Electronic Journal of Differential Equations, Vol. 2010(2010), No. 13, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MULTIPLICITY OF SOLUTIONS FOR SOME FOURTH-ORDER M-POINT BOUNDARY-VALUE PROBLEMS

HAITAO LI, YANSHENG LIU

Abstract. Using the theory of the fixed point index in a cone and the Leray-Schauder degree, this paper investigates the existence and multiplicity of nontrivial solutions for a class of fourth order m-point boundary-value problems.

1. Introduction

Consider the following fourth order m-point boundary-value problem

$$\begin{gathered} u^{(4)}(t) = f(u(t), -u''(t)), \quad t \in (0,1); \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i); \\ u'''(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} \alpha_i u''(\eta_i), \end{gathered} \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given sign-changing continuous function, $m \geq 3$, $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$ and $\alpha_i > 0$ for $i = 1, \dots, m-2$, with

$$\sum_{i=1}^{m-2} \alpha_i < 1. \tag{1.2}$$

The multi-point boundary-value problems for ordinary differential equations arise in many areas of applied mathematics and physics. The existence of solutions of the fourth order two-point boundary-value problems and the second order m-point boundary-value problems have been studied intensively because of their interest to physics(see [1,2,6,7,9-11] and [5,8,13,14], resp.). However, to our best knowledge, the multiplicity of nontrivial solutions of the nonlinear multi-point boundary-value problems for fourth order differential equations has not been studied intensively.

2000 *Mathematics Subject Classification.* 34B16.
Key words and phrases. Fixed point index; Leray-Schauder degree; fourth order; m-point boundary-value problems; sign-changing solution.

Submitted December 17, 2009. Published January 21, 2010.
Supported by grants 209072 from Key Project of Chinese Ministry of Education, and ZR2009AM006 from the Natural Science Foundation of Shandong Province.

Recently in [12], Wei and Pang investigated the existence and multiplicity of nontrivial solutions for the following fourth order m-point boundary-value problems

$$\begin{gathered} x^{(4)}(t) = f(x(t), -x''(t)), \quad t \in (0,1); \\ x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i); \\ x''(0) = 0, \quad x''(1) = \sum_{i=1}^{m-2} \alpha_i x''(\eta_i), \end{gathered} \tag{1.3}$$

where $m \geq 3$, $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$ are constants and $\alpha_i \in (0,1)$ for $i = 1, \dots, m-2$ satisfies (1.2). $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

- (S0) The sequence of positive solutions of
$$\sin(\sqrt{s}) = \sum_{i=1}^{m-2} \alpha_i \sin(\eta_i \sqrt{s})$$
is $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$.
- (S1) $f(0,0) = 0$; and for $u > 0$, $v > 0$, $f(u,v) \geq 0$; for $u < 0$, $v < 0$, $f(u,v) \leq 0$; for $uv > 0$, $f(u,v)$ does not vanished.
- (S2) $f(u,v)$ has a continuous partial derivative at the point $(0,0)$, and there exists a positive integer n_0 such that $\mu_{2n_0} < 1 < \mu_{2n_0+1}$, where $\mu_n = \frac{\lambda_n^2}{a_0 + b_0\lambda_n}$, $a_0 = f'_u(0,0) > 0$, $b_0 = f'_v(0,0) > 0$.
- (S3) There exist $a_1 > 0$, $b_1 > 0$ such that
$$\lim_{|u|+|v| \to +\infty} \frac{|f(u,v) - a_1 u - b_1 v|}{|u| + |v|} = 0,$$
and there exists a positive integer n_1 such that $\gamma_{2n_1} < 1 < \gamma_{2n_1+1}$, where $\gamma_n = \frac{\lambda_n^2}{a_1 + b_1\lambda_n}$.
- (S4) There exists a constant $T > 0$ such that $|f(u,v)| < W^{-1}T$, for all $0 < |u| \leq T$, $0 < |v| \leq T$, where
$$W = \frac{1}{2} + \frac{\sum_{i=1}^{m-2} \alpha_i}{6(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)}.$$

Using the theory of the fixed point index in a cone and the Leray-Schauder degree, we obtain the following results.

