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NONLINEAR SCALAR TWO-POINT BOUNDARY-VALUE PROBLEMS ON TIME SCALES

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Abstract. We establish sufficient conditions for the solvability of scalar nonlinear boundary-value problems on time scales. Our attention will be focused on problems where the solution space for the corresponding linear homogeneous boundary-value problem is nontrivial. As a consequence of our results we are able to provide easily verifiable conditions for the existence of periodic behavior for dynamic equations on time scales.

1. Introduction

This paper is devoted to the study of scalar nonlinear boundary-value problems on time scales. We examine the problem

$$u^{\Delta^n}(t) + a_{n-1}(t)u^{\Delta^{n-1}}(t) + \cdots + a_0(t)u(t) = q(t) + g(u(t)), \quad t \in [a,b]_{\mathbb{T}} \tag{1.1}$$

subject to

$$\sum_{j=1}^{n} b_{ij}u^{\Delta^{j-1}}(a) + \sum_{j=1}^{n} d_{ij}u^{\Delta^{j-1}}(b) = 0, \tag{1.2}$$

for $i = 1, 2, \dots, n$. Throughout this paper we will assume that $\mathbb{T}$ is a time scale and $[a,b]_{\mathbb{T}} \subset \mathbb{T}^{\kappa^n}$ where $[a,b]_{\mathbb{T}}$ will denote $\{t \in \mathbb{T} : a \le t \le b\}$. The functions $a_0, a_1, \dots, a_{n-1}$ and q are real-valued, rd-continuous functions defined on $\mathbb{T}$. The nonlinear term g is continuous, real-valued, and defined on $\mathbb{R}$. We will assume the solution space for the corresponding homogeneous boundary-value problem, namely,

$$u^{\Delta^n}(t) + a_{n-1}(t)u^{\Delta^{n-1}}(t) + \cdots + a_0(t)u(t) = 0, \quad t \in [a,b]_{\mathbb{T}} \tag{1.3}$$

subject to

$$\sum_{j=1}^{n} b_{ij}u^{\Delta^{j-1}}(a) + \sum_{j=1}^{n} d_{ij}u^{\Delta^{j-1}}(b) = 0, \quad \text{for } i = 1, 2, \dots, n, \tag{1.4}$$

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has dimension 1. Let $A(t)$ be the $n \times n$ matrix-valued function given by

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \dots & -a_{n-1}(t) \end{bmatrix}.$$

Clearly A is rd-continuous, and we assume A is also regressive. Let the matrices B and D be defined by $B = (b_{ij})$ and $D = (d_{ij})$. It should be observed that linear independence of the boundary conditions is equivalent to the matrix $[B|D]$ having full rank. To analyze the boundary-value problem (1.1)–(1.2) we will look at the equivalent $n \times n$ system,

$$x^{\Delta}(t) = A(t)x(t) + h(t) + f(x(t)), \quad t \in [a, b]_{\mathbb{T}} \tag{1.5}$$

subject to

$$Bx(a) + Dx(b) = 0 \tag{1.6}$$

where

$$[f(x)]_i = \begin{cases} 0 & \text{for } i = 1, 2, \dots n-1 \\ g([x]_1) & \text{for } i = n \end{cases}$$

and

$$[h(t)]_i = \begin{cases} 0 & \text{for } i = 1, 2, \dots n-1 \\ q(t) & \text{for } i = n \end{cases}.$$

Note that the solution space of

$$x^{\Delta}(t) = A(t)x(t), \quad t \in [a, b]_{\mathbb{T}} \tag{1.7}$$

subject to

$$Bx(a) + Dx(b) = 0 \tag{1.8}$$

has dimension one as a result of the assumption on (1.3)–(1.4). Through use of the Lyapunov-Schmidt Procedure conditions will be established to guarantee the existence of solutions to the boundary-value problem (1.5)–(1.6) and thus (1.1)–(1.2).

We will pay particular attention to second-order equations subject to periodic boundary conditions. We obtain results which significantly extend previous work by Etheridge and Rodríguez concerning the periodic behavior of nonlinear discrete dynamical systems[5].

