Electronic Journal of Differential Equations, Vol. 2010(2010), No. 119, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ILL-POSEDNESS FOR PERIODIC NONLINEAR DISPERSIVE EQUATIONS

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Abstract. In this article, we establish new results about the ill-posedness of the Cauchy problem for the modified Korteweg-de Vries and the defocusing modified Korteweg-de Vries equations, in the periodic case. The lack of local well-posedness is in the sense that the dependence of solutions upon initial data fails to be continuous. We also develop a method for obtaining ill-posedness results in the periodic and non-periodic cases for the equations in the hierarchies of these equations and also in the case of the Benjamin-Ono equation.

1. Introduction

The purpose of this article is to investigate ill-posedness of the periodic Cauchy problem for some models of Korteweg-de Vries type in the periodic Sobolev space $H^s_{\rm per}$. The models that we are interested are, the modified Korteweg-de Vries (mKdV) in (3.1), the defocusing modified Korteweg-de Vries (dmKdV) in (6.1) and the Benjamin-Ono (BO) in (1.3). Also, we develop a new technique to obtain ill-posedness of the periodic and non periodic Cauchy problem associated with the higher order equations in the hierarchies of these models.

Before describing our results, it is convenient to define the notion of well-posedness (and consequently ill-posedness) related to a general evolution equation

$$u_t = \partial_x I'(u(t)) \tag{1.1}$$

where $I(u)$ is a generic conservation law for the flow generated by (1.1), namely, $I(u(t_1)) = I(u(t_2))$ for all times t_1, t_2. Here I' represents the gradient of I, defined by

$$\langle I'(u), v\rangle \equiv \langle \frac{\delta I(u)}{\delta u}, v\rangle \equiv \frac{d}{d\epsilon} I(u+\epsilon v)\big|_{\epsilon=0}, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on L^2.

Throughout this paper we shall say that a Cauchy problem associated to (1.1) is locally well-posed (also called C^0-well-posed) in some normed function space $\mathcal{X}$ if, for any initial data $u_0 \in \mathcal{X}$ there exist a time $T = T(\|u_0\|_{\mathcal{X}}) > 0$, a function space Y continuously embedded in $C([-T,T];\mathcal{X})$ and a unique solution $u(t)$ such that

(1) $u \in C([-T,T];\mathcal{X}) \cap Y \equiv Z_T$

2000 *Mathematics Subject Classification.* 76B25, 35Q51, 35Q53.

Key words and phrases. Ill-posedness, mKdV equation; defocusing mKdV equation; BO equation; higher order evolutions equations.

Submitted January 26, 2010. Published August 24, 2010.

(2) the mapping data-solution $u_0 \to u$ from $\mathcal{X}$ to Z_T is continuous.

A Cauchy problem associated to (1.1) is globally well-posed in $\mathcal{X}$ if T above can be chosen as $T = +\infty$. Finally, a Cauchy problem will be said to be ill-posed if it is not C^0-well-posed.

Here is a more precise description of the problems of local well-posedness and global well-posedness in the periodic case for the models mKdV, dmKdV and BO. In [27] the local well-posedness for the mKdV (and defocusing mKdV) was obtained for $s \geq \frac{1}{2}$, and in [21, 22] it was shown that it is globally well posed for $s \geq \frac{1}{2}$. If one strengthens our notion of well-posedness, requiring that the mapping data-solution is smooth, Bourgain showed in [17] that the known results on mKdV ($s > \frac{1}{4}$ in the line, $s \geq \frac{1}{2}$ in the periodic case) are optimal in the sense of this map to be of class C^3. In Christ, Colliander and Tao [20], ill-posedness for the defocusing mKdV is obtained for $s \in (-1, 1/2)$. Regarding the BO equation, L. Molinet in [33] proved global well-posedness in $H^s_{\rm per}$ for $s \geq 0$ and also showed that the mapping data-solution can not be of class $C^{1+\alpha}$, $\alpha > 0$, from $\dot{H}^s_{\rm per}$ into $\dot{H}^s_{\rm per}$ for $s < 0$ where $\dot{H}^s_{\rm per} = (-\triangle)^{-s/2} L^2_{\rm per}$.

In the non-periodic case the ill-posedness for some classical non-linear dispersive equations (for instance, Korteweg-de Vries equation (KdV), cubic Schrödinger equation, complex KdV, mKdV, and BO equations) is studied in [12, 14, 16, 17, 13, 20, 21, 27, 28]. The approach in [12], [13], and [14] uses the existence and good properties of the solitary wave solutions associated to the equations. In particular, a good behavior of its Fourier transforms is required.

In this paper we extend the technique developed in [14] to the periodic case and to higher order evolutions equations. Our approach is based on the theory of Jacobian elliptic functions, the Poisson summation formula, the Floquet theory and on techniques coming from integrable systems. Our method can be used for studying the ill-posedness of the periodic and non periodic Cauchy problem associated with higher order equations.

The first objective of this work is to apply our approach to the study of the ill-posedness for the mKdV, dmKdV and BO equations and to show that the solutions cannot depend continuously on their initial data in the Sobolev spaces $H^s_{\rm per}$ for $s < -1/2$. In other words, we construct a sequence converging (strongly) to a specific data in $H^s_{\rm per}$ and then we show that the corresponding sequence of solutions does not converge (strongly) in $H^s_{\rm per}$. The specific data will be the Dirac delta periodic distribution. The main point in the analysis is the construction of explicit smooth curves of periodic traveling waves solutions for the mKdV, dmKdV and BO equations with a fixed minimal period and a specific behavior of the associated Fourier transform. To construct such solutions we shall use the theory of elliptic functions, the Poisson summation formula and the implicit function theorem. To obtain the ill-posedness results we shall use the ideas in Birnir, Ponce and Svenstedt [14]. Our results extend the ill-posed results of Christ, Colliander and Tao [20] concerning to the mKdV and dmKdV in the periodic case.

The second objective of this paper is to show that the approach for obtaining ill-posedness for the mKdV, dmKdV and BO equations can be applied to the higher order evolution equations in the hierarchies of these models. So we obtain similar results of local ill-posedness in the spaces $H^s_{\rm per}$ for $s < -1/2$. Indeed, from the ideas of Lax in [30] we develop a general scheme which will imply that the profile given by the periodic (or solitary) travelling wave solutions associated with the mKdV, dmKdV and BO equations, and with a specific speed-wave will be a periodic (or solitary) travelling wave solutions *for every equation* from the mKdV, dmKdV and BO hierarchies respectively. For instance, we consider the BO equation

$$u_t + uu_x - \mathcal{H}u_{xx} = 0, \quad u = u(x,t) \in \mathbb{R}, \tag{1.3}$$

where $\mathcal{H}$ denotes the Hilbert transform on $2l$-periodic functions, f, defined by

$$\mathcal{H}f(x) = \frac{1}{2l} p.v. \int_{-l}^{l} \cot[\frac{\pi(x-y)}{2l}] f(y)\, dy.$$

So, we obtain via the Fourier transform that $\widehat{\mathcal{H}f}(k) = -i\,\mathrm{sgn}(k)\widehat{f}(k)$, $k \in \mathbb{Z}$. Next, let $I(u)$ be a generic conserved quantity for the BO equation and consider the associated hierarchy equation (1.1), then there is a spectral parameter $\lambda_{I,c}$ such that $u_c(x,t) = \chi_c(x + \lambda_{I,c}t)$ is a periodic travelling wave for (1.1), provided that $\chi_c(x - ct)$ is a periodic travelling wave for (1.3). The existence of the speed-wave $\lambda_{I,c}$ is deduced from the property that the kernel of the pseudo-differential operator

$$\mathcal{L}_{BO} = \frac{d}{dx}\mathcal{H} - \chi_c + c.$$

is one-dimensional and generated by $\frac{d}{dx}\chi_c$.