Theorem 1.1. *Suppose* (S0)–(S4) *hold. Then* (1.3) *has at least six nontrivial solutions: Two positive solutions, two sign-changing solutions, and two negative solutions.*

Theorem 1.2. *Suppose* (S0)–(S4) *hold, and f is odd. Then* (1.3) *has at least eight nontrivial solutions.*

Motivated by [12], we investigate the existence and multiplicity of nontrivial solutions for (1.1). Let $X = C[0,1]$ with the norm $\|u\|_0 = \max_{t\in[0,\ 1]} |u(t)|$,

$$Y = \{u \in C^2[0,1] : u'(0) = 0, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}$$

with the norm $\|u\| = \max\{\|u\|_0, \|u'\|_0, \|u''\|_0\}$,

$$E = \{u \in C^3[0,1] \cap Y : u'''(0) = 0, u''(1) = \sum_{i=1}^{m-2} \alpha_i u''(\eta_i)\}$$

with the norm $\|u\| = \max\{\|u\|_0, \|u'\|_0, \|u''\|_0, \|u'''\|_0\}$. Then X, Y, E are Banach spaces. We define a cone in E as

$$P = \{x \in E : x(t) \geq 0, -x''(t) \geq 0, \forall t \in [0,1]\}.$$

Let

$$\Gamma(s) = \cos(\sqrt{s}) - \sum_{i=1}^{m-2} \alpha_i \cos(\eta_i \sqrt{s}), \quad s \in \mathbb{R}.$$

Then we can list the sequence of positive solutions of the equation $\Gamma(s) = 0$ as follows:

$$0 < s_1 < s_2 < \cdots < s_n < s_{n+1} < \dots .$$

Regarding the nonlinearity $f(u,v)$, we assume that it satisfies the following conditions:

- (H1) $f(0,0) = 0$; and for $u > 0$, $v > 0$, $f(u,v) \geq 0$; for $u < 0$, $v < 0$, $f(u,v) \leq 0$; for $uv > 0$, $f(u,v)$ does not vanish.
- (H2) There exist $a_0 > 0$, $b_0 > 0$, such that

  $$f(u,v) = a_0 u + b_0 v + o(|(u,v)|), \quad \text{as } |(u,v)| \to 0,$$

  where $(u,v) \in \mathbb{R} \times \mathbb{R}$, and $|(u,v)| := \max\{|x|,\ |y|\}$. And there exists a positive integer n_0 such that $\mu_{n_0} < 1 < \mu_{n_0+1}$, where $\mu_n = \frac{s_n^2}{a_0 + b_0 s_n}$.
- (H3) There exist $a_1 > 0$, $b_1 > 0$, such that

  $$f(u,v) = a_1 u + b_1 v + o(|(u,v)|), \quad \text{as } |(u,v)| \to +\infty,$$

  where $(u,v) \in \mathbb{R} \times \mathbb{R}$, and $|(u,v)| := \max\{|x|,\ |y|\}$. And there exists a positive integer n_1 such that $\gamma_{n_1} < 1 < \gamma_{n_1+1}$, where $\gamma_n = \frac{s_n^2}{a_1 + b_1 s_n}$.
- (H4) There exists a constant $T > 0$ such that $|f(u,v)| < M^{-1}T$, for all (u,v) satisfying $0 < |u| \leq T$, $0 < |v| \leq T$, where $M = \max\{1, N, N^2\}$ and $N = \frac{1}{2}(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i})$.

This paper is organized as follows. In Section 2, we present some basic properties of the fixed point index, and make use of these properties to obtain some important lemmas. In Section 3, we shall give our main results and their proofs.

2. Preliminaries

Let us list some properties of the fixed point index in a cone (for details, [3, 4]).

Lemma 2.1 ([4])**.** *Let P be a cone of the Banach space E, and $A : P \to P$ be completely continuous, suppose that A is differential at θ and ∞ along P and 1 is not an eigenvalue of $A'_+(\theta)$ and $A'_+(\infty)$ corresponding to a positive eigenfunction.*

(i) If $A'_+(\theta)$ has a positive eigenfunction corresponding to an eigenvalue greater than 1, and $A\theta = \theta$, then there exists $\tau > 0$ such that $i(A, P \cap B(\theta, r), P) = 0$ for any $0 < r < \tau$.

