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A REDUCED MODELLING APPROACH TO THE PRICING OF MORTGAGE BACKED SECURITIES

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Abstract. We consider a pricing model for mortgage backed securities formulated as a non-linear partial differential equation. We show that under certain feasible assumptions this model can be greatly simplified. We prove the well posedness of the simplified PDE.

1. Introduction

Reduced modelling is of great importance in the applied sciences. More often than not, models representative of complex real world phenomenon can be difficult and not pragmatic to deal with. The reasons for this are many fold. Large number of variables or parameters in a model, inherent non-linearities, and inconsistencies in initial data are some pertinent ones that come to mind. It is of utmost practical interest then to approach complex or non-linear problems with a view towards simplification, when possible. One practise is to consider various limiting cases of a parameter or variable of interest in a model. The equations in the limit, albeit unrealistic, might be easier to analyse or perform numerical computations on per se. Recently this approach has been carried out successfully in fluid convection problems, [14], and fluid convection in a porous media, [7].

In the current manuscript we derive a reduced model for the pricing of mortgage backed securities (abbreviated MBS). These securities have been criticised as the primary cause of the recent economic recession in the United States, [6]. They constitute over a trillion dollar issuance in the United States debt markets alone, [5].

Given the events of the past year, it is beneficial for both academics and practitioners to further understand the dynamics of mortgage backed securities. Since the seminal work of Black and Scholes [1], much importance has been given to pricing of derivative securities as partial differential equations.

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Brianni and Papi [11] derive a partial differential equation for the price $u(\mathbf{x}, t)$ of a mortgage backed security

$$\begin{gathered}\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+\mu(\mathbf{x},t)\nabla u-\rho\frac{|\sigma^T(\mathbf{x},t))\nabla u|^2}{u+h(\mathbf{x},t)+\xi(t)}-(r(t)-\tau)h(t)-r(t)u,\\ (\mathbf{x},t)\in\mathbb{R}^N\times(0,T),\\ u(\mathbf{x},0)=0.\end{gathered}\tag{1.1}$$

Here

$$r(t)=\delta(T-t),\quad \xi(t)=A_0e^{\int_0^{T-t}\delta(s)ds}.\tag{1.2}$$

Where δ is a deterministic discount rate and $\mathbf{x}$ are the various economic factors that the price of a MBS could depend on.

The reader is referred to [11] and [9] for a detailed derivation of (1.1). Briani and Papi [11] show that (1.1) posesses well defined viscosity solutions. See [2] for a in depth treatment of viscosity theory . They then show via certain sophisticated techniques, see [10], that these viscosity solutions are classical weak solutions. To this end, they need to assume a high degree of regularity for the coefficients and the prepayment function $h(\mathbf{x}, t)$. Note, the quadratic non-linearity in (1.1) is difficult to deal with and probably dissuades Briani and Papi from attempting the well posedness of (1.1) via a standard Galerkin truncation method in the first place. We will summarize the result of interest from [11].

Theorem 1.1 (Briani and Papi, 2004)**.** *Assume that the risk free rate δ is continuous and there exists a collection of stochastic processes $\{X_t^x : t \in [0,T]\}$, for $x \in \mathbb{R}^N$ which represents all the economic factors affecting MBS prices, satisfying*

$$dX_t^x=\mu(X_t^x,T-t)dt+\sigma(X_t^x,T-t)dB_t,\tag{1.3}$$

where $X_0^x = x$ and the coefficients μ and σ are continuous in $\mathbb{R}^N \times [0,T]$ and x-Lipschitz continuous uniformly in time. Furthermore assume

$$h(.,t)\in W^{4,\infty}(\Omega)\cap H^1(0,T;L^\infty(\Omega))\quad\text{and}\quad \frac{\partial}{\partial t}h(.,t)\in W^{2,\infty}(\Omega).\tag{1.4}$$

Then (1.1) admits a unique solution $u \in L^\infty(0,T;H^2(\Omega)) \cap H^1(0,T;L^\infty(\Omega))$.

Here $\Omega \subset \mathbb{R}^N$.

We show that under certain assumptions on $h(\mathbf{x}, t)$ and assuming constant μ and σ, (1.1) can be simplified to a diffusion equation, without the quadratic non-linearity. This is our reduced model. Furthermore we pose our problem on a bounded domain in $\mathbb{R}^3$, not on the whole space. This is easier for the purposes of numerics, which is our goal in a work in preparation, [8]. We also prove that the reduced model is well posed via application of the Banach fixed point theorem. In all our estimates C is a generic constant that can change in its value from line to line, and sometimes within the same line if so required.

