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PERIODIC SOLUTIONS FOR A SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATION WITH VARIABLE PARAMETER AND MULTIPLE DEVIATING ARGUMENTS

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Abstract. By employing the continuation theorem of coincidence degree theory developed by Mawhin, we obtain periodic solution for a class of neutral differential equation with variable parameter and multiple deviating arguments.

1. Introduction

Neutral functional differential equations (in short NFDEs) are an important research subject of functional differential equations and provide good models in many fields including physics, mechanics, biology and economics (see [1, 2, 3, 4, 5]). With such clear indications of the importance of NFDEs in the applications, it is not surprising that the subject has undergone a rapid development in the previous twenty years. Particularly, in recent years the problems of periodic solution for second-order NFDEs have been studied by many authors. In [6], by employing the continuation theorem of coincidence degree theory, Lu and Ge studied the following second-order NFDE:

$$(x(t) + cx(t-r))'' + f(x'(t)) + g(x(t-\tau(t))) = p(t).$$

After that, Lu and Gui [7] went still one step further to study the above equation in the critical case and obtained more profound results. Furthermore, in [8] Lu and Ren investigated the second-order NFDE with multiple deviating arguments as follows:

$$\frac{d^2}{dt^2}(u(t) - ku(t-\tau)) = f(u(t))u'(t) + \alpha(t)g(u(t)) + \Sigma_{j=1}^n \beta_j(t)g(u(t-\gamma_j(t))) + p(t).$$

The authors used new techniques and methods for multiple deviating arguments and obtained some new results. In very recent years, p-Laplacian NFDEs were studied by some researchers. In [9]-[10], Zhu and Lu studied the following p-Laplacian NFDEs:

$$(\varphi_p[(x(t) - cx(t-\sigma))'])' + g(t, x(t-\tau(t))) = e(t)$$

and

$$(\varphi_p[(x(t) - cx(t-\sigma))'])' = f(x(t))x'(t) + \Sigma_{j=1}^n \beta_j(t)g(x(t-\gamma_j(t))) + p(t).$$

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However, for all the above papers they obtained the existence of periodic solution to NFDEs based on the properties of neutral operator A. In 1995, Zhang [11] obtained the following results. Define A on C_T

$$A : C_T \to C_T, [Ax](t) = x(t) - cx(t-\tau), \forall t \in \mathbb{R},$$

where $C_T = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t)\}$, c is constant. When $|c| \neq 1$, then A has a unique continuous bounded inverse A^{-1} satisfying

$$[A^{-1}f](t) = \begin{cases} \sum_{j\geq 0} c^j f(t-j\tau), & \text{if } |c| < 1, \ \forall f \in C_T, \\ -\sum_{j\geq 1} c^{-j} f(t+j\tau), & \text{if } |c| > 1, \ \forall f \in C_T. \end{cases}$$

Obviously, we have

(1) $\|A^{-1}\| \leq \frac{1}{|1-|c||}$;
(2) $\int_0^T |[A^{-1}f](t)|dt \leq \frac{1}{|1-|c||} \int_0^T |f(t)|dt, \forall f \in C_T$;
(3) $\int_0^T |[A^{-1}f](t)|^2 dt \leq \frac{1}{|1-|c||} \int_0^T |f(t)|^2 dt, \forall f \in C_T$.

When c is a variable $c(t)$, we have obtained the properties of the neutral operator $A : C_T \to C_T$, $[Ax](t) = x(t) - c(t)x(t-\tau)$ in [12]. We note that there are few results on the existence of periodic solutions to second-order neutral equations for the cases of a variable $c(t)$. The purpose of this article is to investigate the existence of periodic solution for the second-order NFDE with variable parameter and multiple deviating arguments by using the properties of the operator A in [12] and Mawhin's continuation theorem. Here we use the same technique, but our results extend and complement the existing ones. We will study the following NFDE:

$$(x(t) - c(t)x(t-\tau))'' + \sum_{j=1}^{n} \beta_j(t) g(x(t-\gamma_j(t))) = e(t), \tag{1.1}$$

where $g \in C(\mathbb{R}, \mathbb{R})$; $c \in C^2(\mathbb{R}, \mathbb{R})$ with $c(t) = c(t+T)$ and $|c(t)| \neq 1$; $e(t), \beta_j(t), \gamma_j(t)$ are $T-$periodic functions on $\mathbb{R}$ $(j = 1, 2, \dots, n)$; $\tau, T > 0$ are given constants.