(ii) If $A'_+(\infty)$ has a positive eigenfunction which corresponds to an eigenvalue greater than 1, then there exists $\zeta > 0$ such that $i(A, P \cap B(\theta, R), P) = 0$ for any $R > \zeta$.

Lemma 2.2 ([4])**.** *Let $\theta \in \Omega$ and $A : P \cap \overline{\Omega} \to P$ be condensing. Suppose that $Ax \neq \mu x$, for all $x \in P \cap \partial\Omega$ and $\mu \geq 1$. Then $i(A, P \cap \Omega, P) = 1$.*

We first transform (1.1) into another form. Suppose $u(t)$ is a solution of (1.1). Let $v(t) = -u''(t)$. Note that

$$\begin{gathered} u''(t) + v(t) = 0, \quad t \in (0,1); \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{gathered} \tag{2.1}$$

thus $u(t)$ can be written as

$$u(t) = Lv(t), \tag{2.2}$$

where the operator L is defined by $Lv(t) = \int_0^1 H(t,s)v(s)ds$, for all $v \in Y$, and

$$H(t,s) = G(t,s) + \frac{\sum_{i=1}^{m-2} \alpha_i G(\eta_i, s)}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} t,$$

$$G(t,s) = \begin{cases} 1 - t, & 0 \leq s \leq t \leq 1; \\ 1 - s, & 0 \leq t \leq s \leq 1. \end{cases}$$

Therefore, we obtain the following equivalent form of (1.1):

$$\begin{gathered} v''(t) + f((Lv)(t), v(t)) = 0, \quad t \in (0,1); \\ v'(0) = 0, \quad v(1) = \sum_{i=1}^{m-2} \alpha_i v(\eta_i). \end{gathered} \tag{2.3}$$

Similar to (2.1) and (2.2), $v(t)$ can be written as

$$v(t) = (LF)u(t), \tag{2.4}$$

where $(Fu)(t) = f(u(t), -u''(t))$, $t \in (0,1)$, for all $u \in E$. From (2.2) and (2.4) we obtain $u(t) = (L^2F)u(t)$. Define $A = L^2F$, then it is easy to get the following lemma.

Lemma 2.3. *$u(t)$ is a solution of (1.1) if and only if $u(t)$ is a solution of the operator equation*

$$u(t) = Au(t). \tag{2.5}$$

Lemma 2.4. *Suppose (H1) holds. Then $A : P \to P$ is completely continuous.*

Proof. By the continuity of f, it is easy to see that $A : E \to E$ is completely continuous. Suppose $x(t) \in P$, condition (H1) implies

$$Ax(t) = (L^2F)x(t) \geq 0, \quad -(Ax)''(t) = (LF)x(t) \geq 0, \quad \forall t \in [0,1].$$

Therefore ,$Ax(t) \in P$. □

Remark 2.5. Similarly to the above, if f satisfies (H1), then $A : -P \to -P$ is completely continuous.

Set

$$Kx(t) = L^2x(t), \tag{2.6}$$

$$Qx(t) = L^2(-x'')(t). \tag{2.7}$$

Lemma 2.6. (i) *$K : C[0,1] \to E$ is a completely continuous linear operator;*
(ii) *$F : E \to C[0,1]$ is a continuous bounded operator, and $A = KF$;*
(iii) *$Q : E \to E$ is a completely continuous linear operator;*
(iv) *the sequences of all eigenvalues of the operators $a_0K + b_0Q$ and $a_1K + b_1Q$ are $\{\frac{1}{\mu_n}\}$, and $\{\frac{1}{\gamma n}\}$, respectively, where μ_n and γ_n are respectively defined by* (H2) *and* (H3).