2. Preliminaries and Assumptions

A MBS is formed by pooling together a group of mortgages and then selling this pool as a security to investors. The investors receive paymets via the monthly mortgage payments of the mortgagees, much like dividend payments from bonds. What makes MBS modelling interesting is that the cash flows from a MBS are not guaranteed due to a mortgagee having the option to prepay his or her mortgage at any time. This often happens due to fluctuations in interest rates, death of a spouse, divorce etc. See [4],[5], for more on MBS. We begin with a closer inspection of

$$h(\mathbf{x}, t) = MB(t)e^{-S(\mathbf{x},t)}. \tag{2.1}$$

Here $MB(t)$ is the remaining principal on a mortgage at time t. When there are no prepayments we have

$$MB(t) = MB(0)\frac{e^{\tau' T} - e^{\tau' t}}{e^{\tau' T} - 1}, \tag{2.2}$$

where τ' is the fixed rate paid by the mortgagor, while the investor receives $\tau < \tau'$. $S(\mathbf{x}, t)$ is the so called preypayment function. Generally, there is no closed form for this function. Most times practitioners use empirical data for its construction, [4]. Various models have been proposed for the form of $S(\mathbf{x}, t)$. A popular approach is to use a proportional hazards model, See [12] and [13].

$$S(\mathbf{x}, t) = g(t)\exp\Big(\sum_{j=1}^{n} \beta_j x_j\Big). \tag{2.3}$$

Here $g(t)$ is a log logistic hazard function given by

$$g(t) = \frac{p\gamma(\gamma t)^{P-1}}{1 + (\gamma t)^p}. \tag{2.4}$$

Where γ and p are appropriately chosen parameter values. x_j are the various other economic factors the mortgage preypayment could depend on such as interest rates, death, divorce etc. The interest rates are of primary importance in the current manuscript. These are the rates a mortgagee pays on his/her mortgage, such as the 15 year fixed or variable rate, or the 30 year fixed or variable rate. β_j are the regression coefficients between the control variate S and the input variables x_j. We next introduce the following assumptions as a first step towards deriving our reduced model.

Assumptions

(H1) The price u depends on 4 economic factors and time.

$$u(\mathbf{x}, t) = u(x_1, x_2, x_3, x_4, t). \tag{2.5}$$

(H2) Of primary concern is the interest rate represented by x_4. The preypayment function depends only on interest rate and time

$$S(\mathbf{x}, t) = S(x_4, t). \tag{2.6}$$

We want to consider a economic scenario of decreasing interest rates

$$x_4 \searrow 0. \tag{2.7}$$

The interest rate cannot hit 0 in reality, but our assumption is merely a theoretical construct to gain some insight into the behavior of MBS prices.

(H3) We consider the limiting situation

$$S(x_4, t) = \lim_{x_4 \to 0} g(t)e^{-\beta_4 x_4} = g(t) = S(t). \tag{2.8}$$

(H4) Constant mean and volatility are assumed

$$\mu(\mathbf{x}, t) = \mu, \quad \sigma(\mathbf{x}, t) = \sigma. \tag{2.9}$$

Also, we want to consider the non degenerate case, thus we assume

$$\sigma\sigma^T = I. \tag{2.10}$$

(H5) We assume $u(\mathbf{x},0) \in L^2(U)$. The function u is assumed locally bounded in [11]. We further assume u is compactly supported on some bounded domain $U \subset \mathbb{R}^3$, and takes 0 boundary values.

(H6) It is assumed in [11] that $u(\mathbf{x},t) + h(\mathbf{x},t) > 0$, we only require
$$u(\mathbf{x},t) > 0. \tag{2.11}$$

(H7) We assume $h(t) \in H^1(0,T)$.

(H8) Lastly we assume constant discount rates, so $r(t) = r$.

Remark 2.1. Assumption (H2) has been seen to an extent in the U.S. markets. See Figure 1. During the time period from August 2008 to April 2009, the 15 year fixed interest mortgage rate fell from about 6 percent to about 4.44 percent and the 30 year fixed interest mortgage rate fell from about 6.32 percent to about 4.87. See bankrate (http://www.bankrate.com) for these and similar figures. These were some of the sharpest declines witnessed in recent years due in large part to the ongoing financial crisis. More importantly, an actual economic scenario of sharply falling interest rates was realized.

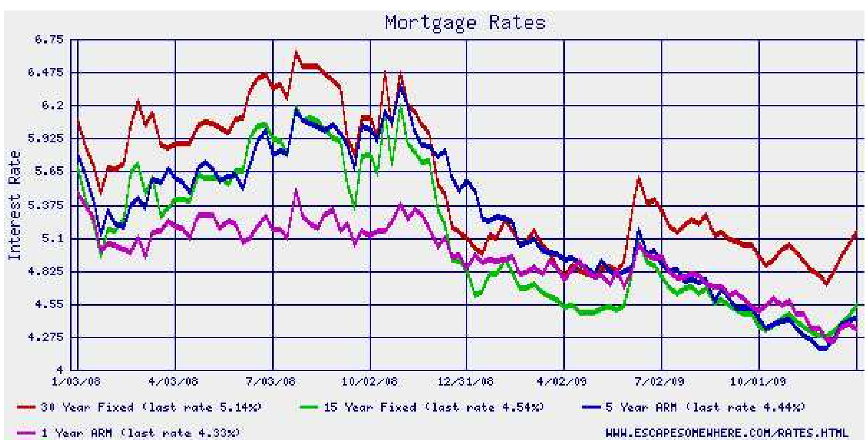

Figure 1. Various interest rates in 2008-2009

Note that (H2) is also feasible during a period when preypayments are fairly constant. This is actually an assumption when practitioners use the Bond market association preypayment model, see [4],[5].