2. Preliminaries

The notation and preliminary results presented here are a straightforward generalization of previous work in differential equations and discrete time systems [5, 15, 13, 14, 7, 6, 10]. We provide references concerning general information on time scales[2, 1, 3] as well as boundary-value problems[9, 16]. Let

$$X = \{x \in C[a, b]_{\mathbb{T}} : Bx(a) + Dx(b) = 0\},$$

and

$$Y = C_{\rm rd}[a, b]_{\mathbb{T}}$$

where $C_{\rm rd}[a, b]_{\mathbb{T}}$ denotes the space of rd-continuous $\mathbb{R}^n$-valued maps on $[a, b]_{\mathbb{T}}$, and $C[a, b]_{\mathbb{T}}$ denotes the subspace of $C_{\rm rd}[a, b]_{\mathbb{T}}$ where the maps are continuous. $|\cdot|$ will denote the Euclidean norm on $\mathbb{R}^n$. The operator norm will be used for matrices, and the supremum norm will be used for $x \in Y \cup X$, that is,

$$\|x\| = \sup_{t\in[a,b]_{\mathbb{T}}} |x(t)|.$$

It is clear that X and Y are Banach spaces with this norm. We define the norm of a product space, $V_1 \times V_2 \times \cdots \times V_m$, by

$$\|(v_1, v_2, \dots, v_m)\| = \sum_{i=1}^{m} \|v_i\|_i$$

where $\|\cdot\|_i$ denotes the norm on V_i.

We define the operator $L : D(L) \to Y$ where $D(L) = X \cap C^1_{\mathrm{rd}}([a, b]_{\mathbb{T}} \to \mathbb{R}^n)$ by

$$(Lx)(t) = x^{\Delta}(t) - A(t)x(t), \quad t \in [a, b]_{\mathbb{T}}$$

and the operator $F : X \to Y$ by

$$(Fx)(t) = f(x(t)), \quad t \in [a, b]_{\mathbb{T}}.$$

Clearly x is a solution to (1.5)–(1.6) if and only if $Lx = h + Fx$. Φ will denote the fundamental matrix solution for $x^{\Delta}(t) = A(t)x(t)$, $t \in [a, b]_{\mathbb{T}}$ where $\Phi(a) = I$.

Proposition 2.1. *The solution space for the homogeneous boundary-value problem* (1.7)–(1.8) *and the kernel of* $(B + D\Phi(b))$ *have the same dimension.*

Proof. The the solution space of (1.7)–(1.8) and kernel of L have the same dimension. $x \in \ker(L)$ if and only if $x^{\Delta}(t) = A(t)x(t)$, $t \in [a, b]_{\mathbb{T}}$ and x satisfies the boundary conditions. This is true if and only if there is a c in $\mathbb{R}^n$ such that $x(t) = \Phi(t)c$ for all $t \in [a, b]_{\mathbb{T}}$ and $Bc + D\Phi(b)c = 0$. It follows that the kernel of L and the kernel of $(B + D\Phi(b))$ have the same dimension. $\square$

Let d be a unit vector which spans the kernel of $(B+D\Phi(b))$. Define $S : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ by

$$S(t) = \Phi(t)d.$$

The following result is obvious.

Corollary 2.2. *labelcoro1 The kernel of* L *consists of* x *such that* $x(t) = S(t)\alpha$ *for some real number* α.

3. Main Result

We will now construct projections onto the kernel and image of L in order to use the Lyapunov-Schmidt Procedure [4, 5]. Define $P : X \to X$ by

$$(Px)(t) = S(t)d^T x(a), \quad t \in [a, b]_{\mathbb{T}}.$$

Proposition 3.1. *P is a projection onto the kernel of L.*

Proof. The fact that P is a bounded linear map is self-evident. The fact that P is idempotent can be shown through direct computation. It remains to be shown that $\mathrm{Im}(P) = \ker(L)$. Let $x \in X$. $(Px)(t) = S(t)d^T x(a) = S(t)\alpha$ where $\alpha = d^T x(a)$. Therefore $\mathrm{Im}(P) \subset \ker(L)$.