In general, to determine the exact value of $\lambda_{I,c}$ can be difficult and tedious. Naturally, our general scheme is applicable to the case of travelling waves of solitary wave type and so we can also obtain ill-posedness results for higher order evolution equations in the hierarchies of the models above in Sobolev space $H^s(\mathbb{R})$. We do not find an effective algorithm which give the parameter $\lambda_{I,c}$ for every conserved quantity I given. Here we calculate it explicitly only in the cases of the second equation from the mKdV, dmKdV and BO hierarchies (see (5.1), (6.2) and (5.2), respectively). Of course, with a little more of work, one can yield an ill-posedness result for the third equation from the hierarchy of these models and so on (see [19], [32] and Remarks after Theorem 5.2 below).

2. Notation

For $s \in \mathbb{R}$, the Sobolev space $H^s_{\mathrm{per}}([0,\ell])$ consists of all periodic distributions f such that $\|f\|^2_{H^s} = \ell\sum_{k=-\infty}^{\infty}\sum(1+k^2)^s|\widehat{f}(k)|^2 < \infty$. For simplicity, we will use the notation H^s_{per} in several places and $H^0_{\mathrm{per}} = L^2_{\mathrm{per}}$. We denote $\|f\|_{L^2} = \|f\|$ and $\langle f, g\rangle_{L^2} = \int_0^\ell f(x)g(x)dx = \langle f, g\rangle$. $[H^s_{\mathrm{per}}]'$, the topological dual of H^s_{per}, is isometrically isomorphic to H^{-s}_{per} for all $s \in \mathbb{R}$. The duality is implemented concretely by the pairing

$$(f,g) = \ell \sum_{k=-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}, \quad \text{for } f \in H^{-s}_{\mathrm{per}},\ g \in H^s_{\mathrm{per}}.$$

Thus, if $f \in L^2_{\mathrm{per}}$ and $g \in H^s_{\mathrm{per}}$ with $s \geq 0$, it follows that $(f,g) = \langle f, g\rangle$. The normal elliptic integral of first type is defined by

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\varphi, k)$$

where $y = \sin\varphi$ and $k \in (0,1)$. k is called the modulus and φ the argument. When $y = 1$, we denote $F(\pi/2, k)$ by $K = K(k)$. The three basic Jacobian elliptic functions are denoted by $\mathrm{sn}(u;k)$, $\mathrm{cn}(u;k)$ and $\mathrm{dn}(u;k)$ (called; snoidal, cnoidal and dnoidal, respectively), and are defined via the previous elliptic integral. More precisely, let

$$u(y;k) := u = F(\varphi, k) \tag{2.1}$$

then $y = \sin\varphi := \mathrm{sn}(u;k) = \mathrm{sn}(u)$ and

$$\begin{aligned} \mathrm{cn}(u;k) &:= \sqrt{1-y^2} = \sqrt{1-\mathrm{sn}^2(u;k)} \\ \mathrm{dn}(u;k) &:= \sqrt{1-k^2y^2} = \sqrt{1-k^2\,\mathrm{sn}^2(u;k)}. \end{aligned} \tag{2.2}$$

The following asymptotic formulas are obtained: $\mathrm{sn}(x;1) = \tanh(x)$, $\mathrm{cn}(x;1) = \mathrm{sech}(x)$ and $\mathrm{dn}(x;1) = \mathrm{sech}(x)$.

3. Ill-posedness for the mKdV

We start this section by presenting some results about periodic travelling wave solutions associated to the mKdV equation,

$$u_t + 3u^2u_x + u_{xxx} = 0, \quad u = u(x,t) \in \mathbb{R}, \tag{3.1}$$

which are essential in our analysis. Let $u_c(x,t) = \varphi_c(x-ct)$ be a periodic travelling wave solution for (3.1), so after integration and by choosing the integration constant being zero we have that φ_c needs to satisfy the nonlinear differential equation

$$\varphi_c'' + \varphi_c^3 - c\varphi_c = 0. \tag{3.2}$$

Next, by following the ideas in Angulo [6] and Angulo & Natali [8] (see also Angulo [7]) will obtain an explicit family of periodic solution, $c \to \varphi_c$, for (3.2) via the Poisson summation formula. The method is as follows: for $\omega > 0$ we consider the positive solitary wave solution for the mKdV equation on $\mathbb{R}$, namely,

$$\phi_\omega(x) = \sqrt{2\omega}\,\mathrm{sech}(\sqrt{\omega}x). \tag{3.3}$$

Then ϕ_ω satisfies the elliptic equation $\phi_\omega'' + \phi_\omega^3 - \omega\phi_\omega = 0$. Now, since the Fourier transform of ϕ_ω is given by

$$\widehat{\phi_\omega}(\xi) = \sqrt{2}\pi\,\mathrm{sech}\big(\frac{\pi\xi}{2\sqrt{\omega}}\big),$$

we obtain from Poisson summation formula the following periodic function ψ_ω with a minimal period L,

$$\psi_\omega(\xi) = \sum_{n\in\mathbb{Z}} \phi_\omega(\xi + nL) = \frac{\sqrt{2}\pi}{L}\sum_{n=0}^{\infty} \epsilon_n\,\mathrm{sech}\big(\frac{\pi n}{2\sqrt{\omega}L}\big)\cos\big(\frac{2\pi n\xi}{L}\big), \tag{3.4}$$

where

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n = 1,2,3,\dots \end{cases} \tag{3.5}$$

Next, we consider the Fourier expansion of the Jacobi elliptic dnoidal function with minimal period L,

$$\frac{2K}{L}\,\mathrm{dn}\big(\frac{2K\xi}{L};k\big) = \frac{\pi}{L} + \frac{4\pi}{L}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos\big(\frac{2n\pi\xi}{L}\big),$$

where $K(k)$ is the complete elliptic integral of the first kind, $q = e^{-\frac{\pi K'}{K}}$, $K'(k) = K(\sqrt{1-k^2})$, $k \in (0,1)$. From here we conclude that

$$\frac{q^n}{1+q^{2n}} = \frac{1}{2}\,\mathrm{sech}\big(\frac{n\pi K'}{K}\big).$$

Therefore,

$$\frac{2K}{L}\,\mathrm{dn}\,\big(\frac{2K\xi}{L};k\big)=\frac{\pi}{L}+\frac{2\pi}{L}\sum_{n=1}^{\infty}\mathrm{sech}\,\big(\frac{n\pi K'}{K}\big)\cos\big(\frac{2n\pi\xi}{L}\big). \tag{3.6}$$

So, from (3.4)-(3.6) let $\varphi_c(\xi)=\eta\,\mathrm{dn}\,\big(\frac{\eta\xi}{\sqrt{2}};k\big)$ be a periodic solution of (3.2) for $c>0$ with minimal period L and $\eta>0$. Then the following identities should be satisfied

$$\begin{gathered} c=\frac{\eta^2}{\sqrt{2}}(1+k^{'2}),\quad k^{'2}=1-k^2\\ \eta=2\sqrt{2}\frac{K(k)}{L},\quad k\in(0,1),\quad \eta\in(\sqrt{c},\sqrt{2c}). \end{gathered} \tag{3.7}$$

Therefore, from the asymptotic properties of K, we need to have $c\in\mathcal{I}=(\frac{2\pi^2}{L^2},+\infty)$. In this point we are ready to build a solution of (3.2) from the formula (3.4). Indeed, for $c\in\mathcal{I}$ and by choosing the speed-wave, ω, of the solitary wave ϕ_ω as being

$$\omega(c)=\frac{c}{16(2-k^2)K'^2(k)},$$

we obtain from (3.4), (3.6) and (3.7) that

$$\varphi_c(\xi)=\psi_{\omega(c)}(\xi). \tag{3.8}$$

Finally, from [6, 8] we obtain by using the implicit function theorem that for every $c>\frac{2\pi^2}{L^2}$ there is a unique $\eta=\eta(c)\in(\sqrt{c},\sqrt{2c})$ such that the fundamental period of the solution φ_c in (3.8) is L and the mapping $c\in\mathcal{I}\to\varphi_c\in H^n_{\rm per}([0,L])$ is a smooth function.