Remark 2.2. In [11] it is required that $\frac{\partial h}{\partial t} \in L^\infty[0,T]$. Via the Sobolev embedding $H^2[0,T] \hookrightarrow L^\infty[0,T]$, we have that
$$\left|\frac{\partial h}{\partial t}\right|_{L^\infty[0,T]} \le C\left|\frac{\partial h}{\partial t}\right|_{H^2[0,T]} \le C|h|_{H^3[0,T]}. \tag{2.12}$$
Thus the authors in [11] require $H^3[0,T]$ control in time on h. Our requirement is less stringent.

Remark 2.3. Hypothesis (H5) is achieved via defining an appropriate trace operator
$$T : H^1(U) \to L^2(\partial U). \tag{2.13}$$

We then require $Tu = 0$, and use a trace Theorem, see [3]. This yields $u = 0$ on ∂U. Various other techniques are adaptable to this end. For example the method of cut off functions could also be used, [3]. It is also possible to prescribe other forms of boundary conditions.

Once the above assumptions are implemented, (1.1) takes the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \mu\nabla u - \rho\frac{|\sigma^T\nabla u|^2}{u + h(t) + \xi(t)} - (r-\tau)h(t) - ru \quad \text{in } U, \tag{2.14}$$

$$u = 0 \quad \text{on } \partial U, \tag{2.15}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}). \tag{2.16}$$

2.1. **Derivation of the Reduced Model.**

Lemma 2.4. *The MBS equation* (2.14)-(2.16) *can be reduced to a diffusion equation of the type*

$$\begin{gathered} \frac{\partial V}{\partial t} = \frac{1}{2}\Delta V + \tilde{K}(V,t), \\ V = 0 \quad \text{on } \partial\, U, \\ V(\mathbf{x},0) = V_0(\mathbf{x}), \end{gathered}$$

where

$$\begin{gathered} \tilde{K}(V,t) = (1-2C\rho)F(t)\Big(e^{(\frac{1}{2C\rho-1})s}\Big)(V+l)^{(\frac{2C\rho}{2C\rho-1})} - \frac{\partial l}{\partial t} + \frac{1}{2}\Delta l, \\ F(t) = \tau h(t) + r\xi(t) + h'(t) + \xi'(t), \\ l(\mathbf{x},t) = (h(t)+\xi(t))^{(1-2C\rho)} e^{-\Big(\Big((2C\rho-1)r - \frac{(\mu)^2}{2}\Big)t + \mu\mathbf{x}\Big)}. \end{gathered}$$

Proof. We begin by making the substitution

$$v(\mathbf{x},t) = u(\mathbf{x},t) + h(t) + \xi(t). \tag{2.17}$$

Inserting $v(\mathbf{x},t)$ in (2.14) yields

$$\frac{\partial v}{\partial t} - \frac{1}{2}\Delta v - \mu\nabla v + C\rho\frac{|\nabla v|^2}{v} + rv = F(t). \tag{2.18}$$

Here

$$F(t) = \tau h(t) + r\xi(t) + h'(t) + \xi'(t). \tag{2.19}$$

Next we set

$$v(\mathbf{x},t) = e^{f(\mathbf{x},t)}, \tag{2.20}$$

and insert this into (2.18) to yield

$$e^f\frac{\partial f}{\partial t} - \frac{1}{2}e^f\Delta f - \frac{1}{2}e^f|\nabla f|^2 - \mu e^f\nabla f + C\rho\frac{|e^f\nabla f|^2}{e^f} + re^f = F(t). \tag{2.21}$$

This yields

$$\frac{\partial f}{\partial t} - \frac{1}{2}\Delta f + (C\rho - \frac{1}{2})|\nabla f|^2 - \mu\nabla f = F(t)e^{-f} - r. \tag{2.22}$$

We first want to eliminate the nonlinear term $|\nabla f|^2$. To this end we make the logarithmic substitution

$$f(\mathbf{x},t) = \frac{1}{1-2C\rho}\ln(w(\mathbf{x},t)). \tag{2.23}$$

Inserting this into (2.22) yields

$$\frac{\partial w}{\partial t}-\frac{1}{2}\Delta w-\mu\nabla w+(1-2C\rho)r\ w=(1-2C\rho)F(t)w^{\left(\frac{2C\rho}{2C\rho-1}\right)}. \tag{2.24}$$

We now want to eliminate the convective term ∇w and the linear damping term w. Thus we introduce a function

$$s(\mathbf{x},t)=\Big((2C\rho-1)r-\frac{(\mu)^2}{2}\Big)t+\mu\mathbf{x}. \tag{2.25}$$

We then make the substitution

$$w(\mathbf{x},t)=e^{s(\mathbf{x},t)}k(\mathbf{x},t). \tag{2.26}$$

Once this is inserted into (2.24) we obtain

$$\begin{aligned}&\frac{\partial k}{\partial t}-\frac{1}{2}\Delta k\\&=\Big[(1-2C\rho)F(t)\Big(e^{\left(\frac{1}{2C\rho-1}\right)\left(\left((2C\rho-1)r-\frac{(\mu)^2}{2}\right)t+\mu\mathbf{x}\right)}\Big)(k)^{\left(\frac{2C\rho}{2C\rho-1}\right)}\Big].\end{aligned} \tag{2.27}$$