In this article, we assume that $e(t)$ is not a constant function on $\mathbb{R}$. Furthermore, we suppose that $\gamma_j \in C^1(\mathbb{R}, \mathbb{R})$ with $\gamma_j'(t) < 1, \forall t \in \mathbb{R}, (j = 1, 2, \dots, n)$. It is obvious that the function $t - \gamma_j(t)$ has a unique inverse denoted by $\mu_j(t)$, $(j = 1, 2, \dots, n)$. Let

$$\Gamma(t) = \Sigma_{j=1}^{n} \frac{\beta_j(\mu_j(t)}{1-\gamma_j'(\mu_j(t))}, \quad \bar{h} = \frac{1}{T}\int_0^T h(s)ds.$$

2. Preliminary

In this section, we give some lemmas which will be used in this paper.

Lemma 2.1 ([12]). *If $|c(t)| \neq 1$, then operator A has continuous inverse A^{-1} on C_T, satisfying: (1)*

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} c(t-(i-1)\tau) f(t-j\tau), & c_0 < 1, \forall f \in C_T, \\ -\frac{f(t+\tau)}{c(t+\tau)} - \sum_{j=1}^{\infty} \prod_{i=1}^{j+1} \frac{1}{c(t+i\tau)} f(t+j\tau+\tau), & \sigma > 1, \forall f \in C_T, \end{cases}$$

(2)

$$\int_0^T |[A^{-1}f](t)|dt \leq \begin{cases} \frac{1}{1-c_0} \int_0^T |f(t)|dt, & c_0 < 1, \forall f \in C_T, \\ \frac{1}{\sigma-1} \int_0^T |f(t)|dt, & \sigma > 1, \forall f \in C_T, \end{cases}$$

where

$$c_0 = \max_{t\in[0,T]} |c(t)|, \quad \sigma = \min_{t\in[0,T]} |c(t)|, \quad c_1 = \max_{t\in[0,T]} |c'(t)|.$$

Let X and Y be two Banach spaces and let $L : D(L) \subset X \to Y$ be a a linear operator, Fredholm operator with index zero (meaning that $\operatorname{Im} L$ is closed in Y and $dim \ker L = codim \operatorname{Im} L < +\infty$. If L is a Fredholm operator with index zero, then there exist continuous projectors $P : X \to X, Q : Y \to Y$ such that $\operatorname{Im} P = \ker L, \operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$ and $L_{D(L)\cap \ker P} : (I - P)X \to \operatorname{Im} L$ is invertible. Denote by K_p the inverse of L_P.

Let Ω be an open bounded subset of X, a map $N : \bar{\Omega} \to Y$ is said to be L-compact in $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and the operator $K_p(I - Q)N(\bar{\Omega})$ is relatively compact. We first give the famous Mawhin's continuation theorem.

Lemma 2.2 ([13]). *Suppose that X and Y are Banach spaces, and $L : D(L) \subset X \to Y$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$. if all the following conditions hold:*

1. *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, and all $\lambda \in (0, 1)$,*
2. *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \ker L$,*
3. *$\deg\{QN, \Omega \cap \ker L, 0\} \neq 0$,*

Then the equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap D(L)$.

Define the linear operator $L : D(L) \subset C_T \to C_T$ as $Lx = (Ax)''$, and a nonlinear operator $N : C_T \to C_T$,

$$Nx = -\sum_{j=1}^{n} \beta_j(t)g(x(t - \gamma_j(t))) + e(t),$$

where $D(L) = \{x|x \in C_T^1\}$. For $x \in \ker L$, we have $(x(t) - c(t)x(t - \tau))'' = 0$. Then

$$x(t) - c(t)x(t - \tau) = \tilde{c_1}t + \tilde{c_2},$$

where $\tilde{c_1}, \tilde{c_2} \in \mathbb{R}$. Since $x(t) - c(t)x(t - \tau) \in C_T$, then $\tilde{c_1} = 0$. Let $\varphi(t)$ be a solution of $x(t) - c(t)x(t - \tau) = 1$ and $\int_0^T \varphi^2(t)dt \neq 0$. We get

$$\ker L = \{a_0\varphi(t), a_0 \in \mathbb{R}\}, \operatorname{Im} L = \{y|y \in C_T, \int_0^T y(s)ds = 0\}.$$