From (3.2), φ_c satisfies the first-order equation

$$[\varphi_c']^2=\frac{1}{2}[-\varphi_c^4+2c\varphi_c^2+4B_{\varphi_c}] \tag{3.9}$$

where B_{φ_c} is an integration constant determined uniquely as follows: For $c\in(\frac{2\pi^2}{L^2},\infty)$ there is a unique $\eta=\eta(c)\in(\sqrt{c},\sqrt{2c})$ such that for $\beta^2\equiv 2c-\eta^2$ and

$$4B_{\varphi_c}=-\eta^2\beta^2,$$

we have that η and β are the positive zeros of the even polynomial $F_{\varphi_c}(t)=-t^4+2ct^2+4B_{\varphi_c}$.

We also note from the first and third relations in (3.7) that $\eta(c)\to+\infty$ and $K(k)\to+\infty$, as $c\to+\infty$. Hence $k(c)\to 1$, as $c\to+\infty$. From here we conclude that $\omega(c)\to+\infty$, as $c\to+\infty$.

Next we have the following lemmas for obtaining our ill-posed result associated with the mKdV.

Lemma 3.1. *The $H^s_{\rm per}$ norms of $u_0(x)=\varphi_c(x)$ and $u_c(x,t)=\varphi_c(x-ct)$ are finite for $s<-1/2$, and*

$$\lim_{c\to+\infty}\|u_0\|_s=\sqrt{2}\pi\|\delta_L\|_s$$
$$\lim_{c\to+\infty}\|u_c(\cdot,t)\|_s=\sqrt{2}\pi\|\delta_L\|_s,$$

where δ_L represents the Dirac periodic distribution centered in zero, namely, for $f\in C^\infty_{\rm per}([0,L])$ we have $\delta_L(f)=(\delta_L,f)=f(0)$ (see [24]).

Proof. From the Parseval identity and (3.4)-(3.8) we obtain

$$\|\varphi_c\|_s^2 = \frac{2\pi^2}{L} \sum_{n=-\infty}^{+\infty} (1+n^2)^s \operatorname{sech}^2\big(\frac{n\pi}{2\sqrt{\omega}L}\big) \leq \frac{2\pi^2}{L} \sum_{n=-\infty}^{+\infty} (1+n^2)^s.$$

Hence, if $s < -1/2$, then the series on the right-hand side of the above inequality is uniformly convergent. Therefore, from above analysis we obtain

$$\lim_{c\to+\infty} \|\varphi_c\|_s^2 = \frac{2\pi^2}{L} \sum_{n=-\infty}^{+\infty} (1+n^2)^s = 2\pi^2 \|\delta_L\|_s^2,$$

since $\widehat{\delta_L}(n) = 1/L$.

Now, for the solution $u_c(x,t) = \varphi_c(x-ct) = \tau_{ct}\varphi_c$ (where $\tau_{ct} f(x) = f(x-ct)$), we have from (3.4)-(3.8),

$$\begin{aligned}\|u_c(\cdot,t)\|_s^2 &= L \sum_{n=-\infty}^{+\infty} \frac{1}{(1+n^2)^{-s}} |\widehat{u_c}(n)|^2 = L \sum_{n=-\infty}^{+\infty} \frac{1}{(1+n^2)^{-s}} |\widehat{\tau_{ct}\varphi_c}(n)|^2 \\ &= L \sum_{n=-\infty}^{+\infty} \frac{1}{(1+n^2)^{-s}} |\widehat{\varphi_c}(n)|^2 < +\infty,\end{aligned}$$

and so

$$\|u_c(\cdot,t)\|_s^2 \to 2\pi^2 \|\delta_L\|_s^2, \quad \text{as } c \to +\infty.$$

This completes the proof of the lemma. □

Lemma 3.2. *The initial data $u_0(x) \equiv \varphi_c(x)$ converges weakly to $\sqrt{2}\pi\delta_L$ as $c \to +\infty$.*

Proof. Let $\phi \in C^\infty_{\rm per}([0,L])$ (where $C^\infty_{\rm per}([0,L])$ denotes the space of smooth periodic function with period L). Then we have (see [24])

$$(u_0, \phi) = L \frac{\sqrt{2}\pi}{L} \sum_{n=-\infty}^{+\infty} \operatorname{sech}\big(\frac{n\pi}{2\sqrt{\omega}L}\big) \widehat{\phi}(n). \tag{3.10}$$

Since

$$\big|\operatorname{sech}\big(\frac{n\pi}{2\sqrt{\omega}L}\big)\widehat{\phi}(n)\big| \leq |\widehat{\phi}(n)|$$

and the series $\sum_{n=-\infty}^{+\infty} |\widehat{\phi}(n)|$ converges, it follows from the M-Weierstrass Theorem that

$$\begin{aligned}\lim_{c\to+\infty}(u_0,\phi) &= \sqrt{2}\pi \sum_{n=-\infty}^{+\infty} \widehat{\phi}(n) \lim_{c\to+\infty} \operatorname{sech}\big(\frac{n\pi}{2\sqrt{\omega}L}\big) \\ &= \sqrt{2}\pi \sum_{n=-\infty}^{+\infty} \widehat{\phi}(n) = \sqrt{2}\pi\phi(0) = (\sqrt{2}\pi\delta_L, \phi).\end{aligned}$$

This shows that u_0 converges weakly to $\sqrt{2}\pi\delta_L$ in $H^s_{\rm per}$ $(s < -1/2)$. □

We can now prove the main result of this section.

Theorem 3.1. *The initial value problem for the mKdV is locally ill-posed in $H^s_{\rm per}$ for $s < -12$.*

Proof. From Lemma 3.1 the $H^s_{\rm per}$-norm of $u_c(x,0)=\varphi_c(x)$ converges to the $H^s_{\rm per}$-norm of $\sqrt{2}\pi\delta_L$ and by Lemma 3.2, $u_c(x,0)$ converges weakly to $\sqrt{2}\pi\delta_L$. Consequently $u_c(x,0)$ converges strongly to $\sqrt{2}\pi\delta_L$ in $H^s_{\rm per}$. Next, from (3.4)-(3.8) we have that

$$\widehat{u_c(x,0)}(n)=\widehat{\varphi_c}(n)=\frac{\sqrt{2}\pi}{L}\operatorname{sech}\Big(\frac{n\pi}{2\sqrt{\omega}L}\Big)\to\frac{\sqrt{2}\pi}{L}\quad\text{as } c\to+\infty.$$

On the other hand, we have that $\widehat{u_c(x,t)}(n)=\widehat{\tau_{ct}\varphi_c}(n)=e^{ictn}\widehat{\varphi_c}(n)$, which not converge as $c\to+\infty$ for all $n\neq 0$. This shows that $u_c(x,t)$ can not converge weakly in $H^s_{\rm per}$. □