Note that via the above transforms we have

$$u(\mathbf{x},t)=\Big(e^{\left(\frac{1}{1-2C\rho}\right)\left(\left((2C\rho-1)r-\frac{(\mu)^2}{2}\right)t+\mu\mathbf{x}\right)}\Big)(k(\mathbf{x},t))^{\frac{1}{1-2C\rho}}-h(t)-\xi(t). \tag{2.28}$$

Hence on ∂U we have

$$u(\mathbf{x},t)=0=\Big(e^{\left(\frac{1}{1-2C\rho}\right)\left(\left((2C\rho-1)r-\frac{(\mu)^2}{2}\right)t+\mu\mathbf{x}\right)}\Big)(k(\mathbf{x},t))^{\frac{1}{1-2C\rho}}-h(t)-\xi(t). \tag{2.29}$$

This implies that on ∂U,

$$k(\mathbf{x},t)=(h(t)+\xi(t))^{(1-2C\rho)}e^{-\left(\left((2C\rho-1)r-\frac{(\mu)^2}{2}\right)t+\mu\mathbf{x}\right)}. \tag{2.30}$$

We homogenize by setting

$$l(\mathbf{x},t)=(h(t)+\xi(t))^{(1-2C\rho)}e^{-\left(\left((2C\rho-1)r-\frac{(\mu)^2}{2}\right)t+\mu\mathbf{x}\right)}, \tag{2.31}$$

and considering the function

$$V(\mathbf{x},t)=k(\mathbf{x},t)-l(\mathbf{x},t). \tag{2.32}$$

Inserting this into (2.27) yields

$$\frac{\partial V}{\partial t}=\frac{1}{2}\Delta V+\tilde{K}(V,t), \tag{2.33}$$

$$V=0 \quad \text{on } \partial U, \tag{2.34}$$

$$V(\mathbf{x},0)=V_0(\mathbf{x}). \tag{2.35}$$

Here

$$\begin{aligned}&\tilde{K}(V,t)\\&=(1-2C\rho)F(t)\Big(e^{\left(\frac{1}{2C\rho-1}\right)\left(\left((2C\rho-1)r-\frac{(\mu)^2}{2}\right)t+\mu\mathbf{x}\right)}\Big)(V+l)^{\left(\frac{2C\rho}{2C\rho-1}\right)}-\frac{\partial l}{\partial t}+\frac{1}{2}\Delta l.\end{aligned}$$

This proves the Lemma. We call (2.33)-(2.35) a reduced MBS model. $\square$

3. Well Posedness of the reduced MBS Model

In this section we show the well posedness of (2.33)-(2.35). We decide not to take the standard approach of performing a priori estimates on a Galerkin truncation aimed at extracting appropriate subsequences. Instead we use a more elegant fixed point method. The fixed point method depends upon demonstrating that the nonlinear term $\tilde{K}(V,t)$ satisfies certain properties. We address these by deriving certain Lemmas ultimately crucial to the proof of our main result.

3.1. **Apriori Estimates.** We begin with a definition.

Definition 3.1. A function V such that

$$V \in L^2(0,T;H_0^1(U)), \quad V' \in L^2(0,T;H^{-1}(U)), \tag{3.1}$$

is said to be a weak solution to (2.33)-(2.35) if

$$\langle V', v\rangle + B[V,v] = (K(V),v), \quad \text{a.e } 0 \le t \le T, \text{ for all } v \in H_0^1(U). \tag{3.2}$$

Here $\langle , \rangle$ denotes the pairing of $H^{-1}(U)$ and $H_0^1(U)$ and

$$B[V,v] = \int_U \nabla V \cdot \nabla v d\mathbf{x}. \tag{3.3}$$

In order to proceed we will derive certain properties that $\tilde{K}(V,t)$ satisfies. The first of these is Lipscitz continuity in V, in the L^2 norm. This is stated via the following Lemma.

Lemma 3.2. *Consider the function*

$$\tilde{K}(V,t) = (1-2C\rho)F(t)\Big(e^{\left(\frac{1}{2C\rho-1}\right)s}\Big)(V+l)^{\left(\frac{2C\rho}{2C\rho-1}\right)} - \frac{\partial l}{\partial t} + \frac{1}{2}\Delta l. \tag{3.4}$$

If $C\rho < 1/2$ then $\tilde{K}(V,t)$ is Lipschitz continuous in the L^2 norm with respect to the variable V.