Let $x \in \ker(L)$. There exists a $\beta \in \mathbb{R}$ such that $x(t) = S(t)\beta$. $(Px)(t) = S(t)d^T x(a) = S(t)d^T S(a)\beta = S(t)\beta = x(t)$. Therefore $\ker(L) \subset \mathrm{Im}(P)$. $\square$

Let k be a vector that spans the kernel of $((B + D\Phi(b))^T)$. Define the map $\Psi : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ by

$$\Psi(t) = [D\Phi(b)\Phi^{-1}(\sigma(t))]^T k, \quad t \in [a, b]_{\mathbb{T}}.$$

Proposition 3.2. *y is in the image of L if and only if $\int_a^b y^T(\tau)\Psi(\tau)\Delta\tau = 0$.*

Proof. Using the variation of constants formula [2] and the boundary conditions it is clear that $y \in \operatorname{Im}(L)$ if and only if there exists $x \in X$ such that $(B+D\Phi(b))x(a)+ D\int_a^b \Phi(b)\Phi^{-1}(\sigma(\tau))y(\tau)\Delta\tau = 0$, which is equivalent to

$$-x^T(a)(B + D\Phi(b))^T = \Big[\int_a^b D\Phi(b)\Phi^{-1}(\sigma(\tau))y(\tau)\Big]^T \Delta\tau.$$

This holds if and only if $\int_a^b \big[D\Phi(b)\Phi^{-1}(\sigma(\tau))y(\tau)\big]^T \Delta\tau\beta = 0$ where β is an element of the kernel of $(B + D\Phi(b))^T$ and therefore must be a multiple of k. Therefore, $\int_a^b y^T(\tau)\Psi(\tau)\Delta\tau = 0$. □

Define the operator W from Y into Y by

$$(Wy)(t) = \Psi(t)\Big[\int_a^b |\Psi(\tau)|^2\Delta\tau\Big]^{-1} \int_a^b \Psi^T(\tau)y(\tau)\Delta\tau, \quad t \in [a, b]_{\mathbb{T}}.$$

Proposition 3.3. *E, defined by $E = I - W$, is a projection onto the image of L.*

Proof. First we will show that E is a projection. Since W is a bounded linear map E is also a bounded map. To prove $E^2 = E$ it will be sufficient to show that $W^2 = W$. Let $y \in Y$.

$$\begin{aligned}
&(W(Wy))(t)\\
&= W\Big(\Psi(\cdot)\Big[\int_a^b |\Psi(\tau)|^2\Delta\tau\Big]^{-1}\int_a^b \Psi^T(\tau)y(\tau)\Delta\tau\Big)(t), \quad t \in [a, b]_{\mathbb{T}}\\
&= \Psi(t)\Big[\int_a^b |\Psi(\tau)|^2\Delta\tau\Big]^{-1}\int_a^b \Psi^T(\tau)\Psi(\tau)\Delta\tau\Big[\int_a^b |\Psi(\nu)|^2\Delta\nu\Big]^{-1}\int_a^b \Psi^T(\nu)y(\nu)\Delta\nu\\
&= \Psi(t)\Big[\int_a^b |\Psi(\nu)|^2\Delta\nu\Big]^{-1}\int_a^b \Psi^T(\nu)y(\nu)\Delta\nu = (Wy)(t).
\end{aligned}$$

Finally we will prove that $\operatorname{Im}(E) = \operatorname{Im}(L)$. It is clear that $Ey \in \operatorname{Im}(E)$.

$$\begin{aligned}
&\int_a^b \Psi^T(\tau)(Ey)(\tau)\Delta\tau\\
&= \int_a^b \Psi^T(\tau)(y - Wy)(\tau)\Delta\tau\\
&= \int_a^b \Psi^T(\tau)y(\tau)\Delta\tau - \int_a^b \Psi^T(\tau)\Psi(\tau)\Delta\tau\Big[\int_a^b |\Psi(\nu)|^2\Delta\nu\Big]^{-1}\int_a^b \Psi^T(\nu)y(\nu)\Delta\nu = 0.
\end{aligned}$$

Therefore $Ey \in \operatorname{Im}(L)$, and $\operatorname{Im}(E) \subset \operatorname{Im}(L)$.

Now suppose $y \in \operatorname{Im}(L)$.

$$(Ey)(t) = y(t) - \Psi(t)\Big[\int_a^b |\Psi(\tau)|^2\Delta\tau\Big]^{-1}\int_a^b \Psi^T(\tau)y(\tau)\Delta\tau = y(t),$$

for all $t \in [a, b]_{\mathbb{T}}$. Therefore $y \in \operatorname{Im}(E)$, and $\operatorname{Im}(L) \subset \operatorname{Im}(E)$. □