4. Ill-posedness for the BO equation

In this section we consider the ill-posedness for the Benjamin-Ono equation (1.3). As in the previous section, first we will obtain a periodic solution with minimal period $2L$ for the BO equation by using the Poisson summation formula. So, we consider the family of solitary wave solutions for the BO equation, $u(x,t)=\phi_\omega(x-\omega t)$, where ϕ_ω satisfies the nonlocal differential equation

$$\mathcal{H}\phi_\omega'+\omega\phi_\omega-\frac{1}{2}\phi_\omega^2=0$$

with a profile given by

$$\phi_\omega(x)=\frac{4\omega}{1+\omega^2x^2},\quad \omega>0. \tag{4.1}$$

So, we obtain that the Fourier transform of ϕ_ω is given by

$$\widehat{\phi_\omega}(\xi)=4\pi e^{-2\pi|\xi|/\omega}.$$

By the Poisson summation formula, we obtain the following periodic function ρ_ω with a minimal period $2L$ (see [10, 6, 7, 8])

$$\begin{aligned}\rho_\omega(x)&=\sum_{n=-\infty}^{+\infty}\phi_\omega(x+2Ln)=\frac{2\pi}{L}\sum_{n=-\infty}^{+\infty}e^{-\frac{\pi|n|}{\omega L}}e^{\frac{i\pi nx}{L}}\\&=\frac{2\pi}{L}\sum_{n=0}^{+\infty}\epsilon_n e^{-\frac{\pi|n|}{\omega L}}\cos\Big(\frac{n\pi x}{L}\Big)=\frac{2\pi}{L}\operatorname{Re}\Big[\coth\Big(\frac{\pi}{2\omega L}+\frac{i\pi x}{2L}\Big)\Big]\\&=\frac{2\pi}{L}\frac{\sinh\big(\frac{\pi}{\omega L}\big)}{\cosh\big(\frac{\pi}{\omega L}\big)-\cos\big(\frac{\pi x}{L}\big)},\end{aligned}\tag{4.2}$$

where ϵ_n is defined in (3.5). Now, let $\chi_c(x-ct)$ be a smooth periodic travelling wave solution of the BO equation, with $c>0$ and minimal period $2L$. Then by considering the Fourier expansion series

$$\chi_c(x)=\sum_{n=-\infty}^{+\infty}a_ne^{\frac{i\pi nx}{L}}$$

and by substituting this expression in

$$\mathcal{H}\chi_c'+c\chi_c-\frac{1}{2}\chi_c^2=0,$$

we obtain

$$\Big(\frac{\pi|n|}{L}+c\Big)=\frac{1}{2}\sum_{m=-\infty}^{+\infty}a_{n-m}a_n. \tag{4.3}$$

By (4.2), we define $a_n=\frac{2\pi}{L}e^{-\gamma|n|}$ with $n\in\mathbb{Z}$ and $\gamma\in\mathbb{R}$ to be chosen. So, from equality (4.3) we have the basic relation

$$c+\frac{\pi|n|}{L}=\frac{2\pi}{2L}(|n|+\coth\gamma),$$

which implies

$$c=\frac{\pi}{L}\coth\gamma.$$

Then, for $\gamma\equiv\frac{\pi}{\omega L}$ and $c>\frac{\pi}{L}$ we chose the speed-wave for the solitary wave solution ϕ_ω in (4.1), $\omega=\omega(c)$, such that $\tanh(\gamma)=\frac{\pi}{cL}$. Therefore, from (4.2) we have

$$\chi_c(x)=\rho_{\omega(c)}(x)=\frac{2\pi}{L}\Big(\frac{\sinh\gamma}{\cosh\gamma-\cos\frac{\pi x}{L}}\Big) \tag{4.4}$$

is a periodic solution of the BO equation, with period $2L$ and Fourier coefficients given by

$$\widehat{\chi_c}(n)=\frac{2\pi}{L}e^{-\gamma|n|}.$$

Note that $\gamma\to 0$ as $c\to+\infty$. So we have our ill-posedness result for the BO equation.

Theorem 4.1. *The initial value problem for the BO equation is locally ill-posed in $H^s_{\rm per}$ for $s<-1/2$.*

Proof. As in the case of the mKdV we have

$$\|\chi_c\|_s^2=\frac{8\pi^2}{L}\sum_{n=0}^{+\infty}(1+n^2)^se^{-2\gamma|n|}\leq\frac{8\pi^2}{L}\sum_{n=0}^{+\infty}(1+n^2)^s.$$

If $s<-1/2$, then the series on the right-hand side of the above inequality converges uniformly and therefore

$$\|\chi_c\|_s^2\to16\pi^2\|\delta_{2L}\|_s^2,\quad \text{as } c\to+\infty.$$

This shows that the $H^s_{\rm per}$ norm of $u_0(x)=\chi_c(x)$ converges to the $H^s_{\rm per}$ norm of $4\pi\delta_{2L}$.

Now, for $\phi\in C^\infty_{\rm per}([0,2L])$, we have

$$\langle u_0,\phi\rangle=\int_0^L\chi_c(x)\phi(x)dx=4\pi\sum_{n=-\infty}^{+\infty}e^{-\gamma|n|}\widehat{\phi}(n)\to4\pi\phi(0),\quad \text{as } c\to+\infty.$$

From the above, we obtain that $u_0(x)$ converges strongly to $4\pi\delta_{2L}$ in $H^s_{\rm per}$ for $s<-1/2$. On the other hand, for $u_c(x,t)=\chi_c(x-ct)$, we have

$$\|u_c(\cdot,t)\|_s^2=2L\sum_{n=-\infty}^{+\infty}(1+n^2)^s|\widehat{\chi_c}(n)|^2<+\infty,$$

and so

$$\|u_c(\cdot,t)\|_s^2\to16\pi^2\|\delta_{2L}\|_s^2.$$

Moreover, since $\widehat{u_c(x,t)}(n)=\widehat{\tau_{ct}\chi_c}(n)=e^{ictn}\widehat{\chi_c}(n)$ we get that the rest of the proof is the same as the one for Theorem 3.1. This completes the proof. ∎

We remark that recently Molinet [34] showed ill-posedness of the BO equation in $H^s_{\rm per}$ for $s<0$.

5. Ill-posedness for higher order evolution equations

In this section we develop a general scheme which shows that every travelling wave solution (periodic or solitary wave) for the mKdV and BO equations (3.1) and (1.3), respectively, is also a travelling wave solution (with a different speed wave) of every equation belonging to the hierarchy generated by these two basic equations. So, we can deduce ill-posedness results in the periodic and non-periodic cases, for instance, for the fifth order modified Korteweg-de Vries equation (5-mKdV)

$$u_t - u_{xxxxx} - 30u^4u_x - 10u^2u_{xxx} - 10(u_x)^3 - 40uu_xu_{xx} = 0, \tag{5.1}$$

and for the third order Benjamin-Ono equation (3-BO)

$$u_t - 4u_{xxx} + 3u^2u_x - 3(u\mathcal{H}u_x)_x - 3\mathcal{H}(uu_x)_x = 0. \tag{5.2}$$

5.1. **The Method.** Initially we set an abstract hamiltonian system of the form

$$u_t = \partial_x E'(u(t)), \tag{5.3}$$

where E is a conserved quantity for (5.3) with $E''(u)$ being a self-adjoint linear operator. We assume that (5.3) is invariant under the symmetry of translation. More specifically, let $\{T(\gamma)\}_{\gamma\in\mathbb{R}}$ be the one-paramenter group of unitary operators on L^2 defined for $\gamma\in\mathbb{R}$ as

$$T(\gamma)f(x) = f(x+\gamma).$$

So, for $u(\cdot,t)$ being a solution of (5.3) with initial data $u(x,0)=u_0(x)$ we obtain that for $\gamma\in\mathbb{R}$, $T(\gamma)u(\cdot,t)=u(\cdot+\gamma,t)$ is solution of (5.3) with initial data $T(\gamma)u(x,0)=u_0(x+\gamma)$. Next we denote by $T'(0)$ the infinitesimal generator of the group of translations, then $T'(0)=\frac{d}{dx}$.