Obviously, $\operatorname{Im} L$ is a closed in C_T and $\dim \ker L = \operatorname{codim} \operatorname{Im} L = 1$, So L is a Fredholm operator with index zero. Define continuous projectors P, Q

$$P : C_T \to \ker L, \quad (Px)(t) = \frac{\int_0^T x(t)\varphi(t)dt}{\int_0^T \varphi^2(t)dt}\varphi(t),$$

$$Q : C_T \to C_T/\operatorname{Im} L, \quad Qy = \frac{1}{T}\int_0^T y(s)ds.$$

Let

$$L_P = L|_{D(L)\cap \ker P} : D(L) \cap \ker P \to \operatorname{Im} L,$$

then

$$L_P^{-1} = K_p : \operatorname{Im} L \to D(L) \cap \ker P.$$

Since $\operatorname{Im} L \subset C_T$ and $D(L) \cap \ker P \subset C_T^1$, so K_p is an embedding operator. Hence K_p is a completely operator in $\operatorname{Im} L$. By the definitions of Q and N, it follows that $QN(\bar{\Omega})$ is bounded on $\bar{\Omega}$. Hence nonlinear operator N is L-compact on $\overline{\Omega}$.

3. Existence of periodic solution for (1.1)

For convenience when applying Lemma 2.1 and Lemma 2.2, we introduce some notation and sate some assumptions:

$$C_T = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t),\ \forall t \in \mathbb{R}\},$$
$$|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|, \quad \forall \varphi \in C_T,$$
$$C_T^1 = \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t),\ \forall t \in \mathbb{R}\},$$
$$\|\varphi\| = \max_{t\in[0,T]} \{|\varphi|_0, |\varphi'|_0\}, \quad \forall \varphi \in C_T^1,$$

where $|\cdot|_0$ and $\|\cdot\|$ are the norms of C_T and C_T^1 respectively. Obviously, C_T, C_T^1 are both Banach space.

(H1) $\Gamma(t) > 0$, for all $t \in \mathbb{R}$;
(H2) $\lim_{|x|\to+\infty} \frac{|g(x)|}{|x|} \le r \in [0, \infty)$;
(H3) There exists a positive constant d such that $xg(x) > 0$, whenever $|x| > d$.

Theorem 3.1. *Suppose that $\int_0^T e(s)ds = 0$, $\int_0^T \varphi^2(s)ds \neq 0$, $|c(t)| \neq 1$ for all $t \in \mathbb{R}$, and assumptions* (H1)–(H3) *hold, where $\varphi(t)$ is a solution of $x(t)-c(t)x(t-\tau) = 1$. Then* (1.1) *has at least one T-periodic solution, if*

$$\frac{T^{1/2}}{1-c_0}\sqrt{T\sum_{j=1}^{n} |\beta_j|_0(1+c_0)r} + \frac{c_1 T}{1-c_0} < 1 \quad \text{for } c_0 < \frac{1}{2},$$

or if

$$\frac{T^{1/2}}{\sigma-1}\sqrt{T\sum_{j=1}^{n} |\beta_j|_0(1+c_0)r} + \frac{c_1 T}{\sigma-1} < 1 \quad \text{for } \sigma > 1.$$

Proof. Take $\Omega_1 = \{x \in D(L) : Lx = \lambda Nx, \lambda \in (0, 1)\}$. For $x \in \Omega_1$, we have

$$(x(t) - c(t)x(t-\tau))'' + \lambda \sum_{j=1}^{n} \beta_j(t)g(x(t-\gamma_j(t))) = \lambda e(t). \tag{3.1}$$

We claim that there exists a point $\xi \in \mathbb{R}$ such that

$$|x(\xi)| \le d, \tag{3.2}$$

where d is a constant which is independent with λ. Integrating two sides of (3.1) over the interval $[0, T]$,

$$\sum_{j=1}^{n} \int_0^T \beta_j(t)g(x(t-\gamma_j(t)))dt = 0;$$

i.e.,

$$\int_0^T \Gamma(t)g(x(t))dt = 0.$$

By mean value theorem for integrals, there exists a point $\xi_1 \in [0, T]$ such that

$$g(x(\xi_1))\bar{\Gamma}T = 0.$$

By $\bar{\Gamma} \neq 0$, then $g(x(\xi_1)) = 0$. From the assumption (H3), the inequality (3.2) holds. Furthermore we have

$$|x(t)| \leq d + \int_0^T |x'(t)|dt. \tag{3.3}$$

From the conditions

$$\frac{T^{1/2}}{1-c_0}\sqrt{T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)r} + \frac{c_1 T}{1-c_0} < 1,$$

and

$$\frac{T^{1/2}}{\sigma-1}\sqrt{T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)r} + \frac{c_1 T}{\sigma-1} < 1,$$

there exists a constant $\varepsilon_1 > 0$ such that

$$\frac{T^{1/2}}{1-c_0}\sqrt{T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)(r+\varepsilon_1)} + \frac{c_1 T}{1-c_0} < 1, \tag{3.4}$$

or

$$\frac{T^{1/2}}{\sigma-1}\sqrt{T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)(r+\varepsilon_1)} + \frac{c_1 T}{\sigma-1} < 1. \tag{3.5}$$