By constructing the projections P and E we are now able to analyze the existence of solutions to (1.5)–(1.6) using the classic Lyapunov-Schmidt Procedure. We provide a self-contained presentation of our approach, but offer references [4, 8, 10, 11, 12] for a more general formulation and for applications to differential and difference equations. We can utilize the fact that P and E are projections and write

$$X = \mathrm{Im}(P) \oplus \mathrm{Im}(I-P) \quad \text{and} \quad Y = \mathrm{Im}(I-E) \oplus \mathrm{Im}(E).$$

For all $x \in X$ there exists $u \in \ker(L)$ and $v \in \mathrm{Im}(I-P)$ such that $x = u + v$. It is clear that $L : \mathrm{Im}(I-P) \cap D(L) \to \mathrm{Im}(L)$ is a bijection, and therefore there exists a bounded linear map $M : \mathrm{Im}(L) \to \mathrm{Im}(I-P) \cap D(L)$ such that

$$LMy = y, \forall y \in \mathrm{Im}(L) \quad \text{and} \quad MLx = v, \forall x \in X.$$

Define $H_1 : \mathbb{R} \times \mathrm{Im}(I-P) \to \mathbb{R}$ by

$$H_1(\alpha, v) = \alpha - \int_a^b g([\alpha S(\tau) + Mh(\tau) + MEF(S\alpha + v)(\tau)]_1)[\Psi(\tau)]_n \Delta\tau,$$

Define $H_2 : \mathbb{R} \times \mathrm{Im}(I-P) \to \mathrm{Im}(I-P)$ by

$$H_2(\alpha, v) = Mh + MEF(S\alpha + v).$$

Define $H : \mathbb{R} \times \mathrm{Im}(I-P) \to \mathbb{R} \times \mathrm{Im}(I-P)$ by

$$H(\alpha, v) = (H_1(\alpha, v), H_2(\alpha, v)).$$

Proposition 3.4. *$Lx = h + Fx$ if and only if there exists $(\alpha, v) \in \mathbb{R} \times \mathrm{Im}(I-P)$ such that $H(\alpha, v) = (\alpha, v)$.*

Proof. Let $x \in X$. There exist $\alpha \in \mathbb{R}$ and $v \in \mathrm{Im}(I-P)$ such that $x = S\alpha + v$ and

$$\begin{aligned}
Lx = h + Fx &\iff \begin{cases} E[Lx - h - Fx] = 0 \\ (I-E)[Lx - h - Fx] = 0 \end{cases} \\
&\iff \begin{cases} Lv - h - EF(x) = 0 \\ (I-E)F(x) = 0 \end{cases} \\
&\iff \begin{cases} v = Mh + MEF(S\alpha + v) \\ \int_a^b g([\alpha S(\tau) + Mh(\tau) + MEF(S\alpha + v)(\tau)]_1)[\Psi(\tau)]_n \Delta\tau = 0 \end{cases} \\
&\iff H(\alpha, v) = (\alpha, v).
\end{aligned}$$

□

Define $g(\pm\infty)$ as follows, provided the corresponding limits exist,

$$\lim_{x \to \pm\infty} g(x) = g(\pm\infty).$$

Proposition 3.5. *Assume g is continuous, $g(\infty)$ and $g(-\infty)$ exist, $[S(t)]_1$ is of one sign, and $g(\infty)g(-\infty) \int_a^b [\Psi(\tau)]_n \Delta\tau \neq 0$. Then*

$$\int_a^b g([\pm\alpha S(\tau) + Mh(\tau) + MEFx(\tau)]_1)[\Psi(\tau)]_n \Delta\tau \to g(\pm\infty) \int_a^b [\Psi(\tau)]_n \Delta\tau$$

as $\alpha \to \infty$.

Proof. We will show that

$$\int_a^b g([\alpha S(\tau) + Mh(\tau) + MEFx(\tau)]_1)[\Psi(\tau)]_n \Delta\tau \to g(\infty) \int_a^b [\Psi(\tau)]_n \Delta\tau$$

as $\alpha \to \infty$. The proof for the corresponding result with the opposite sign follows an analogous argument.