Proof. We notice that

$$\begin{gathered} l(x,t) = (h(t)+\xi(t))^{(1-2C\rho)}e^{-s(x,t)} > 0,\\ |F(t)|_\infty \le C,\\ |s(\mathbf{x},t)|_\infty \le C. \end{gathered}$$

Since we have assumed $C\rho < 1/2$, we must have

$$\frac{C\rho}{2C\rho-1} < 0. \tag{3.5}$$

Thus

$$|\tilde{K}(V,t)|_2 \le C|V(\mathbf{x},t) + l(\mathbf{x},t)^{\frac{C\rho}{2C\rho-1}}|_2 \le C|V(\mathbf{x},t)^{\frac{C\rho}{2C\rho-1}}|_2. \tag{3.6}$$

Hence it suffices to prove the Lipschitz continuity of $V(\mathbf{x},t)^{\frac{C\rho}{2C\rho-1}}$ with respect to the variable V. We take the derivative of $V(\mathbf{x},t)^{\frac{C\rho}{2C\rho-1}}$ with respect to V to yield

$$\frac{d}{dV}V(\mathbf{x},t)^{\frac{C\rho}{2C\rho-1}} = \frac{C\rho}{2C\rho-1}V(\mathbf{x},t)^{\frac{C\rho}{2C\rho-1}-1} \le C. \tag{3.7}$$

This follows as

$$\begin{aligned} V(\mathbf{x},t) &= k(\mathbf{x},t) - (h(t)+\xi(t))^{1-2C\rho}e^{-s(\mathbf{x},t)}\\ &= (u(\mathbf{x},t)+h(t)+\xi(t))^{1-2C\rho}e^{-s(\mathbf{x},t)} - (h(t)+\xi(t))^{1-2C\rho}e^{-s(\mathbf{x},t)} > 0. \end{aligned}$$

Notice that the last inequality follows as $u > 0$ via assumption (H6). The boundedness of the derivative w.r.t V in turn implies that K is Lipschitz in V. Thus there exists a C such that

$$|K(V) - K(\tilde{V})| \le C|V - \tilde{V}|. \tag{3.8}$$

The L^2 inequality follows trivially

$$|K(V) - K(\tilde{V})|_2 \le C|V - \tilde{V}|_2. \tag{3.9}$$

□

The next property is an a priori estimate in $L^2(0,T;L^2(U))$. This is stated via the following Lemma.

Lemma 3.3. *The function $\tilde{K}(V,t)$ satisfies the following a priori estimates for $1/6 < C\rho < 1/4$,*

$$|\tilde{K}(V,t)|_{L^\infty(0,T;L^2(U))} \le C. \tag{3.10}$$

$$|\tilde{K}(V,t)|_{L^2(0,T;L^2(U))} \le C. \tag{3.11}$$

The constant C depends only on the L^2 norm of the initial data. If $1/4 < C\rho < 1/2$ then we require the initial data to be in L^α where

$$\alpha = \frac{2C\rho}{1-2C\rho} > 1. \tag{3.12}$$

Proof. We consider the case when $1/6 < C\rho < 1/4$. The proof for the case with $1/4 < C\rho < 1/2$ is analogous. We just require more smoothness of the initial data than $L^2(U)$ in this case. Consider

$$\frac{\partial V}{\partial t} - \frac{1}{2}\Delta V = \tilde{K}(V,t) \quad \text{in } U, \tag{3.13}$$

$$V = 0 \quad \text{on } \partial U, \tag{3.14}$$

$$V(\mathbf{x},0) = V_0(\mathbf{x}). \tag{3.15}$$

We multiply (3.13) by V^α, where $\alpha = \frac{2C\rho}{1-2C\rho}$ to yield

$$\frac{\partial V}{\partial t}V^\alpha - \frac{1}{2}\Delta V V^\alpha = \tilde{K}(V,t)V^\alpha \le \left(C(V+l)^{-\alpha}\right)V^\alpha \le C\ V^{-\alpha}V^\alpha = C. \tag{3.16}$$

This follows via

$$l(\mathbf{x},t) = (h(t)+\xi(t))^{(1-2C\rho)}e^{-s(\mathbf{x},t)} > 0, \tag{3.17}$$

and

$$|\tilde{K}(V,t)|_\infty \le C(V+l)^{-\alpha}. \tag{3.18}$$

We now integrate the above by parts over U. Since $V = 0$ on ∂U there are no boundary terms. Thus we obtain

$$\frac{1}{1+\alpha}\frac{\partial}{\partial t}|V|_{1+\alpha}^{1+\alpha} + \frac{\alpha}{2}\int_U |\nabla V|^2 V^{\alpha-1} d\mathbf{x} \le C. \tag{3.19}$$

Applying Poincaire's inequality yields

$$\frac{\partial}{\partial t}|V|_{1+\alpha}^{1+\alpha} + \frac{\alpha(\alpha+1)}{2}\int_U V^{\alpha+1} d\mathbf{x} \le C. \tag{3.20}$$

We can now multiply the above by $e^{\frac{\alpha(\alpha+1)}{2}t}$ and integrate in the time interval $[0,T]$ to yield

$$|V(T)|_{1+\alpha}^{1+\alpha} \le e^{-\frac{\alpha(\alpha+1)}{2}T}|V(0)|_{1+\alpha}^{1+\alpha} + Ce^{-\frac{\alpha(\alpha+1)}{2}t}\int_0^T e^{\frac{\alpha(\alpha+1)}{2}t}dt \le C. \tag{3.21}$$

