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LIÉNARD TYPE P-LAPLACIAN NEUTRAL RAYLEIGH EQUATION WITH A DEVIATING ARGUMENT

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Abstract. Based on Manásevich-Mawhin continuation theorem, we prove the existence of periodic solutions for Liénard type p-Laplacian neutral Rayleigh equations with a deviating argument,

$$(\phi_p(x(t)-cx(t-\sigma))')'+f(x(t))x'(t)+g(t,x(t-\tau(t)))=e(t).$$

An example is provided to illustrate our results.

1. Introduction

The existence of periodic solutions for Liénard type p-Laplacian equation with a deviating argument

$$(\phi_p(x'(t)))'+f(x(t))x'(t)+g(t,x(t-\tau(t)))=e(t) \tag{1.1}$$

has been studied using the coincidence degree theory [1]. Zhu and Lu [6], studied the existence of periodic solution for p-Laplacian neutral functional differential equation with a deviating argument when $p>2$

$$(\phi_p(x(t)-cx(t-\sigma))')'+g(t,x(t-\tau(t)))=e(t). \tag{1.2}$$

They obtained some results by transforming (1.2) into a two-dimensional system to which Mawhin's continuation theorem was applied.

Peng [4] discussed the existence of periodic solution for p-Laplacian neutral Rayleigh equation with a deviating argument

$$(\phi_p(x(t)-cx(t-\sigma))')'+f(x'(t))+g(t,x(t-\tau(t)))=e(t) \tag{1.3}$$

and obtained the existence of periodic solutions under the assumption $f(0)=0$ and $\int_0^T e(t)dt=0$.

Throughout this paper, $2<p<\infty$ is a fixed real number. The conjugate exponent of p is denoted by q; i.e., $\frac{1}{p}+\frac{1}{q}=1$. Let $\phi_p:\mathbb{R}\to\mathbb{R}$ be defined by $\phi_p(s)=|s|^{p-2}s$ for $s\neq 0$, and $\phi_p(0)=0$. In this article, we will investigate the existence of periodic solution to the Liénard type p-Laplacian neutral Rayleigh equation

$$(\phi_p(x(t)-cx(t-\sigma))')'+f(x(t))x'(t)+g(t,x(t-\tau(t)))=e(t) \tag{1.4}$$

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where f, e and τ are real continuous functions on $\mathbb{R}$. τ and e are periodic with period T, $T > 0$ is fixed. g is continuous function defined on $\mathbb{R}^2$ and T-periodic in the first argument, c and σ are constants such that $|c| \neq 1$.

2. Preliminaries

Let $\mathcal{C}_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}$ and $\mathcal{C}_T^1 = \{x \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}$. $\mathcal{C}_T$ is a Banach space endowed with the norm $\|x\|_\infty = \max|x(t)|_{t\in[0,T]}$. $\mathcal{C}_T^1$ is a Banach space endowed with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$. In what follows, we will use $\|.\|_p$ to denote the L^P-norm. We also define a linear operator $A : \mathcal{C}_T \to \mathcal{C}_T$,

$$(Ax)(t) = x(t) - cx(t-\sigma).$$

Lemma 2.1 ([2, 5]). *If $|c| \neq 1$, then A has continuous bounded inverse on $\mathcal{C}_T$, and*

(1) $\|A^{-1}x\|_\infty \le \frac{\|x\|_\infty}{|1-|c||}$, *for all $x \in \mathcal{C}_T$;*

(2)
$$(A^{-1}x)(t) = \begin{cases} \sum_{j\ge 0} c^j x(t-j\sigma), & |c| < 1 \\ -\sum_{j\ge 1} c^{-j} x(t+j\sigma), & |c| > 1. \end{cases}$$

(3)
$$\int_0^T |(A^{-1}x)(t)|dt \le \frac{1}{|1-|c||}\int_0^T |x(t)|dt, \quad \forall x \in \mathcal{C}_T.$$

Lemma 2.2 ([4]). *If $|c| \neq 1$ and $p > 1$, then*

$$\int_0^T |(A^{-1}x)(t)|^p dt \le \frac{1}{|1-|c||^p}\int_0^T |x(t)|^p dt, \quad \forall x \in \mathcal{C}_T. \tag{2.1}$$

For the T-periodic boundary value problem

$$(\phi_p(x'(t)))' = \widetilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{2.2}$$

where $\widetilde{f} \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, we have the following result.

Lemma 2.3 ([3]). *Let Ω be an open bounded set in $\mathcal{C}_T^1$, and let the following conditions hold:*

(i) *For each $\lambda \in (0, 1)$, the problem*
$$(\phi_p(x'(t)))' = \lambda\widetilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T)$$
has no solution on $\partial\Omega$.

(ii) *The equation*
$$F(a) = \frac{1}{T}\int_0^T \widetilde{f}(t, a, 0)dt = 0$$
has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) *The Brouwer degree of F,* $\deg(F, \Omega \cap \mathbb{R}, 0) \neq 0$.