Let $\epsilon > 0$. Since Mh and MEF are bounded on $[a,b]_{\mathbb{T}}$ and S achieves its minimum on the set there exists $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$

$$|g(\infty) - g([\alpha S(t) + Mh(t) + MEFx(t)]_1)| < \epsilon.$$

Let $\alpha > \alpha_0$. Then

$$\begin{aligned}
&\Big| g(\infty) \int_a^b [\Psi(\tau)]_n \Delta\tau - \int_a^b g([\alpha S(\tau) + Mh(\tau) + MEFx(\tau)]_1)[\Psi(\tau)]_n \Delta\tau \Big| \\
&\leq \int_a^b |g(\infty) - g([\alpha S(\tau) + Mh(\tau) + MEFx(\tau)]_1)[\Psi(\tau)]_n| \, \Delta\tau \\
&\leq \epsilon \|\Psi\|(b-a).
\end{aligned}$$

Therefore, $\int_a^b g([\pm\alpha S(\tau)+Mh(\tau)+MEFx(\tau)]_1)[\Psi(\tau)]_n\Delta\tau \to g(\pm\infty)\int_a^b [\Psi(\tau)]_n\Delta\tau$ as $\alpha \to \infty$. □

Theorem 3.6. *Suppose that the kernel of $(B + D\Phi(b))$ is one dimensional. If*

(i) $[S(t)]_1$ *is of one sign for all* $t \in [a,b]_{\mathbb{T}}$,
(ii) $g : \mathbb{R} \to \mathbb{R}$ *is continuous,*
(iii) $g(\infty)$ *and* $g(-\infty)$ *exist,*
(iv) $g(\infty)g(-\infty)\big|\int_a^b [\Psi(\tau)]_n \Delta\tau\big| < 0$, *and*
(v) $\int_a^b h^T(\tau)\Psi(\tau)\Delta\tau = 0$

then there is at least one solution to the boundary-value problem (1.1)–(1.2).

Proof. For simplicity we will assume that $g(\infty) > g(-\infty)$ and $\int_a^b [\Psi(\tau)]_n \Delta\tau > 0$. Let $r = \sup_{z\in\mathbb{R}} |g(z)|$. Using Proposition 3.5 there is an $\alpha_0 > 0$ such that for $\alpha > \alpha_0$

$$\int_a^b g([S(\tau)\alpha + Mh(\tau) + MEF(S\alpha + v)(\tau)]_1)[\Psi(\tau)]_n \Delta\tau > 0,$$
$$\int_a^b g([S(\tau)(-\alpha) + Mh(\tau) + MEF(S\alpha + v)(\tau)]_1)[\Psi(\tau)]_n \Delta\tau < 0$$

for $v \in \operatorname{Im}(I - P)$. We now use Schauder's Fixed Point Theorem to prove the existence of a solution to (1.5)–(1.6). Let

$$\mathcal{B} = \{(v,\alpha) : \|v\| \leq \|Mh\| + \|ME\|r, \quad \text{and} \quad |\alpha| \leq \delta$$

where $\delta = \alpha_0 + r(b-a)\|\Psi\|\}$. Note that

$$\Big| \int_a^b g([S(\tau)(-\alpha) + Mh(\tau) + MEF(S\alpha + v)(\tau)]_1)[\Psi(\tau)]_n \Delta\tau \Big| \leq r(b-a)\|\Psi\|.$$

For $\alpha \in [0,\delta]$, we have

$$\begin{gathered}
-\delta \leq -r(b-a)\|\Psi\| \leq H_1(\alpha, v) \leq \alpha \leq \delta, \\
-\delta \leq -\alpha \leq H_1(-\alpha, v) \leq r(b-a)\|\Psi\| \leq \delta.
\end{gathered}$$

Now let $(v,\alpha) \in \mathcal{B}$. Then

$$\|H_2(v,\alpha)\| = \|Mh + MEF(S\alpha + v)\| \leq \|Mh\| + \|ME\|r.$$

Since $H(\mathcal{B}) \subset \mathcal{B}$ by the Schauder fixed point theorem there is at least one fixed point of H in $\mathcal{B}$. If $(\hat\alpha, \hat v)$ is this fixed point, then $\hat v = Mh + MEF\hat v$ and $\int_a^b g([\hat\alpha S(\tau) +$

$Mh(\tau) + MEF(\hat{\alpha}S + \hat{v})(\tau)]_1)[\Psi(\tau)]_n = 0$. By Proposition 3.4, $L(\hat{\alpha}S + \hat{v}) = h + F(\hat{\alpha}S + \hat{v})$, and therefore the boundary-value problem (1.5)–(1.6) has at least one solution. Thus (1.1)–(1.2) has at least one solution. □