Note since $C\rho < 1/4$, we have

$$\alpha = \frac{2C\rho}{1-2C\rho} < 1. \tag{3.22}$$

Thus via the compact embedding $L^2(U) \hookrightarrow L^{1+\alpha}(U)$, we obtain

$$|V(T)|_{1+\alpha}^{1+\alpha} \le e^{-\frac{\alpha(\alpha+1)}{2}T}|V(0)|_2^2 + Ce^{-\frac{\alpha(\alpha+1)}{2}t}\int_0^T e^{\frac{\alpha(\alpha+1)}{2}t}dt \le C. \tag{3.23}$$

Taking the supremum in time, in the interval $[0,T]$, implies

$$|V|_{L^\infty(0,T;L^{1+\alpha}(U))} \le C. \tag{3.24}$$

Via the compact embedding $L^{1+\alpha}(U) \hookrightarrow L^\alpha(U)$, this implies

$$|V|_{L^\infty(0,T;L^\alpha(U))} \le C. \tag{3.25}$$

We now multiply (3.13) by $V^{2\alpha}$,

$$\begin{aligned}\frac{\partial V}{\partial t}V^{2\alpha} - \frac{1}{2}\Delta V V^{2\alpha} &= \tilde{K}(V,t)V^{2\alpha}\\ &\le C(V+l)^{-\alpha}V^{2\alpha}\\ &\le CV^{-\alpha}V^{2\alpha} = CV^\alpha.\end{aligned} \tag{3.26}$$

We integrate by parts over U. Since $V=0$ on ∂U, there are no boundary terms. Thus we obtain

$$\frac{1}{1+2\alpha}\frac{\partial}{\partial t}|V|_{1+2\alpha}^{1+2\alpha} + \frac{\alpha}{2}\int_U |\nabla V|^2 V^{2\alpha-1}d\mathbf{x} \le C|V|_\alpha^\alpha. \tag{3.27}$$

Applying Poincaire's inequality yields

$$\frac{\partial}{\partial t}|V|_{1+2\alpha}^{1+2\alpha} + \frac{\alpha(\alpha+1)}{2}\int_U V^{2\alpha+1}d\mathbf{x} \le C|V|_\alpha^\alpha. \tag{3.28}$$

We can now multiply the above by $e^{\frac{\alpha(2\alpha+1)}{2}t}$ and integrate in the time interval $[0,T]$ to yield

$$|V(T)|^{1+2\alpha} \le e^{-\frac{\alpha(\alpha+1)}{2}T}|V(0)|^{1+2\alpha} + Ce^{-\frac{\alpha(2\alpha+1)}{2}t}\int_0^T e^{\frac{\alpha(2\alpha+1)}{2}t}C|V|_{L^\alpha}^\alpha dt \le C.$$

This follows via the estimate derived in (3.25). Now we take the supremum in time, in the interval [0,T], to obtain

$$|V|_{L^\infty(0,T;L^{1+2\alpha}(U))} \le C. \tag{3.29}$$

Via the compact Sobolev embedding $L^{1+2\alpha}(U) \hookrightarrow L^2(U)$, we obtain

$$|V|_{L^\infty(0,T;L^2(U))} \le C. \tag{3.30}$$

Thus via the compact Sobolev embedding $L^\infty(0,T;L^{1+2\alpha}(U)) \hookrightarrow L^2(0,T;L^2(U))$, we obtain

$$|\tilde{K}(V,t)|_{L^2(0,T;L^2(U))} \le C|V|_{L^\infty(0,T;L^{1+2\alpha}(U))} \le C. \tag{3.31}$$

and

$$|\tilde{K}(V,t)|_{L^\infty(0,T;L^2(U))} \le C|V|_{L^\infty(0,T;L^{1+2\alpha}(U))} \le C. \tag{3.32}$$

This proves the Lemma. ∎

Proposition 3.4. *Consider the partial differential equation*

$$\frac{\partial g(\mathbf{x},t)}{\partial t} = \frac{1}{2}\Delta g + H(t) \quad \text{in } U, \tag{3.33}$$

$$g = 0 \quad \text{on } \partial U, \tag{3.34}$$

$$g(\mathbf{x},0) = g_0(\mathbf{x}), \tag{3.35}$$

with $g \in L^2(0,T;H_0^1(U))$, $g' \in L^2(0,T;H^{-1}(U))$, $H \in L^2(0,T;L^2(U)$ *and* $g_0(\mathbf{x}) \in L^2(U)$*. There exists a unique weak solution to* (3.33)-(3.35)*. Thus the following is satisfied*

$$\langle g', v\rangle + B[g,v] = (K(g),v), \quad \text{a.e. } 0 \le t \le T, \text{ for all } v \in H_0^1(U). \tag{3.36}$$

This follows via the standard theory for parabolic PDE, see [3].