Now, we suppose that $F(u)=\frac{1}{2}\|u\|^2$ is also a conserved quantity for (5.3) and E is invariant under translation. So, we have our first hypothesis:

(H1) (Existence of travelling wave) Suppose the existence of travelling wave type solutions $u_c(x,t)=\phi_c(x-ct)$ of (5.3) such that the mapping $c\in I\subset\mathbb{R}\to\phi_c$ is smooth and for every $c\in I$, the profile ϕ_c is a critical point for the functional $H\equiv E+cF$, namely,

$$H'(\phi_c) = E'(\phi_c) + c\phi_c = 0. \tag{5.4}$$

Let us call the set $\Omega_{\phi_c}=\{T(\gamma)\phi_c:\gamma\in\mathbb{R}\}$ the ϕ_c-orbit. Then from the invariance of H under translation and from (5.4) we obtain that every point of Ω_{ϕ_c} is a critical point of H, $H'(T(\gamma)\phi_c)=0$ for all $\gamma\in\mathbb{R}$. Therefore,

$$0 = \frac{d}{d\gamma}H'(T(\gamma)\phi_c)\big|_{\gamma=0} = H''(\phi_c)(T'(0)\phi_c) = H''(\phi_c)\Big(\frac{d}{dx}\phi_c\Big). \tag{5.5}$$

Hence $\frac{d}{dx}\phi_c$ belongs to the kernel of the unbounded and self-adjoint linear operator

$$\mathcal{L}_c = E''(\phi_c) + c. \tag{5.6}$$

Next we have our second hypothesis:

(H2) (One-dimensional kernel) The operator $\mathcal{L}_c$ has kernel spanned by $T'(0)\phi_c = \frac{d}{dx}\phi_c$.

Now, from (5.3) we consider the linear variational equation

$$v_t = V(u)v, \tag{5.7}$$

where $V(u)$ denotes the derivative of $K(u) \equiv \partial_x E'(u)$ (see (1.2)), namely, $V(u)v = K'(u)v$ is a linear function of v given by

$$V(u)v = \partial_x(E''(u)(v)).$$

Let $I(u)$ be any conservation law for (5.3) with derivative $G(u)$, namely,

$$I'(u)(v) = \langle G(u), v\rangle.$$

Therefore, we obtain that for any solution $u(t)$ of (5.3) and $v(t)$ of (5.7)

$$\frac{d}{dt}\langle G(u(t)), v(t)\rangle = 0. \tag{5.8}$$

Then for $u(t)$ being a travelling wave of the model in (5.3) we get from (5.8) that

$$\langle G(\phi_c(x-ct)), v(t)\rangle = \langle G(\phi_c(x)), v(x+ct,t)\rangle$$

is independent of t. Therefore, for $w(x,t) \equiv v(x+ct,t)$, we obtain that for every t,

$$0 = \langle G(\phi_c), w_t(t)\rangle = \langle G(\phi_c), [c\partial_x + V(\phi_c)]w\rangle = \langle [-c\partial_x + (V(\phi_c))^*]G(\phi_c), w(t)\rangle,$$

where V^* represents the adjoint operator associated to V. So, since the value $w(0)$ can be arbitrary we have

$$0 = [-c\partial_x + (V(\phi_c))^*]G(\phi_c) = -[c + E''(\phi_c)](\partial_x I'(\phi_c)) = -\mathcal{L}_c \partial_x I'(\phi_c). \tag{5.9}$$

By hypothesis (H2) above we obtain from (5.9) that there is $\lambda = \lambda(c, I)$ such that

$$\partial_x I'(\phi_c) = \lambda \frac{d}{dx}\phi_c. \tag{5.10}$$

Relation (5.10) contains the most important information in our study. Indeed, if we consider the evolution equation

$$z_t = \partial_x I'(z(t)), \tag{5.11}$$

then $z_\lambda(x,t) = \phi_c(x+\lambda t)$ is a travelling wave solution for (5.11). In general, the value of λ depending on I and c, is not easy to find out. Note that from (5.10) we have $I'(\phi_c) = \lambda\phi_c + \beta$, with β being an integration constant. If ϕ_c is a solitary wave solution ($\lim_{|\xi|\to\infty} \phi(\xi) = 0$) then $\beta = 0$. In the case of periodic travelling waves solutions we will also assume $\beta = 0$.

Remark. Hypothesis (H2), it is a delicate issue to be verified in the periodic setting, but the techniques developed in Angulo and Natali [8, 9] can be useful for this purpose.

Next, we apply the foregoing to the 3-BO and 5-mKdV equations in (5.2) and (5.1), respectively.

5.2. **The 3-BO case.** Let ψ denote the travelling wave solutions of type solitary wave or periodic wave for the BO (1.3). This equation can be written in the Hamiltonian form

$$u_t = \partial_x E'_{BO}(u(t))$$

with

$$E_{BO}(u) = \int u\mathcal{H}u_x - \frac{1}{6}u^3\,dx.$$

So, the hypothesis (H1) above follows from Section 4 (formulas (4.1) and (4.4)). Now, for verifying hypothesis (H2) we need to study the kernel of the pseudo-differential operator

$$\mathcal{L}_{BO} = \frac{d}{dx}\mathcal{H} - \psi + c.$$

For ψ being the solitary wave solution in (4.1) or the periodic travelling wave in (4.4), the works of Bennett *et al.* [11], Albert [1], Albert *et al.* [3] and Angulo *et al.* [8], show that $\ker(\mathcal{L}_{BO}) = [\frac{d}{dx}\psi]$ (see Angulo [7] for a summary of these results). Thus, there is a constant λ_{BO} such that $\psi(x + \lambda_{BO}t)$ is a travelling wave solution of (5.2) according to the results established in Subsection 5.1.

Next, we obtain the exact value of λ_{BO}. We start establishing some nontrivial facts about the solutions of the pseudo-differential equation

$$\mathcal{H}\psi' + c\psi - \frac{1}{2}\psi^2 = 0, \quad c > 0. \tag{5.12}$$

In Albert [2] it was shown an alternative method of proof of uniqueness of the solitary waves solutions for the intermediate long wave equation (ILW) and the BO (see (4.1)), which does not use complex analysis (see [4], [5]). His method make use of positive-operator theory and suitable identities associated to the dispersion operator in the ILW and BO equations. In the case of the BO was used the well-known product formula

$$fg + \mathcal{H}(f \cdot \mathcal{H}g + g \cdot \mathcal{H}f) - \mathcal{H}f \cdot \mathcal{H}g = 0, \tag{5.13}$$

valid for $f, g \in L^2(\mathbb{R})$. Inspecting his proof, one can observe that the key equality in Lemma 3 in [2], established on the line for $N = N_H$ being the "dispersion operator" defined by $\widehat{N_H f}(\xi) = \xi coth(\xi H)\widehat{f}(\xi)$, $\xi \in \mathbb{R}$, it is also true in the periodic setting with N replaced by $M = \partial_x \mathcal{H}$. Indeed, since the formula (5.13) is true for $f, g \in L^2_{\rm per}$, differentiation of (5.13) yields the main equality

$$f'g + fg' + M\Big[f\Big(\int_0^x Mg\Big) + g\Big(\int_0^x Mf\Big)\Big] - Mf\Big(\int_0^x Mg\Big) - Mg\Big(\int_0^x Mf\Big) = 0.$$

Hence following the ideas in [2], we obtain that every positive periodic solution ψ of (5.12) satisfies

$$\mathcal{H}\psi' = -2\Big(\frac{\psi'}{\psi}\Big)'.$$

Therefore, from (5.12) the following ordinary differential equation holds:

$$\Big(\frac{\psi'}{\psi}\Big)' = \frac{c}{2}\psi - \frac{1}{4}\psi^2. \tag{5.14}$$

Then, it is easy to see that ϕ_c in (4.1) (with $\omega = c > 0$) and χ_c defined in (4.4), satisfy (5.14). Now, from (5.14) it follows that ψ satisfies

$$[\psi']^2 = \psi^2[c\psi - \frac{1}{4}\psi^2 + D], \tag{5.15}$$

where D is the constant of integration. The value of this constant in the case $\psi = \phi_c$ in (4.1) it is $D = 0$, and for $\psi = \chi_c$ in (4.4), it is given by $D = -\frac{\pi^2}{L^2}$.