4. Periodic Boundary Conditions

In this section we establish the existence of solutions to periodic boundary-value problems. We consider

$$u^{\Delta\Delta}(t) + \beta u^{\Delta}(t) + \gamma u(t) = q(t) + g(u(t)) \quad t \in [a, b]_{\mathbb{T}} \tag{4.1}$$

subject to

$$u(a) - u(a+T) = 0 \quad \text{and} \quad u^{\Delta}(a) - u^{\Delta}(a+T) = 0 \tag{4.2}$$

where $[a, a+T]_{\mathbb{T}} \subset \mathbb{T}^{\kappa^2}$ and $\beta, \gamma \in \mathbb{R}$ where $\gamma\mu - \beta$ is regressive. We will assume that the solution space of

$$u^{\Delta\Delta}(t) + \beta u^{\Delta}(t) + \gamma u(t) = 0 \quad t \in [a, a+T]_{\mathbb{T}} \tag{4.3}$$

subject to

$$u(a) - u(a+T) = 0 \quad \text{and} \quad u^{\Delta}(a) - u^{\Delta}(a+T) = 0 \tag{4.4}$$

is one-dimensional. Let

$$A = \begin{bmatrix} 0 & 1 \\ -\gamma & -\beta \end{bmatrix}.$$

It is easily verified that the kernel of $(I - \Phi(b))$ is one dimensional if and only if A has at least one zero eigenvalue.

First suppose A has real distinct eigenvalues, zero and λ. Now the solution to the corresponding homogeneous problem is $u(t) = c_1 + c_2 e_\lambda(t, a)$, where $e_\lambda(\cdot, a)$ denotes the time scale exponential function [2]. If we impose the boundary conditions we find that the solution space of this scalar homogeneous boundary-value problem is spanned by $u(t) = 1$ for $t \in [a, a+T]_{\mathbb{T}}$. Consequently the constant function $[1, 0]^T$ spans $\ker(L)$.

Now suppose A has a repeated eigenvalue of zero. The solution to the corresponding homogeneous problem is $u(t) = c_1 + c_2 t$. If we impose the boundary conditions we find that the solution space of this scalar homogeneous boundary-value problem is spanned by $u(t) = 1$ for $t \in [a, a+T]_{\mathbb{T}}$. Consequently the constant function $[1, 0]^T$ spans the $\ker(L)$ in this case as well.

We can now say that the solutions to the corresponding homogeneous boundary-value problem of (4.1)–(4.2) are real multiples of $[1, 0]^T$. Therefore, $[S(t)]_1$ is of one sign for all $t \in [a, a+T]_{\mathbb{T}}$.

Theorem 4.1. *If*

$$u^{\Delta\Delta}(t) + \beta u^{\Delta}(t) + \gamma u(t) = q(t) \quad t \in [a, a+T]_{\mathbb{T}}$$

subject to

$$u(a) - u(a+T) = 0 \quad \textit{and} \quad u^{\Delta}(a) - u^{\Delta}(a+T) = 0$$

has a solution and $g(\infty)$ and $g(-\infty)$ exist where $g(\infty)g(-\infty) < 0$ then there is at least one solution to equation (4.1)–(4.2).

The proof of this theorem follows from Theorem 3.6. It is easy to verify that the most significant results in Etheridge and Rodríguez [5] are a direct consequence of Theorem 4.1.

Corollary 4.2. *Suppose the conditions in Theorem 4.1 are satisfied. If*

(i) *q is periodic with period T*
(ii) *$\mathbb{T}$ is a periodic time scale with period T, meaning if $t \in \mathbb{T}$ then $t+T \in \mathbb{T}$*

then there exists at least one periodic solution to equation (4.1)–(4.2).

Proof. Let x be a solution to (4.1)–(4.2). Since g is bounded and q is periodic it is clear that the solution x exists on all of $\mathbb{T}$. Let $x(t+T) = y(t)$. y satisfies the dynamic equation (4.1), $y(a) = x(a+T) = x(a)$, and $y^{\Delta}(a) = x^{\Delta}(a+T) = x^{\Delta}(a)$. Therefore by uniqueness $x(t) = x(t+T)$. □