For such a constant ε_1, by (H2), there exists a constant $\rho > 0$ such that

$$|g(u)| \leq (r+\varepsilon_1)|u|, \quad |u| > \rho > d. \tag{3.6}$$

Let

$$E_{1j} = \{t | t \in [0,T], |x(t-\gamma_j(t))| \leq \rho\}, \quad E_{2j} = \{t | t \in [0,T], |x(t-\gamma_j(t))| > \rho\},$$

for $j = 1, 2\dots, n$. Multiplying both sides of (3.1) by $(Ax)(t)$ and integrating over $[0,T]$, from (3.3) and (3.6), we obtain

$$\begin{aligned}
&\int_0^T |(Ax)'(t)|^2 dt \\
&= \lambda\int_0^T \sum_{j=1}^{n}\beta_j(t)g(x(t-\gamma_j(t)))(Ax)(t)dt - \lambda\int_0^T e(t)(Ax)(t)dt \\
&\leq |Ax|_0 \int_{E_{1j}} \sum_{j=1}^{n} |\beta_j(t)||g(x(t-\gamma_j(t)))|dt \\
&\quad + |Ax|_0 \int_{E_{2j}} \sum_{j=1}^{n} |\beta_j(t)||g(x(t-\gamma_j(t)))|dt + T|Ax|_0|e|_0 \\
&\leq T\sum_{j=1}^{n}|\beta_j|_0 g_\rho |Ax|_0 + T\sum_{j=1}^{n}|\beta_j|_0|Ax|_0(r+\varepsilon_1)|x|_0 + T|Ax|_0|e|_0 \\
&\leq \Big(T\sum_{j=1}^{n}|\beta_j|_0 g_\rho(1+c_0) + T|e|_0(1+c_0)\Big)|x|_0 + T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)(r+\varepsilon_1)|x|_0^2 \\
&\leq k_1\int_0^T |x'(t)|dt + k_2\Big(\int_0^T |x'(t)|dt\Big)^2 + k_3,
\end{aligned} \tag{3.7}$$

where

$$g_\rho = \max_{|x(t-\gamma_j(t))|\le\rho} |g(x(t-\gamma_j(t)))|,$$

$$k_1 = T\sum_{j=1}^{n} |\beta_j|_0 g_\rho(1+c_0) + T|e|_0(1+c_0) + 2T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)(r+\varepsilon_1)d,$$

$$k_2 = T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)(r+\varepsilon_1),$$

$$k_3 = T\sum_{j=1}^{n}|\beta_j|_0 g_\rho(1+c_0)d + T|e|_0(1+c_0)d + T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)(r+\varepsilon_1)d^2.$$

From $(Ax')(t) = (Ax)'(t) + c'(t)x(t-\tau)$, (3.7) and Lemma 2.1, if $c_0 < \frac{1}{2}$, we have

$$\begin{aligned}
\int_0^T |x'(t)|dt &= \int_0^T |(A^{-1}Ax')(t)|dt \\
&\le \frac{1}{1-c_0}\int_0^T |(Ax')(t)|dt \\
&\le \frac{1}{1-c_0}\int_0^T |(Ax)'(t)|dt + \frac{c_1T}{1-c_0}|x|_0 \\
&\le \frac{T^{1/2}}{1-c_0}\Big(\int_0^T |(Ax)'(t)|^2dt\Big)^{1/2} + \frac{c_1T}{1-c_0}\int_0^T |x'(t)|dt + \frac{c_1Td}{1-c_0} \\
&\le \frac{T^{1/2}}{1-c_0}\Big[k_1\int_0^T |x'(t)|dt + k_2\Big(\int_0^T |x'(t)|dt\Big)^2 + k_3\Big]^{1/2} \\
&\quad + \frac{c_1T}{1-c_0}\int_0^T |x'(t)|dt + \frac{c_1Td}{1-c_0}.
\end{aligned}$$