Now, by denoting $G(u) = 4u_{xx} + 3\mathcal{H}(uu_x) + 3u\mathcal{H}u_x - u^3$, we can write (5.2) in the hamiltonian form

$$u_t = \partial_x G(u).$$

Here $G(u) = I'(u)$ with $I(u)$ being the following conservation law for the BO,

$$I(u) = \int 2(u_x)^2 - \frac{3}{2}u^2\mathcal{H}u_x + \frac{1}{4}u^4\,dx.$$

Then for $u_c(x,t) = \psi(x + \lambda_{BO}t)$ with $\lambda_{BO} = D - 3c^2$, we obtain after some computations based on relations (5.12) and (5.15) that

$$G(\psi) = \lambda_{BO}\psi.$$

Hence, $u_c(x,t)$ is a travelling wave solution (solitary or periodic) for the 3-BO (5.2).

Finally, from the approach in [14], Section 3, and Section 4 above, we obtain the following result.

Theorem 5.1. *The initial value problem for the third order BO equation* (5.2) *is locally ill-posed in* $H^s(\mathbb{R})$ *and in* $H^s_{\rm per}$ *for* $s < -1/2$.

5.3. **The 5-mKdV case.** We start with the following scaling for u being solution of the mKdV (3.1). For $v(x,t) = \frac{\sqrt{2}}{2}u(x,t)$ we have that

$$v_t + 6v^2v_x + v_{xxx} = 0. \tag{5.16}$$

So, we have that (5.1) is the second equation from the mKdV (5.16) hierarchy. Now, for $v(x,t) = \zeta_c(x - ct)$ we have

$$\zeta_c'' + 2\zeta_c^3 - c\zeta_c = 0, \quad \text{and} \quad [\zeta_c']^2 = -\zeta_c^4 + c\zeta_c^2 + B_c, \tag{5.17}$$

with B_c the integration constant. Then for ϕ_c in (3.3) ($c = \omega > 0$) we have that $\zeta_c = \frac{\sqrt{2}}{2}\phi_c$ satisfies (5.17) with $B_c = 0$ and with $B_c = B_{\varphi_c}$ for $\zeta_c = \frac{\sqrt{2}}{2}\varphi_c$, and φ_c being the dnoidal wave solution satisfying (3.9) for $c > \frac{2\pi^2}{L^2}$.

Equation (5.16) can be written in the Hamiltonian form as

$$v_t = \partial_x E'_{mKdV}(v(t))$$

with

$$E_{mKdV}(v) = \frac{1}{2}\int (v_x)^2 - v^4\,dx.$$

Moreover, the family of travelling wave ζ_c satisfies $E'_{mKdV}(\zeta_c) + c\zeta_c = 0$. So, we obtain the hypothesis (H1) in Subsection 5.1.

For obtaining hypothesis (H2) we study the kernel of the second order differential operator

$$\mathcal{L}_{mKdV} = -\frac{d^2}{dx^2} - 6\zeta_c^2 + c.$$

For ζ_c being a solitary wave solution we have that an elementary application of the Oscillation theory of the Sturm-Liouville theory implies that zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\zeta_c$. For $\zeta_c = \frac{\sqrt{2}}{2}\varphi_c$ and φ_c being the dnoidal wave solution defined in Section 3, the analysis is more delicate. In this case, the Floquet theory can be used (see Angulo [6] or the proof of Theorem 6.5 below) for obtaining the desired property for $\mathcal{L}_{mKdV}$. We note in this point that by using the new technique in Angulo&Natali [8] (see also Angulo [7]) which is based in positive properties of the Fourier transform of φ_c, we can also deduce hypothesis (H2).

Then for $G(u) = u_{xxxx} + 6u^5 + 10u^2u_{xx} + 10u(u_x)^2$, one can write (5.1) in the form

$$u_t = \partial_x G(u)$$

with $G(u) = M'(u)$ and M being the following conservation law for the mKdV in (5.16),

$$M(u) = \int \frac{1}{2}(u_{xx})^2 + u^6 - 5u^2(u_x)^2\,dx.$$

Therefore, for $u_c(x,t) = \zeta_c(x + \lambda_{mKdV}t)$ with

$$\lambda_{mKdV} = c^2 - 2B_c,$$

we obtain after some computations based in the relations in (5.17), that $G(\zeta_c) = \lambda_{mKdV}\zeta_c$. Hence, $u_c(x,t)$ represents a travelling wave solution for the 5-mkdv (5.1).

From Section 3 in [14] and Section 3 above we obtain the following result.

Theorem 5.2. *The initial value problem for the fifth order modified Korteweg-de Vries equation* (5.1) *is not locally well-posed in* $H^s(\mathbb{R})$ *and* $H^s_{\rm per}$ *for any* $s < -1/2$.

Remarks. **(a)** Recently Kwon in [29] showed that the initial value problem (IVP) for the following general fifth order mKdV equation

$$u_t - u_{xxxxx} + c_0u^4u_x + c_1(u^3)_{xxx} + c_2uu_xu_{xx} + c_4u^2u_{xxx} = 0,$$

with c_j constants, is local well-posedness in $H^s(\mathbb{R})$ for $s \geq 3/4$ and that the solution map from data to the solutions, fails to be uniformly continuous for $s \in (-\frac{7}{24}, \frac{3}{4})$. Theorem 5.2 above shows that the IVP is also ill-posed in $H^s(\mathbb{R})$ for $s < -1/2$.
(b) The third equation from the mKdV (5.16) hierarchy is given by

$$\begin{aligned} &u_t - \partial_x^7u - 84u\partial_xu\partial_x^4u - 560u^3\partial_xu\partial_x^2u - 14u^2\partial_x^5u - 140u\partial_x^2u\partial_x^3u \\ &- 126\partial_x^3u(\partial_xu)^2 - 182\partial_xu(\partial_x^2u)^2 - 70u^4\partial_x^3u - 420u^2(\partial_xu)^3 - 140u^6\partial_xu = 0, \end{aligned} \tag{5.18}$$

which is coming from the conservation law

$$N(u) = \int -\frac{1}{2}(u_{xxx})^2 - 35u^4(u_x)^2 + 7u^2(u_{xx})^2 - \frac{7}{2}(u_x)^4 + \frac{5}{2}u^8\,dx.$$

Then we have that (5.18) has the hamiltonian form $u_t = \partial_xN'(u)$. Therefore we obtain that $u_c(x,t) = \zeta_c(x + \lambda_{N,c}t)$ with

$$\lambda_{N,c} = c^3 - 6cB_c,$$

is a travelling wave to (5.18). Then we get that the IVP for the seventh order modified Korteweg-de Vries equation (5.18) is ill-posed in $H^s(\mathbb{R})$ and $H^s_{\rm per}$ for $s < -1/2$.
(c) The third equation from the BO hierarchy is

$$u_t = \partial_xS(u) \tag{5.19}$$

where

$$\begin{aligned} S(u) &= u^4 - 4u^2\mathcal{H}u_x - 4u\mathcal{H}(uu_x) + 2(\mathcal{H}u_x)^2 + 4\mathcal{H}(u\mathcal{H}u_x)_x \\ &\quad - 6(u_x)^2 - 12uu_{xx} + 8\mathcal{H}u_{xxx}. \end{aligned}$$

