big(\frac{2}{1+t}\big)^9 + \frac{1}{100}\big(\frac{v}{1000}\big), & u \ge 2.\end{cases}$$

We choose $k=2$, $\beta=1$, $\delta(t)=t/2$ and so

$$\int_0^{+\infty} \varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds = 4, \int_{\frac12}^2 \varphi^{-1}\Big(\int_{\frac12}^s a(\tau)d\tau\Big)ds = 2.$$

It is easy to see by calculating that $\mu=1$, $\gamma=1/64$, $\gamma_1=1/16$ and

$$\begin{gathered}\frac{1}{m} = (1+\mu)\int_0^{+\infty}\varphi^{-1}\Big(\int_0^s a(\tau)d\tau\Big)ds = 8,\\ \frac{1}{M} = \delta\big(\frac12\big)\beta\int_{\frac12}^2 \varphi^{-1}\Big(\int_{\frac12}^s a(\tau)d\tau\Big)ds = \frac12.\end{gathered}$$

Thus $m=1/8$, $M=2$ and let $\rho_1=1/2$, $\rho_2=128$, $\rho_3=320$. After some simple calculation we have

$$f(t,(1+t)u,u') \le 10^{-5} + \frac{1}{512} + \frac12\times 10^{-5} < \frac{1}{256} = \varphi(m\rho_1) = \varphi(m)\varphi(\rho_1),$$

for all $(t,u,u') \in [0,+\infty) \times [0,1/2] \times [0,\frac{1}{2}]$. Therefore, $f_0^{\rho_1} < \varphi(m)$. On the other hand,

$$f(t,(1+t)u,u') \geq 2^9 = 512 > 16 = \varphi(M\gamma\rho_2) = \varphi(M)\varphi(\gamma\rho_2),$$

for all $(t,u,u') \in [1/2,2] \times [2,128] \times [0,128]$. We have $f_{\gamma\rho_2}^{\rho_2} > \varphi(M\gamma)$. At last

$$f(t,(1+t)u,u') \leq 10^{-5} + 2^9 + 320 \times 10^{-5} < 513 < 1600 = \varphi(m\rho_3) = \varphi(m)\varphi(\rho_3),$$

for all $(t,u,u') \in [0,+\infty) \times [0,320] \times [0,320]$. Thus we have $f_0^{\rho_3} < \varphi(m)$. Then the condition (C3) in Theorem 4.1 is satisfied. So boundary-value problem (5.1) has at least three positive solutions in K.

Remark 5.1. From (5.2), we can see that φ is not odd, therefore the boundary-value problem with p-Laplacian operator [10, 12, 13, 14] do not apply to (5.2). So we generalize a p-Laplace operator for some $p > 1$ and the function φ which we defined above is more comprehensive and general than p-Laplace operator.

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