5. Example

In this section we examine the following second-order nonlinear boundary-value problem on several time scales. consider

$$u^{\Delta\Delta}(t) + \beta u^{\Delta}(t) + \gamma u(t) = g(u(t)) \quad t \in [a,b]_{\mathbb{T}} \tag{5.1}$$

subject to

$$B\begin{bmatrix} u(a) \\ u^{\Delta}(a) \end{bmatrix} + D\begin{bmatrix} u(b) \\ u^{\Delta}(b) \end{bmatrix} = 0 \tag{5.2}$$

where $\beta, \gamma \in \mathbb{R}$ and $\gamma\mu - \beta$ is regressive, $[a,b]_{\mathbb{T}} \in \mathbb{T}^{\kappa^2}$, B and D are 2×2 real matrices, and $g : \mathbb{R} \to \mathbb{R}$ is continuous. The scalar boundary-value problem (5.1)–(5.2) is equivalent to the 2×2 system

$$x^{\Delta}(t) = Ax(t) + f(x(t)) \quad t \in [a,b]_{\mathbb{T}} \tag{5.3}$$

subject to

$$Bx(a) + Dx(b) = 0 \tag{5.4}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\gamma & -\beta \end{bmatrix}, \quad f(x) = \begin{bmatrix} 0 \\ g(x_1) \end{bmatrix}, \quad x = \begin{bmatrix} u \\ u^{\Delta} \end{bmatrix}.$$

Suppose d is the vector that spans the kernel of $(B + D\Phi(b))$ and A has real, distinct eigenvalues, λ_1 and λ_2, where $\lambda_1 > \lambda_2$ and both are positively regressive; i.e., $1 + \lambda_k\mu > 0$. Further assume that the eigenpairs for A are given by (λ_1, v) and (λ_2, w). Let

$$\hat{\Phi}(t) = \begin{bmatrix} v_1 e_{\lambda_1}(t,a) & w_1 e_{\lambda_2}(t,a) \\ v_2 e_{\lambda_1}(t,a) & w_2 e_{\lambda_2}(t,a) \end{bmatrix}.$$

It is clear that

$$S(t) = \hat{\Phi}(t)\hat{\Phi}^{-1}(a)d = \hat{\Phi}(t)\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} v_1 d_1 e_{\lambda_1}(t,a) + w_1 d_2 e_{\lambda_2}(t,a) \\ v_2 d_1 e_{\lambda_1}(t,a) + w_2 d_2 e_{\lambda_2}(t,a) \end{bmatrix}.$$

We will provide conditions under which S_1 will be of one sign. It is clear that if v_1, w_1, d_1, or d_2 are zero then $S_1(t)$ is either identically zero or of one sign. Now we investigate the case when v_1, w_1, d_1, and d_2 are all nonzero. $S_1(t)$ will be of one sign on $[a,b]_{\mathbb{T}}$ if and only if $v_1 d_1 e_{\lambda_1}(t,a) + w_1 d_2 e_{\lambda_2}(t,a)$ is of one sign for all $t \in [a,b]_{\mathbb{T}}$. This holds when either

$$\frac{e_{\lambda_1}(t,a)}{e_{\lambda_2}(t,a)} > -\frac{w_1 d_2}{v_1 d_1}, \quad \text{for all } t \in [a,b]_{\mathbb{T}}$$

or

$$\frac{e_{\lambda_1}(t,a)}{e_{\lambda_2}(t,a)} < -\frac{w_1 d_2}{v_1 d_1}, \quad \text{for all } t \in [a,b]_{\mathbb{T}}.$$

It is easy to see that $\dfrac{e_{\lambda_1}(t,a)}{e_{\lambda_2}(t,a)} > 1$ for any time scale. To obtain further results we consider specific time scales.

The first time scale we will discuss is given by

$$\mathbb{T}_1 = \big\{[1-\frac{1}{2^{2n}}, 1-\frac{1}{2^{2n+1}}] : n = 0,1,2,\dots\big\} \cup \{1\}.$$

For simplicity we assume that $a = 0$ and $b = 1$.