Proof. Items (i)-(iii) have obvious proofs. To prove (iv), let μ be a positive eigenvalue of the linear operator $a_0K + b_0Q$, and $y \in E \setminus \{\theta\}$ be an eigenfunction corresponding to the eigenvalue μ. By (2.6) and (2.7), we have

$$\begin{gathered} \mu y^{(4)} = a_0y + b_0(-y''); \\ y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i); \\ y'''(0) = 0, \quad y''(1) = \sum_{i=1}^{m-2} \alpha_i y''(\eta_i). \end{gathered} \tag{2.8}$$

Define $D = \frac{d}{dt}$, $G = \mu D^4 - a_0 + b_0D^2$, then there exist complex constants r_1, r_2 such that

$$Gu = \mu(D^2 + r_1)(D^2 + r_2)u.$$

By the properties of differential operators, if (2.8) has a nonzero solution, then there exists $r_s, s \in \{1,2\}$ such that $r_s = s_k, k \in N_+$. In this case, $\cos t\sqrt{s_k}$ is a nonzero solution of (2.8). On substituting this solution into (2.8), we have

$$\mu s_k^2 - (a_0 + b_0s_k) = 0.$$

Hence, $\{\frac{a_0+b_0s_k}{s_k^2} = \frac{1}{\mu_k}\}$, $k = 1, 2, \dots$ is the sequence of eigenvalues of the operator $a_0K + b_0Q$. Then μ is one of the values

$$\frac{1}{\mu_1} > \frac{1}{\mu_2} > \dots > \frac{1}{\mu_n} > \dots$$

and the eigenfunction corresponding to the eigenvalue $1/\mu_n$ is

$$y_n(t) = C\cos(t\sqrt{s_n}), \quad t \in [0,1],$$

where C is a nonzero constant.

Similarly, we can show that the sequence of eigenvalues of the operator a_1K+b_1Q is $\{1/\mu_n\}$, $n = 1, 2, \dots$. □

Lemma 2.7. *Suppose (H2) and (H3) hold. Then the operator A is Frechet differentiable at θ and ∞. Moreover, $A'(\theta) = a_0K + b_0Q$ and $A'(\infty) = a_1K + b_1Q$.*

Proof. For any $x \in E$, we have

$$\begin{aligned}[Ax - A\theta - (a_0Kx + b_0Qx)](t) &= L^2[f(x(t), -x''(t)) - (\alpha_0 x(t) - \beta_0 x''(t))] \\ &= L^2 Bx(t), \quad \forall t \in [0,1]\end{aligned} \tag{2.9}$$

$$[Ax - A\theta - (a_0Kx + b_0Qx)]'(t) = -\int_0^t LBx(s)ds, \tag{2.10}$$

$$[Ax - A\theta - (a_0Kx + b_0Qx)]''(t) = -LBx(t), \tag{2.11}$$

$$[Ax - A\theta - (a_0Kx + b_0Qx)]'''(t) = \int_0^t Bx(s)ds, \tag{2.12}$$

where $Bx(t) = f(x(t), -x''(t)) - (a_0 x(t) - b_0 x''(t))$.

For each $\varepsilon > 0$, by (H2), there exists a $\delta > 0$ such that for any $0 < |u|,\ |v| < \delta$,

$$|\frac{f(u,v) - a_0u - b_0v}{\sqrt{u^2 + v^2}}| < \varepsilon.$$

This means

$$|f(u,v) - (a_0u + b_0v)| < \varepsilon\sqrt{u^2 + v^2}, \quad \forall 0 < |u|,\ |v| < \delta. \tag{2.13}$$

Then, for any $x \in E$ with $\|x\| < \delta$, by (2.9)-(2.13), we get

$$\|Ax - A\theta - (a_0Kx + b_0Qx)\| \le \sqrt{2}M\varepsilon\|x\|. \tag{2.14}$$

Consequently,

$$\lim_{\|x\|\to 0} \frac{\|Ax - A\theta - (a_0Kx + b_0Qx)\|}{\|x\|} = 0.$$

Therefore, A is Frechet differentiable at θ, and $A'(\theta) = a_0K + b_0Q$.