Then the T-periodic boundary value problem (2.2) *has at least one periodic solution on $\overline{\Omega}$.*

3. Main results

Theorem 3.1. *Suppose that $p > 2$ and there exist constants $r_1 \geq 0$, $r_2 \geq 0$, $d > 0$ and $k > 0$ such that*

- (A1) $|f(x)| \leq k + r_1|x|^{p-2}$ *for* $x \in \mathbb{R}$;
- (A2) $x[g(t,x) - e(t)] < 0$ *for* $|x| > d$ *and* $t \in \mathbb{R}$;
- (A3) $\lim_{x\to-\infty} \frac{|g(t,x)-e(t)|}{|x|^{p-1}} = r_2$.

Then (1.4) *has at least one T-periodic solution if*

$$\frac{1}{2^{p-1}}(1+|c|)T^{p-1}(r_1 + Tr_2) < |1-|c||^p.$$

Proof. Consider the homotopic equation of (1.4) as follows:

$$(\phi_p(x(t)-cx(t-\sigma))')' + \lambda f(x(t))x'(t) + \lambda g(t, x(t-\tau(t))) = \lambda e(t), \quad \lambda \in (0,1). \tag{3.1}$$

We claim that the set of all possible periodic solution of (3.1) are bounded in $\mathcal{C}_T^1$.

Let $x(t) \in \mathcal{C}_T^1$ be an arbitrary solution of (3.1) with period T. By integrating two sides of (3.1) over $[0,T]$, and noticing that $x'(0) = x'(T)$, we have

$$\int_0^T [g(t, x(t-\tau(t))) - e(t)]dt = 0. \tag{3.2}$$

By the integral mean value theorem, there is a constant $\xi \in [0,T]$ such that $g(\xi, x(\xi-\tau(\xi))) - e(\xi) = 0$. So from assumption (A2), we can get $|x(\xi-\tau(\xi))| \leq d$. Let $\xi - \tau(\xi) = mT + \overline{\xi}$, where $\overline{\xi} \in [0,T]$, and m is an integer. Then, we have

$$|x(t)| = |x(\overline{\xi}) + \int_{\overline{\xi}}^t x'(s)ds| \leq d + \int_{\overline{\xi}}^t |x'(s)|ds, \quad t \in [\overline{\xi}, \overline{\xi}+T],$$

and

$$|x(t)| = |x(t-T)| = |x(\overline{\xi}) - \int_{t-T}^{\overline{\xi}} x'(s)ds| \leq d + \int_{t-T}^{\overline{\xi}} |x'(s)|ds, \quad t \in [\overline{\xi}, \overline{\xi}+T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned}
\|x\|_\infty &= \max_{t\in[0,T]} |x(t)| = \max_{t\in[\overline{\xi},\overline{\xi}+T]} |x(t)| \\
&\leq \max_{t\in[\overline{\xi},\overline{\xi}+T]} \Big\{ d + \frac{1}{2}\Big(\int_{\overline{\xi}}^t |x'(s)|ds + \int_{t-T}^{\overline{\xi}} |x'(s)|ds \Big) \Big\} \\
&\leq d + \frac{1}{2}\int_0^T |x'(s)|ds.
\end{aligned} \tag{3.3}$$

In view of $\frac{1}{2^{p-1}}(1+|c|)T^{p-1}(r_1+Tr_2) < |1-|c||^p$, there exist a constant $\varepsilon > 0$ such that

$$\frac{1}{2^{p-1}}(1+|c|)T^{p-1}(r_1 + T(r_2+\varepsilon)) < |1-|c||^p.$$

From assumption (A3), there exist a constant $\rho > d$ such that

$$|g(t, x(t-\tau(t))) - e(t)|dt \leq (r_2+\varepsilon)|x|^{p-1} \quad for\ t \in \mathbb{R}\ \ and\ \ x < -\rho. \tag{3.4}$$

Denote $E_1 = \{t \in [0,T], x(t-\tau(t)) \leq -\rho\}$, $E_2 = \{t \in [0,T], |x(t-\tau(t))| < \rho\}$, $E_3 = \{t \in [0,T], x(t-\tau(t)) \geq \rho\}$. By (3.2), it is easy to see that

$$\Big(\int_{E_1} + \int_{E_2} + \int_{E_3} \Big)[g(t, x(t-\tau(t))) - e(t)]dt = 0. \tag{3.5}$$

Hence

$$\begin{aligned}\int_{E_3} |g(t,x(t-\tau(t)))-e(t)|dt &= -\int_{E_3} [g(t,x(t-\tau(t)))-e(t)]dt \\ &= \Big(\int_{E_1}+\int_{E_2}\Big)[g(t,x(t-\tau(t)))-e(t)]dt \\ &\le \Big(\int_{E_1}+\int_{E_2}\Big)|g(t,x(t-\tau(t)))-e(t)|dt.\end{aligned} \tag{3.6}$$