Here $S(u) = W'(u)$ for W being the following conservation law for the BO equation

$$W(u) = \int \frac{1}{5}u^5 - \frac{4}{3}u^3\mathcal{H}u_x - u^2\mathcal{H}(uu_x) + 2u(\mathcal{H}u_x)^2 + 6u(u_x)^2 + 4u_{xx}\mathcal{H}u_x\,dx.$$

Therefore we obtain that $u_c(x,t) = \psi(x + \lambda_{W,c}t)$ with

$$\lambda_{W,c} = 4c^3 - 4cD,$$

and D given in (5.15), it is a travelling wave to (5.19) satisfying $S(\psi) = \lambda_{W,c}\psi$. Then we deduce that the IVP for (5.19) is ill-posed in $H^s(\mathbb{R})$ and $H^s_{\rm per}$ for $s < -1/2$.
(d) The role of the index $s = -1/2$ in Theorems 5.1 and 5.2 for the spaces $H^s(\mathbb{R})$, can be explained via a scaling argument, that is, if $u(x,t)$ solves the IVP (5.1), then $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^5 t)$, $\lambda > 0$, solves the same equation with data $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$. Then for D^s defined by $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$, we obtain that the equality

$$\|D^s u_{0,\lambda}\| = \lambda^{s+\frac{1}{2}} \|D^s u_0\|$$

implies that this norm is independent of λ only when $s = -1/2$. A similar analysis is carried on with (5.2) via the scaling

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^3 t).$$

6. Ill-posedness for the dmKdV and the fifth order dmKdV

In this section we develop a theory of ill-posedness in the periodic case for the defocusing modified Korteweg-de Vries equation (dmKdV)

$$v_t + 6v^2 v_x - v_{xxx} = 0, \quad v = v(x,t) \in \mathbb{R}, \tag{6.1}$$

and for the fifth order defocusing modified Korteweg-de Vries equation (5-dmKdV)

$$v_t - v_{xxxxx} - 30v^4 v_x + 10v^2 v_{xxx} + 10(v_x)^3 + 40v v_x v_{xx} = 0. \tag{6.2}$$

The use of the theory of elliptic functions and the Floquet theory associated to the Lamé equation will be basic in our analysis.

6.1. **The dmKdV case.** In this section we focus to the ill-posedness result for the defocusing mKdV (6.1). We start obtaining a family of periodic travelling wave solutions of (6.1) in the form

$$v(x,t) = Q_c(x - ct).$$

So, if we substitute this specific solution in the defocusing mKdV and consider the integration constant equal to zero then $Q = Q_c$ satisfies the ordinary differential equation

$$Q'' + cQ - 2Q^3 = 0. \tag{6.3}$$

From this we obtain the first order differential equation (the associated quadrature form)

$$[Q']^2 = Q^4 - cQ^2 + A, \tag{6.4}$$

where A is the integration constant and which need to be different of zero for obtaining periodic profile solutions. Let us suppose that the fourth order polynomial $F(t) = t^4 - ct^2 + A$ has the positive roots $\eta_1 > \eta_2 > 0$. From (6.4) it follows that

$$\begin{gathered} [Q']^2 = (Q^2 - \eta_1^2)(Q^2 - \eta_2^2), \quad -\eta_2 \le Q \le \eta_2, \\ \eta_1^2 + \eta_2^2 = c > 0, \quad \eta_1^2\eta_2^2 = A > 0. \end{gathered} \tag{6.5}$$

Next, we normalize Q by letting $\varphi = Q/\eta_2$, so that (6.5) becomes

$$[\varphi']^2 = \eta_1^2(k^2\varphi^2 - 1)(\varphi^2 - 1) \tag{6.6}$$

with $k^2 = \eta_2^2/\eta_1^2$. Next, by letting now $\varphi(\xi) = sin(\psi(\xi))$ with $\psi(0) = 0$ and ψ continuous, the substitution of it into (6.6), yields the equation

$$[\psi']^2 = \eta_1^2(1 - k^2 sin^2\psi).$$

We may solve for ψ implicitly to obtain

$$F(\psi;k)=\int_0^{\psi(\xi)}\frac{dt}{\sqrt{1-k^2\sin^2 t}}=\eta_1\,\xi. \tag{6.7}$$

The left-hand side of (6.7) is just the standard elliptic integral of the first kind and so, for fixed k, the elliptic function snoidal $\operatorname{sn}(\xi;k)$ is defined in terms of the inverse of the mapping $\psi\longmapsto F(\psi;k)$. Hence, (6.7) implies that

$$\operatorname{sn}(\eta_1\xi;k)=\sin\psi(\xi).$$

Therefore, we obtain the snoidal periodic profile for (6.3)

$$Q_c(\xi)=\eta_2\operatorname{sn}(\eta_1\xi;k),$$

which is determined by the elliptic modulus $k^2=\eta_2^2/\eta_1^2\in(0,1)$.

Next we establish the main information for obtaining a smooth curve, $c\to Q_c$, of periodic travelling waves for (6.3) with minimal period L. Since sn has minimal period $4K(k)$ then the minimal period of Q_c, T_{Q_c}, is given by $T_{Q_c}=4K(k)/\eta_1$. Moreover, from the relations in (6.5) it follows

$$0<\eta_2<\sqrt{c/2}<\eta_1<\sqrt{c},$$

and k and T_{Q_c} can be seen as functions of c and η_1, namely,

$$k^2(\eta_1,c)=\frac{c-\eta_1^2}{\eta_1^2},\quad T_{Q_c}(\eta_1,c)=\frac{4}{\eta_1}K(k(\eta_1,c)).$$

Therefore, from the properties of $K(k)$ and from the implicit function theorem we have the following result (see Angulo [6, 7]).

Theorem 6.1. *Let $L>0$ be a fixed number and n any positive integer. Then there exists a smooth branch of snoidal waves, $c\in(\frac{4\pi^2}{L^2},+\infty)\mapsto Q_c\in H^n_{\rm per}([0,L])$, such that*

$$Q_c''(\xi)+cQ_c(\xi)-2Q^3(\xi)=0,\quad \text{for all } \xi\in\mathbb{R}, \tag{6.8}$$

where

$$Q_c(\xi)=\eta_2\operatorname{sn}(\eta_1\xi;k). \tag{6.9}$$

Here, η_1, η_2 and k are smooth functions of c, satisfying the relations

$$\begin{gathered}\eta_1\in(\sqrt{c/2},\sqrt{c}),\quad \eta_2=\sqrt{c-\eta_1^2},\quad k^2=\frac{\eta_2^2}{\eta_1^2}\\ \frac{4K(k(c))}{\eta_1(c)}=L,\quad \text{for all } c>\frac{4\pi^2}{L^2}.\end{gathered} \tag{6.10}$$

Moreover, $k(c)\to 1$ as $c\to\infty$.

The following theorem is the main piece in our study of ill-posedness for the defocusing mKdV.

Theorem 6.2. *The Fourier coefficients $\{\widehat{Q_c}(m)\}_{m\in\mathbb{Z}}$ for Q_c defined in (6.9) satisfy*

$$\lim_{c\to+\infty}\widehat{Q_c}(m)=\frac{4\pi}{L},\quad \text{for all } m\ge 0.$$

Proof. From [18] we have for $q = e^{-\pi K'/K}$ that the Fourier series of sn is

$$\operatorname{sn} u = \frac{2\pi}{kK} \sum_{m=0}^{\infty} \frac{q^{m+\frac{1}{2}}}{1-q^{2m+1}} \sin\Big[(2m+1)\frac{\pi}{2}\frac{u}{K}\Big].$$

Therefore, from the relations $\eta_1 = 4K/L$ and $\eta_2 = \eta_1 k$ we obtain

$$Q_c(\xi) = \frac{4\pi}{L} \sum_{m=0}^{\infty} \operatorname{sech}\Big[(2m+1)\frac{\pi}{2}\frac{K'}{K}\Big] \sin\Big[(2m+1)\frac{2\pi\xi}{L}\Big].$$

Hence from Theorem 6.1 we finish the proof. □

So, following the method in Section 3 we obtain the main result of this Subsection.