For each $\varepsilon > 0$, by (H3), there exists a constant $R_1 > 0$ such that $|f(u,v) - a_1u - b_1v| < \varepsilon(|u| + |v|)$, for $|u| + |v| > R_1$. Let

$$b = \max_{|u|+|v|\le R_1} |f(u,v) - a_1u - b_1v|,$$

then we have

$$|f(u,v) - a_1u - b_1v| \le \varepsilon(|u| + |v|) + b, \quad \forall u,\ v \in \mathbb{R}.$$

By a consideration similar to (2.14), we get

$$\|Ax - (a_1Kx + b_1Qx)\| \le (2\varepsilon\|x\| + b)M, \quad \forall x \in E.$$

Consequently, $\lim_{\|x\|\to\infty} \frac{\|Ax - (a_1Kx + b_1Qx)\|}{\|x\|} = 0$. This implies that A is Frechet differentiable at ∞, and $A'(\infty) = a_1K + b_1Q$. □

Lemma 2.8. *Suppose that* (H1)–(H3) *hold. Then*

(i) *there exists a constant r_0 such that $0 < r_0 < T$, and for any $0 < r \le r_0$, $i(A, P \cap B(\theta, r), P) = 0$, $i(A, -P \cap B(\theta, r), -P) = 0$;*
(ii) *there exists a constant $R_0 > T$ such that for any $R \ge R_0$, $i(A, P \cap B(\theta, R), P) = 0$, $i(A, -P \cap B(\theta, R), -P) = 0$.*

Proof. We prove conclusion (i) only; conclusion (ii) can be proved in the same way. By Lemma 2.7, $A : P \to P$ is completely continuous and Frechet differentiable along P at θ, and $A'_+(\theta) = a_0K + b_0Q, A\theta = \theta$. By Lemma 2.6 and (H2), $A'_+(\theta)$ has an eigenvalue $\frac{1}{\mu_1} = \frac{a_0+b_0 s_1}{s_1^2} > 1$, and $\frac{1}{\mu_1} > \frac{1}{\mu_2} > \cdots > \frac{1}{\mu_{n_0}} > 1 > \frac{1}{\mu_{n_0+1}} > \cdots > 0$, so 1 is not an eigenvalue of $A'_+(\theta)$ corresponding to a positive eigenfunction.

The eigenfunction corresponding to $\frac{1}{\mu_1}$ is $y(t) = \cos t\sqrt{s_1}, t \in [0, 1]$, where s_1 is the smallest positive solution of the equation

$$\cos(\sqrt{s}) = \sum_{i=1}^{m-2} \alpha_i \cos(\eta_i \sqrt{s}).$$

Since

$$\cos(\sqrt{0}) - \sum_{i=1}^{m-2} \alpha_i \cos 0 = 1 - \sum_{i=1}^{m-2} \alpha_i > 0,$$
$$\cos(\sqrt{(\pi/2)^2}) - \sum_{i=1}^{m-2} \alpha_i \cos(\eta_i \sqrt{(\pi/2)^2}) = -\sum_{i=1}^{m-2} \alpha_i \cos(\frac{\pi}{2}\eta_i) < 0.$$

Then by the mean-value theorem, $s_1 \in (0, (\frac{\pi}{2})^2)$. Consequently

$$y(t) = \cos(t\sqrt{s_1}) \geq 0, \quad t \in [0, 1].$$

And then it follows from Lemma 2.1 that there exists an $\tau_0 > 0$ such that $i(A, P \cap B(\theta, r), P) = 0$ for any $0 < r \leq \tau_0$.

Similarly, we can show that there exists an $\tau_1 > 0$ such that $i(A, -P \cap B(\theta, r), -P) = 0$ for any $0 < r \leq \tau_1$. Let $r_0 < \min\{T, \tau_0, \tau_1\}$, then the conclusion (i) holds and the proof is complete. □

3. Main Results

Now we are ready to give our main results.