4. Main Results

We are now in a position to state our main result

Theorem 4.1. *Consider the reduced MBS model*

$$\frac{\partial V(\mathbf{x},t)}{\partial t} = \frac{1}{2}\Delta V + \tilde{K}(V,t) \quad \text{in } U, \tag{4.1}$$

$$V = 0 \quad \text{on } \partial U, \tag{4.2}$$

$$V(\mathbf{x},0) = V_0(\mathbf{x}). \tag{4.3}$$

For $1/6 < C\rho < 1/2$ *there exists a unique weak solution* V *with*

$$V \in L^2(0,T;H_0^1(U)) \quad \text{and} \quad V' \in L^2(0,T;H^{-1}(U)). \tag{4.4}$$

Proof. We will first prove the existence of a solution. To this end we work the space

$$X = C([0,T];L^2(U)) \tag{4.5}$$

equipped with a supremum type norm

$$|V| = \max_{t \le 0 \le T} |V(t)|_{L^2(U)}. \tag{4.6}$$

The strategy of our proof is as follows. Via Proposition 3.4 there exists a unique solution to (3.33)-(3.35), as long as the forcing function $H(t) \in L^2(0,T;L^2(U))$. Next we define an appropriate operator A as follows

$$A[V] = g. \tag{4.7}$$

We will show that A induces a contraction under the dynamics of (3.33)-(3.35) for T chosen small enough. The key is that for a given $V \in X$, we will set

$$H(t) = \tilde{K}(V,t), \tag{4.8}$$

and proceed via the standard energy method technique. We just insert $\tilde{K}(V,t)$ in place of $H(t)$ where appropriate. Recall, this is feasible as Lemma 3.3 tells us that

$$\tilde{K}(V,t) \in L^2(0,T;L^2(U)). \tag{4.9}$$

The idea becomes transparent in the estimates that follows.

Consider 2 solutions g and $\tilde{g}$. Via the definition of the operator A we have

$$A[V] = g, \ A[\tilde{V}] = \tilde{g}. \tag{4.10}$$

Now g and $\tilde{g}$ satisfy (3.33). Thus their difference satisfies

$$\frac{\partial(g(\mathbf{x},t)-\tilde{g}(\mathbf{x},t))}{\partial t} = \frac{1}{2}\Delta(g(\mathbf{x},t)-\tilde{g}(\mathbf{x},t)) + H(t) - \tilde{H}(t) \quad \text{in } U, \tag{4.11}$$

$$g(\mathbf{x},t)-\tilde{g}(\mathbf{x},t) = 0 \quad \text{on } \partial U, \tag{4.12}$$

$$g(\mathbf{x},0)-\tilde{g}(\mathbf{x},0) = 0. \tag{4.13}$$

We multiply (4.11) by $g-\tilde{g}$ and integrate by parts over U. There are no boundary terms as $g(\mathbf{x},t)-\tilde{g}(\mathbf{x},t) = 0$ on ∂U. Thus we obtain

$$\frac{d}{dt}|g-\tilde{g}|_2^2 + 2|g-\tilde{g}|_{H_0^1}^2 = 2\langle g-\tilde{g}, H-\tilde{H}\rangle. \tag{4.14}$$

We now use the compact embedding $H_0^1(U) \hookrightarrow L^2(U)$ to yield

$$\frac{d}{dt}|g-\tilde{g}|_2^2 + 2|g-\tilde{g}|_2^2 \le 2\langle g-\tilde{g}, H-\tilde{H}\rangle. \tag{4.15}$$

Note that via the Cauchy inequality, with ϵ, we have

$$2\langle g-\tilde{g}, h-\tilde{h}\rangle \le C\epsilon|g-\tilde{g}|_2^2 + \frac{1}{\epsilon}|H-\tilde{H}|_2^2. \tag{4.16}$$

We insert this estimate into (4.15) to yield

$$\begin{aligned}\frac{d}{dt}|g-\tilde{g}|_2^2 + 2|g-\tilde{g}|_2^2 &\le C\epsilon|g-\tilde{g}|_2^2 + \frac{1}{\epsilon}|H-\tilde{H}|_2^2 \\ &= C\epsilon|g-\tilde{g}|_2^2 + \frac{1}{\epsilon}|K(V)-K(\tilde{V})|_2^2 \\ &\le C\epsilon|g-\tilde{g}|_2^2 + \frac{C}{\epsilon}|V-\tilde{V}|_2^2.\end{aligned}$$

Here we have used the Lipschitz property of $\tilde{K}(V,t)$. We now choose ϵ such that $2 > C\epsilon$. This yields

$$\frac{d}{dt}|g-\tilde{g}|_2^2 + (2-C\epsilon)|g-\tilde{g}|_2^2 \le C|V-\tilde{V}|_2^2. \tag{4.17}$$

Using the positivity of $(2-C\epsilon)|g-\tilde{g}|_2^2$ we obtain

$$\frac{d}{dt}|g-\tilde{g}|_2^2 \le C|V-\tilde{V}|_2^2. \tag{4.18}$$

Now recall, via the definition of the operator A, that $g = A[V]$; therefore,

$$\frac{d}{dt}|A[V]-A[\tilde{V}]|_2^2 \le C|V-\tilde{V}|_2^2. \tag{4.19}$$

Integration of the above on the time interval $[0,T]$ yields

$$|A[V]-A[\tilde{V}]|_2 \le C\int_0^T |V-\tilde{V}|_2^2 \ leq (CT)^{\frac{1}{2}}|V-\tilde{V}|_2. \tag{4.20}$$