By (3.4), there exists a constant $M_1 > 0$ which is independent with λ such that

$$\int_0^T |x'(t)|dt \le M_1.$$

Similarly, for $\sigma > 1$, by (3.5), there exists a constant$M_1' > 0$ which is independent with λ such that

$$\int_0^T |x'(t)|dt \le M_1'.$$

Combining (3.3) with the above two inequalities, we obtain

$$|x|_0 \le d + \max\{M_1, M_1'\} := M_2.$$

From

$$(Ax'')(t) = (Ax)''(t) + 2c'(t)x'(t-\tau) + c''(t)x(t-\tau),$$

if $c_0 < 1/2$, we have

$$\begin{aligned}
&\int_0^T |x''(t)|dt \\
&= \int_0^T |[A^{-1}Ax''](t)|dt
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^T \frac{|(Ax'')(t)|}{1-c_0}dt\\
&= \int_0^T \frac{|(Ax)''(t) + 2c'(t)x'(t-\tau) + c''(t)x(t-\tau)|}{1-c_0}dt\\
&\le \frac{1}{1-c_0}\Big(\int_0^T \sum_{j=1}^n |\beta_j(t)||g(x(t-\gamma_j(t)))|dt + \int_0^T |e(t)|dt + 2c_1M_1 + c_2M_2T\Big)\\
&\le \frac{1}{1-c_0}(\sum_{j=1}^n |\beta_j|_0 T g_{M_2} + T|e|_0 + 2c_1M_1 + c_2M_2T) := M_3;
\end{aligned}
$$

if $\sigma > 1$, we have

$$
\int_0^T |x''(t)|dt \le \frac{1}{\sigma - 1}(\sum_{j=1}^n |\beta_j|_0 T g_{M_2} + T|e|_0 + 2c_1M_1 + c_2M_2T) := M_3',
$$

where $g_{M_2} = \max_{|x|\le M_2} |g(x)|, c_2 = \max_{t\in[0,T]} |c''(t)|$. Since $x \in \Omega_1$, so $x(0) = x(T)$ and there exists a point $\eta \in [0,T]$ such that $x'(\eta) = 0$. Then

$$
x'(t) = x'(\eta) + \int_\eta^t x''(s)\mathrm{d}s,
$$

$$
|x'|_0 \le \int_0^T |x''(t)|dt \le \max\{M_3, M_3'\} := M_4.
$$

Then

$$
\|x\| = \max_{t\in[0,T]}\{|x|_0, |x'|_0\} \le \max\{M_2, M_4\}.
$$

Hence Ω_1 is bounded.

Take $\Omega_2 = \{x \in \ker L \cap C_T^1 : Nx \in \operatorname{Im} L\}$, for all $x \in \Omega_2$, then $x(t) = a_0\varphi(t)$, $a_0 \in \mathbb{R}$ satisfying

$$
\int_0^T \Gamma(t)g(a_0\varphi(t))dt = 0. \tag{3.8}
$$

When $c_0 < 1/2$, we have

$$
\begin{aligned}
\varphi(t) = A^{-1}(1) &= 1 + \sum_{j=1}^\infty \prod_{i=1}^j c(t-(i-1)\tau)\\
&\ge 1 - \sum_{j=1}^\infty \prod_{i=1}^j c_0\\
&= 1 - \frac{c_0}{1-c_0}\\
&= \frac{1-2c_0}{1-c_0} := \delta_1 > 0.
\end{aligned}
$$

Then we have $a_0 \le d/\delta_1$. Otherwise, for all $t \in [0,T]$, $a_0\varphi(t) > d$, from assumption (H3), we have

$$
\int_0^T \Gamma(t)g(a_0\varphi(t))dt > 0
$$

which is contradiction to (3.8). When $\sigma > 1$, we have

$$\begin{aligned}\varphi(t) = A^{-1}(1) &= -\frac{1}{c(t+\tau)} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{c(t+i\tau)}\\ &\le -\frac{1}{\sigma} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{\sigma}\\ &= -\frac{1}{\sigma-1} := \delta_2 < 0.\end{aligned}$$

Then we have $a_0 \le -d/\delta_2$. Otherwise, for all $t \in [0,T]$, $a_0\varphi(t) < -d$, from assumption (H3), we have

$$\int_0^T \Gamma(t)g(a_0\varphi(t))dt < 0$$

which is contradiction to (3.8). Then we have

$$|x| = |a_0\varphi(t)| \le \max\{\frac{d}{\delta_1}, -\frac{d}{\delta_2}\}|\varphi|_0.$$

Hence Ω_2 is a bounded set.