Theorem 3.1. *Suppose* (H1)–(H4) *hold. Then* (1.1) *has at least four nontrivial solutions: Two positive solutions, and two negative solutions.*

Proof. For $x \in E$, we have

$$Ax(t) = L^2Fx(t), \quad (Ax)'(t) = -\int_0^t LFx(s)ds;$$
$$(Ax)''(t) = -LFx(t), \quad (Ax)''(t) = \int_0^t Fx(s)ds.$$

As for (2.14) we have

$$\|Ax\| \leq M\|x\|. \tag{3.1}$$

Therefore, for any $x \in E$, $\|x\| = T$, by (H4) and (3.1), $\|Ax\| < T = \|x\|$. Then by Lemma 2.2, we have

$$i(A, P \cap B(\theta, T), P) = 1, \tag{3.2}$$
$$i(A, -P \cap B(\theta, T), -P) = 1. \tag{3.3}$$

By Lemma 2.8, there exists two constants r_0, R_0, $0 < r_0 < T < R_0$, such that
$$i(A, P \cap B(\theta, r_0), P) = 0, \tag{3.4}$$
$$i(A, -P \cap B(\theta, r_0), -P) = 0, \tag{3.5}$$
$$i(A, P \cap B(\theta, R_0), P) = 0, \tag{3.6}$$
$$i(A, -P \cap B(\theta, R_0), -P) = 0. \tag{3.7}$$
Thus by (3.2), (3.4) and (3.6) we have
$$i(A, P \cap (B(\theta, T) \setminus \overline{B(\theta, r_0)}), P) = 1, \tag{3.8}$$
$$i(A, P \cap (B(\theta, R_0) \setminus \overline{B(\theta, T)}), P) = -1. \tag{3.9}$$
Therefore, the operator A has at least two fixed points $x_1 \in P \cap (B(\theta, R_0) \setminus \overline{B(\theta, T)})$ and $x_2 \in P \cap (B(\theta, T) \setminus \overline{B(\theta, r_0)})$, respectively. By Lemma 2.3, x_1 and x_2 are positive solutions of (1.1).

Similarly, by (3.3), (3.5) and (3.7) we have
$$i(A, -P \cap (B(\theta, R_0) \setminus \overline{B(\theta, T)}), -P) = -1, \tag{3.10}$$
$$i(A, -P \cap (B(\theta, T) \setminus \overline{B(\theta, r_0)}), -P) = 1. \tag{3.11}$$
Thus, the operator A has at least two fixed points $x_3 \in (-P) \cap (B(\theta, T) \setminus \overline{B(\theta, r_0)})$ and $x_4 \in (-P) \cap (B(\theta, R_0) \setminus \overline{B(\theta, T)})$, respectively. Obviously by Lemma 2.3, x_3 and x_4 are negative solutions of (1.1). □

By the method used in the proof of Theorem 3.1, it is easy to show the following corollaries.

Corollary 3.2. *Equation* (1.1) *has at least two different nontrivial solutions: One positive and one negative, provided that* (H1), (H2), (H4) *hold.*

Corollary 3.3. *Suppose that* (H1), (H3), (H4) *hold. Then* (1.1) *has at least two different nontrivial solutions: One positive and one negative.*

If the nonlinearity f does not depend on the second order derivative., then (1.1) becomes the following fourth-order m-point boundary-value problem
$$\begin{gathered} u^{(4)}(t) = f(u(t)), \quad t \in (0, 1); \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i); \\ u'''(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} \alpha_i u''(\eta_i). \end{gathered} \tag{3.12}$$

We have the following corollary.

Corollary 3.4. *If f satisfies*

- (H1') *$f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$; and $xf(x) \geq 0$, for $x \in \mathbb{R}$;*
- (H2') *there exists a positive integer n_0 such that $s_{n_0}^2 < a_0 < s_{n_0+1}^2$, where $a_0 = \lim_{x\to 0} \frac{f(x)}{x}$;*
- (H3') *there exists a positive integer n_1 such that $s_{n_1}^2 < a_1 < s_{n_1+1}^2$, where $a_1 = \lim_{x\to\infty} \frac{f(x)}{x}$;*

(H4') *There exists a constant* $T > 0$ *such that* $|f(x)| < M^{-1}T$*, for all* $0 < |x| \le T$*, where* M *is defined as in (H4).*

Then (3.12) *has at least four nontrivial solutions.*

Acknowledgements. The authors want to thank the anonymous referee for the suggestions on this paper.

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Haitao Li
Department of Mathematics, Shandong Normal University, Jinan, 250014, China
E-mail address: haitaoli09@gmail.com

Yansheng Liu
Department of Mathematics, Shandong Normal University, Jinan, 250014, China
E-mail address: yanshliu@gmail.com