Now we choose T such that

$$(CT)^{1/2} \le \gamma \le 1. \tag{4.21}$$

This implies that, for any $t < T_1 = 1/C$,

$$|A[V]-A[\tilde{V}]|_2 \le \gamma|V-\tilde{V}|_2, \quad \gamma < 1. \tag{4.22}$$

Thus for a given $V \in X$, A induces a contraction on the time interval $[0, T_1]$. Via the Banach fixed point theorem, see [3], the operator A must posses a fixed point. Thus there must exist a V^* such that

$$A[V^*] = V^* \tag{4.23}$$

However via the definition of the operator A, $A[V] = g$, so

$$A[V^*] = V^* = g. \tag{4.24}$$

This implies the existence of a V^* which is also a solution to (3.33). The solution is valid on the short time interval $[0, T_1]$. From the existence of a solution we have

$$V^*(T_1) \in H_0^1(U). \tag{4.25}$$

We can now repeat the above argument to extend the solution to say $[T_1, 2T_1]$ and eventually to $[0, T]$, where T is the terminal point in the original time interval.

To demonstrate the uniqueness we consider two different solutions $g = V$ and $\tilde{g} = \tilde{V}$ and insert them into (4.18) and integrate in the time interval $[0, T]$ to yield

$$|V - \tilde{V}|_2^2 \le C \int_0^T |V - \tilde{V}|_2^2 dt. \tag{4.26}$$

Now via the application of Gronwall's Lemma in integral form we have

$$|V - \tilde{V}|_2^2 = 0. \tag{4.27}$$

This implies that $V = \tilde{V}$ which gives us the uniqueness. ☐

We now state a Corrollary which is an immediate consequence of our main result.

Corollary 4.2. *There exists a unique weak solution u to* (2.14) *with*

$$u \in L^2(0, T; H_0^1(U)) \quad \text{and} \quad u' \in L^2(0, T; H^{-1}(U)). \tag{4.28}$$

Proof. Via Theorem 4.1, we have the existence of a unique solution V to (2.33)-(2.35) with

$$V \in L^2(0, T; H_0^1(U)) \quad \text{and} \quad V' \in L^2(0, T; H^{-1}(U)). \tag{4.29}$$

Now u is transformed to V via a series of continuous transformations. Essentially

$$u(\mathbf{x}, t) = e^{\frac{1}{1-2C\rho} s(\mathbf{x},t)} \Big(V(x,t) + (h(t) + \xi(t))^{1-2C\rho} e^{-s(\mathbf{x},t)} \Big) - h(t) - \xi(t). \tag{4.30}$$

Thus via the uniqueness of V and the continuity of the transformations there exists a unique solution u to (2.14)-(2.15) with

$$u \in L^2(0, T; H_0^1(U)) \quad \text{and} \quad u' \in L^2(0, T; H^{-1}(U)) \tag{4.31}$$

☐

4.1. **Concluding Remarks.** We would like to point out certain open directions as well as highlight certain future endeavours. Since we have established the well posedness of our reduced MBS model, our next aim is to solve it numerically. Furthermore we want to test our results against real market prices realized over the previous twelve months. The reduced model would be relatively easier to perform numerical computations on as essentially we are solving

$$\frac{\partial V(x,t)}{\partial t} = \frac{1}{2}\Delta V + V^{-\alpha} \quad \text{in } U,\ V > 0,\ \alpha > 0, \tag{4.32}$$

$$V = 0 \quad \text{on } \partial U, \tag{4.33}$$

$$V(x,0) = V_0(x). \tag{4.34}$$

This is currently under investigation in [8]. It would also be worthwhile to investigate the well posedness in the case $C\rho > 1/2$. This case is trickier as it would entail $-\alpha > 0$ in (4.32). This seems problematic. When $-\alpha = 2$, the model is essentially like

$$\frac{\partial V(x,t)}{\partial t} = \frac{1}{2}\Delta V + V^{2} \quad \text{in } U, \tag{4.35}$$

$$V = 0 \quad \text{on } \partial U, \tag{4.36}$$

$$V(x,0) = V_0(x). \tag{4.37}$$

The above problem is ill posed, see [3] for a detailed proof. One approach might be to use weighted Sobolev spaces to do away with the troublesome exponent α. However, the Theorems derived therein would only be valid in the weighted spaces. This case is crucial to address investors who want less exposure to risk. Recall that ρ is a measure of risk aversion. Our assumption that $C\rho < 1/2$ limits us to the scenario where

$$\rho \ll 1. \tag{4.38}$$

This is typically the case where an investor is risk friendly. It is only fair that investors at the other end of the spectrum are also considered. These are some of the interesting unanswered questions that we can pose at this juncture.

It is our hope that the proposed simplified model is a small step to gain some intuition behind the "breakdown" of the financial machinery over the course of the last year, particularly due to the meltdown in the mortgage backed securities market. We believe that when one considers the ramifications of the above, any effort to first further understand the dynamics of these instruments and then hopefully propose a remedy, is ultimately not futile.

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