Theorem 6.3. *The Cauchy problem for the defocusing mKdV is not locally well-posed in $H^s_{\rm per}$, $s < -1/2$, in the sense that the mapping data-solution $u_0 \to u$ is not continuous.*

Then, from Christ, Colliander and Tao [20, Theorem 8] and from Theorem 6.3 above we obtain the following sharp ill-posedness type result for (6.1).

Theorem 6.4. *The Cauchy problem for the defocusing mKdV is not locally well-posed in $H^s_{\rm per}$, $s < 1/2$, in the sense that the mapping data-solution $u_0 \to u$ is not uniformly continuous.*

6.2. **The fifth order dmKdV case.** In this Subsection we focus to the ill-posedness result for the 5-dmKdV (6.2). We start writing the dmKdV equation (6.1) in the hamiltonian form

$$v_t = \partial_x E'_{dmKdV}(v(t))$$

with

$$E_{dmKdV}(v) = \frac{1}{2}\int (v'(x))^2 + v^4(x)\,dx.$$

Therefore, the snoidal wave solution Q_c in (6.9) is a critical point of the functional $H = E_{dmKdV} - cF$ for $F(u) = \|u\|^2/2$ and for every $c > 4\pi^2/L^2$. Hence from Subsection 5.1 we need to show that the kernel of the linear operator $H''(Q_c) = \mathcal{L}_{dmKdV}$,

$$\mathcal{L}_{dmKdV} = -\frac{d^2}{dx^2} + 6Q_c^2 - c, \tag{6.11}$$

is generated by $\frac{d}{dx}Q_c$. So, we establish the next periodic spectral problem for $\mathcal{L}_{sn} \equiv \mathcal{L}_{dmKdV}$, namely,

$$\begin{gathered} \mathcal{L}_{sn}\chi = \lambda\chi, \\ \chi(0) = \chi(L), \quad \chi'(0) = \chi'(L), \end{gathered} \tag{6.12}$$

and the following result is obtained in this context.

Theorem 6.5. *Let Q_c be the snoidal wave given in Theorem 6.1 for $c \in (\frac{4\pi^2}{L^2}, +\infty)$. Let*

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \cdots,$$

connote the eigenvalues of the problem (6.12). *Then*

$$\lambda_0 < \lambda_1 = 0 < \lambda_2 < \lambda_3 < \lambda_4$$

are all simple whilst, for $j \ge 5$, the λ_j are double eigenvalues. The λ_j only accumulate at $+\infty$.

Proof. Theorem 6.5 is a consequence of Floquet theory (Magnus and Winkler [31]) together with some particular facts about the periodic eigenvalue problem associated to the Lamé equation,

$$\begin{gathered} \frac{d^2}{dx^2}\Lambda + [\rho - 6k^2 sn^2(x;k)]\Lambda = 0 \\ \Lambda(0) = \Lambda(4K(k)), \quad \Lambda'(0) = \Lambda'(4K(k)). \end{gathered} \tag{6.13}$$

Indeed, we certainly know that $\lambda_0 < \lambda_1 \le \lambda_2$. Since $\mathcal{L}_{sn}\frac{d}{dx}Q_c = 0$ and $\frac{d}{dx}Q_c$ has 2 zeros in $[0, L)$, it follows that 0 is either λ_1 or λ_2. We will show that $0 = \lambda_1 < \lambda_2$. First, we perform the change of variable $\Lambda(x) \equiv \chi(x/\eta_1)$. Then, using the explicit form (6.9) for Q_c and that $\eta_1^2 k^2 = \eta_2^2$, the problem (6.12) is equivalent to the eigenvalue problem (6.13) with

$$\rho = \frac{c+\lambda}{\eta_1^2} = \frac{\eta_1^2 + \eta_2^2 + \lambda}{\eta_1^2} = 1 + k^2 + \frac{\lambda}{\eta_1^2}. \tag{6.14}$$

Next, from Floquet theory the Lamé equation in (6.13) (see Angulo [6, 7]) with boundary conditions $\Lambda^j(0) = \Lambda^j(2K(k))$, $j = 0, 1$, has exactly three instability intervals, so the first five eigenvalues for (6.13), $\{\rho_j : 0 \le j \le 4\}$, are simple and for $j \ge 5$, the ρ_j are double eigenvalues. In Angulo [6] the explicit values of that simple eigenvalues and its eigenfunctions are given. Indeed, the eigenvalues are:

$$\begin{gathered} \rho_0 = 2[1 + k^2 - \sqrt{1 - k^2 + k^4}], \quad \rho_1 = 1 + k^2, \quad \rho_2 = 1 + 4k^2, \\ \rho_3 = 4 + k^2, \quad \rho_4 = 2[1 + k^2 + \sqrt{1 - k^2 + k^4}]. \end{gathered} \tag{6.15}$$

Then from (6.14) we obtain the following relations

$$\begin{gathered} \rho_0 \mapsto \lambda_0 < 0, \quad \rho_1 \mapsto \lambda_1 = 0, \quad \rho_2 \mapsto \lambda_2 > 0, \\ \rho_3 \mapsto \lambda_3 > \lambda_2, \quad \rho_4 \mapsto \lambda_4 > \lambda_3. \end{gathered} \tag{6.16}$$

This completes the proof. □

Now, for $P(v) = v_{xxxx} + 6v^5 - 10v(v_x)^2 - 10v^2 v_{xx}$ we write (6.2) in the form

$$v_t = \partial_x P(v)$$

with $P(v) = R'(v)$ and R being the following conservation law for the dmKdV equation (6.1),

$$R(v) = \int \frac{1}{2}(v_{xx})^2 + v^6 + 5v^2(v_x)^2\,dx.$$

Then for $v_c(x,t) = Q_c(x + \lambda_{R,c}t)$ with

$$\lambda_{R,c} = c^2 + 2A,$$

we obtain from (6.4) that $P(Q_c) = \lambda_{R,c}Q_c$. Hence, $v_c(x,t)$ represents a travelling wave solution for the 5-dmKdV (6.2).

Then following the method in Section 3 we obtain the following result.

Theorem 6.6. *The Cauchy problem for the fifth order defocusing mKdV (6.2) is not locally well-posed in $H^s_{\rm per}$, $s < -1/2$, more precisely, the mapping data-solution $u_0 \to u$ fails to be continuous with respect to the $H^s_{\rm per}$.*

Remark: Theorem 6.5 and the property of concavity of the function

$$d(c) = E_{dmKdV}(Q_c) - c\frac{1}{2}\int Q_c^2(x)\,dx$$

($d''(c) < 0$) give us the basic information which could give the initial steps for obtaining a instability theory of the orbit generated by the snoidal wave Q_c, namely, $\Omega_{Q_c} = \{Q_c(\cdot + r) : r \in \mathbb{R}\}$, by the periodic flow generated by the defocusing mKdV. We note that the classical stability theories in [23] and [15] do not give a light for obtaining a conclusive answer about this issue. We plan to discuss this in a subsequent paper.

Acknowledgements. J. Angulo was supported partially by grant from CNPq/Brazil and by Edital Universal MCT/CNPq, 14/2009. S. Hakkaev was supported by FAPESP São Paulo/SP Brazil. The second author would like to express his thanks to the Institute of Mathematics and Statistic (IME) at the University of São Paulo for its hospitality. The authors are grateful to the reviewers for their fruitful remarks.

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