Let $\Omega \supset \Omega_1 \cup \Omega_2$ be a bounded set. For $x \in \partial\Omega \cup D(L)$, $\forall \lambda \in (0,1)$, we have $Lx \ne \lambda Nx$. For all $x \in \partial\Omega \cap \ker L$, we have $Nx \notin \operatorname{Im} L$. Hence the conditions (1) and (2) of Lemma 2.2 hold. It remains to verify conditions (3) of Lemma 2.2. Now, for $x \in \partial\Omega \cap \ker L$, take the homotopy

$$H(x,\mu) = \begin{cases} -\mu x - \frac{1}{T}(1-\mu)\int_0^T \sum_{j=1}^n \beta_j(t)g(x)dt, & \text{if } (\sum_{j=1}^n \bar\beta_j)xg(x) > 0;\\ \mu x - \frac{1}{T}(1-\mu)\int_0^T \sum_{j=1}^n \beta_j(t)g(x)dt, & \text{if } (\sum_{j=1}^n \bar\beta_j)xg(x) < 0.\end{cases}$$

Clearly,

$$H(x,\mu) = \begin{cases} -\mu x - (1-\mu)g(x)\sum_{j=1}^n \bar\beta_j, & \text{if } (\sum_{j=1}^n \bar\beta_j)xg(x) > 0;\\ \mu x - (1-\mu)g(x)\sum_{j=1}^n \bar\beta_j, & \text{if } (\sum_{j=1}^n \bar\beta_j)xg(x) < 0.\end{cases}$$

For $x \in \partial\Omega \cap \ker L$ and $\mu \in [0,1]$, $xH(x,\mu) \ne 0$. So we have

$$\begin{aligned}\deg\{QN, \Omega\cap\ker L, 0\} &= \deg\big\{-\frac{1}{T}\int_0^T \sum_{j=1}^n \beta_j(t)g(x)dt, \Omega\cap\ker L, 0\big\}\\ &= \deg\{-x, \Omega\cap\ker L, 0\} \ne 0.\end{aligned}$$

Applying Lemma 2.2, we reach the conclusion. □

As an application, we consider the following example.

Example 3.1. Consider the equation

$$\begin{aligned}&(x(t) - \frac{1}{10}(2-\sin t)x(t-\tau))'' + (1+\frac{1}{2}\sin t)\frac{u(t-\frac{1}{2}\cos t)}{80000}\\ &+ (1-\frac{1}{2}\sin t)\frac{u(t-\frac{1}{2}\sin t)}{80000} = \sin t,\end{aligned} \tag{3.9}$$

where

$$c(t) = \frac{1}{10}(2-\sin t), \quad \beta_1(t) = 1+\frac{1}{2}\sin t, \quad \beta_2(t) = 1-\frac{1}{2}\sin t,$$

$$\gamma_1(t) = \frac{1}{2}\cos t, \quad \gamma_2(t) = \frac{1}{2}\sin t, \quad e(t) = \sin t, \quad T = 2\pi.$$

From simple calculations, we have

$$c_0 = \frac{3}{10}, \quad c_1 = \frac{1}{10}, \quad |\beta_1|_0 = \frac{3}{2}, \quad |\beta_2|_0 = \frac{3}{2}, \quad r = \frac{1}{80000}.$$

Let $\mu_1(t)$ and $\mu_2(t)$ be the inverses of $t - \frac{1}{2}\cos t$ and $t - \frac{1}{2}\sin t$ respectively. We have

$$\begin{aligned}
\Gamma(t) &= \frac{\beta_1(\mu_1(t))}{1-\gamma_1'(\mu_1(t))} + \frac{\beta_2(\mu_2(t))}{1-\gamma_2'(\mu_2(t))} \\
&= \frac{1+\frac{1}{2}\sin\mu_1(t)}{1+\frac{1}{2}\sin\mu_1(t)} + \frac{1-\frac{1}{2}\sin\mu_2(t)}{1-\frac{1}{2}\cos\mu_2(t)} \\
&= 1 + \frac{1-\frac{1}{2}\sin\mu_2(t)}{1-\frac{1}{2}\cos\mu_2(t)} > 0
\end{aligned}$$

and

$$\frac{T^{1/2}}{1-c_0}\sqrt{T\sum_{j=1}^{n}|\beta_j|_0(1+c_0)r} + \frac{c_1 T}{1-c_0} \approx 0.96 < 1.$$

Applying Theorem 3.1, Equation (3.9) has at least one 2π-periodic solution.

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