$$e_{\lambda_k}(t,0) = \exp\Big\{\lambda_k\Big[t - \sum_{i=0}^{l-1}\frac{1}{2^{2i+2}}\Big]\Big\}\prod_{i=0}^{l-1}(1+\frac{1}{2^{2i+2}}\lambda_k)$$

where $t \in \big[1-\frac{1}{2^{2l}}, 1-\frac{1}{2^{2l+1}}\big]$ and $k = 1,2$. Let $t \in [1-\frac{1}{2^{2l}}, 1-\frac{1}{2^{2l+1}}]$ where $l \in \mathbb{Z}^+ \cup \{0\}$. Observe that

$$\begin{aligned}
1 < \frac{e_{\lambda_1}(t,0)}{e_{\lambda_2}(t,0)} &= \exp\Big\{(\lambda_1-\lambda_2)\Big[t - \sum_{i=0}^{l-1}\frac{1}{2^{2i+2}}\Big]\Big\}\prod_{i=0}^{l-1}\frac{(1+\frac{1}{2^{2i+2}}\lambda_1)}{(1+\frac{1}{2^{2i+2}}\lambda_2)}\\
&< \exp\Big\{(\lambda_1-\lambda_2)\Big[1 - \sum_{i=0}^{\infty}\frac{1}{2^{2i+2}}\Big]\Big\}\big(\frac{1+\lambda_1}{1+\lambda_2}\big)^l\\
&= \exp\big\{(\lambda_1-\lambda_2)(\frac{1}{3})\big\}\big(\frac{1+\lambda_1}{1+\lambda_2}\big)^l.
\end{aligned}$$

Therefore, $S_1(t)$ will be of one sign on $[0,1]$ when

$$1 > -\frac{w_1d_2}{v_1d_1} \quad\text{or}\quad \exp\big\{(\lambda_1-\lambda_2)(\frac{1}{3})\big\}\big(\frac{1+\lambda_1}{1+\lambda_2}\big)^l < -\frac{w_1d_2}{v_1d_1} \quad \text{for } l = 0,1,2\dots.$$

Now we consider the time scale

$$\mathbb{T}_2 = \{[2n, 2n+1] : n = 0,1,2,\dots\}.$$

Let $a = 0$ and $b > 0$ where $b \in [1-\frac{1}{2^{2N}}, 1-\frac{1}{2^{2N+1}}]$ where $N \in \mathbb{Z}^+ \cup \{0\}$.

$$e_{\lambda_k}(t,0) = \exp\{\lambda_k(t-l)\}(1+\lambda_k)^l$$

where $t \in [2l, 2l+1]$ and $k = 1,2$. Let $t \in [1-\frac{1}{2^{2l}}, 1-\frac{1}{2^{2l+1}}]$ where $l \in \mathbb{Z}^+ \cup \{0\}$. Note that

$$\begin{aligned}
\exp\{(\lambda_1-\lambda_2)(b-N)\}\big(\frac{1+\lambda_1}{1+\lambda_2}\big)^N &\ge \frac{e_{\lambda_1}(t,0)}{e_{\lambda_2}(t,0)}\\
&= \exp\{(\lambda_1-\lambda_2)(t-l)\}\big(\frac{1+\lambda_1}{1+\lambda_2}\big)^l > 1.
\end{aligned}$$

Therefore, $S_1(t)$ will be of one sign on $[0,b]$ when

$$1 > -\frac{w_1d_2}{v_1d_1} \quad\text{or}\quad \exp\{(\lambda_1-\lambda_2)(b-N)\}\big(\frac{1+\lambda_1}{1+\lambda_2}\big)^N < -\frac{w_1d_2}{v_1d_1}.$$

Finally consider the time scale

$$\mathbb{T}_3 = \{2^n : n = 0,1,2,\dots\}.$$

Let $a = 1$ and $b = 2^N$ where $N \in \mathbb{Z}^+$.

$$e_{\lambda_k}(t,1) = \prod_{i=0}^{l-1}(1+2^i\lambda_k)$$

where $t = 2^l$ and $k = 1, 2$. Let $t = 2^l$ where $l \in \mathbb{Z}^+ \cup \{0\}$. Observe that

$$(\frac{1 + 2^{N-1}\lambda_1}{1 + 2^{N-1}\lambda_2})^N \geq \prod_{i=0}^{l-1} (\frac{1 + 2^i\lambda_1}{1 + 2^i\lambda_2}) = \frac{e_{\lambda_1}(t, 1)}{e_{\lambda_2}(t, 1)} > 1.$$

Therefore, $S_1(t)$ will be of one sign on $[0, b]$ when

$$1 > -\frac{w_1 d_2}{v_1 d_1} \quad \text{or} \quad \Big(\frac{1 + 2^{N-1}\lambda_1}{1 + 2^{N-1}\lambda_2}\Big)^N < -\frac{w_1 d_2}{v_1 d_1}.$$

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