Therefore, by (3.4) and (3.6), we obtain

$$\begin{aligned}\int_0^T |g(t,x(t-\tau(t)))-e(t)|dt &= \Big(\int_{E_1}+\int_{E_2}+\int_{E_3}\Big)|g(t,x(t-\tau(t)))-e(t)|dt \\ &\le 2\Big(\int_{E_1}+\int_{E_2}\Big)|g(t,x(t-\tau(t)))-e(t)|dt \\ &\le 2\int_{E_1}(r_2+\varepsilon)|x(t-\tau(t))|^{p-1}dt+2\widetilde{g}_\rho T \\ &\le 2(r_2+\varepsilon)T\|x\|_\infty^{p-1}+2\widetilde{g}_\rho T.\end{aligned} \tag{3.7}$$

Where $\widetilde{g}_\rho = \max_{t\in E_2}|g(t,x(t-\tau(t)))-e(t)|$. Multiplying both sides of (3.1) by $(Ax)(t)=x(t)-cx(t-\sigma)$ and integrating them over $[0,T]$, we have

$$\begin{aligned}\|Ax'\|_p^p &= \lambda\int_0^T (Ax)(t)\left[f(x(t))x'(t)+g(t,x(t-\tau(t)))-e(t)\right]dt \\ &\le (1+|c|)\|x\|_\infty\int_0^T \left[|f(x(t))x'(t)|+|g(t,x(t-\tau(t)))-e(t)|\right]dt.\end{aligned} \tag{3.8}$$

From assumption (A1), we obtain.

$$\int_0^T |f(x(t))x'(t)|dt \le k\int_0^T |x'(t)|dt + r_1\int_0^T |x'(t)||x(t)|^{p-2}dt. \tag{3.9}$$

Using Hölder inequality, and substituting (3.3) into (3.9), we obtain

$$\int_0^T |f(x(t))x'(t)|dt \le kT^{1/q}\|x'\|_p + r_1T^{1/q}\|x'\|_p\Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p-2}. \tag{3.10}$$

From (3.3) and (3.7), we have

$$\int_0^T |g(t,x(t-\tau(t)))-e(t)|dt \le 2\,\widetilde{g}_\rho\, T + 2(r_2+\varepsilon)T\Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p-1}. \tag{3.11}$$

Substituting (3.10), (3.11) and (3.3) into (3.8), we obtain

$$\begin{aligned}&\|Ax'\|_p^p \\ &\le (1+|c|)\Big[kT^{1/q}\|x'\|_p\Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big) \\ &\quad + \Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p-1} r_1T^{1/q}\|x'\|_p \\ &\quad + 2(r_2+\varepsilon)T\Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big)^p + 2\widetilde{g}_\rho T\Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big)\Big].\end{aligned} \tag{3.12}$$

Case(1). If $\int_0^T |x'(t)|dt = 0$, from (3.3), we have $\|x\|_\infty < d$.

Case(2). If $\int_0^T |x'(t)|dt > 0$, then

$$\Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p-1} = \Big(\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p-1}\Big(1+\frac{2d}{\int_0^T |x'(t)|dt}\Big)^{p-1}. \tag{3.13}$$

By elementary analysis, there is a constant $\delta > 0$ such that

$$(1+u)^{p-1} \le 1+pu, \quad \forall u \in [0,\delta]. \tag{3.14}$$

If $2d/\int_0^T |x'(t)|dt > \delta$, then $\int_0^T |x'(t)|dt < 2d/\delta$, so from (3.3), we have $\|x\|_\infty < d + (d/\delta)$.

If $2d/\int_0^T |x'(t)|dt \le \delta$, by (3.13) and (3.14),

$$\begin{aligned}
&\Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p-1}\\
&\le \Big(\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p-1}\Big(1+\frac{2pd}{\int_0^T |x'(t)|dt}\Big)\\
&\le \big(\frac{1}{2}\big)^{p-1}\Big(\int_0^T |x'(t)|dt\Big)^{p-1} + \big(\frac{1}{2}\big)^{p-2}pd\Big(\int_0^T |x'(t)|dt\Big)^{p-2}\\
&\le \big(\frac{1}{2}\big)^{p-1}T^{\frac{p-1}{q}}\|x'\|_p^{p-1} + \big(\frac{1}{2}\big)^{p-2}pdT^{\frac{p-2}{q}}\|x'\|_p^{p-2}.
\end{aligned} \tag{3.15}$$

Similarly, from(3.14), there is a constant $\delta' > 0$ such that

$$(1+u)^p \le 1+(1+p)u, \quad \forall u \in [0,\delta'] \tag{3.16}$$

If $2d/\int_0^T |x'(t)|dt > \delta'$, then $\int_0^T |x'(t)|dt < 2d/\delta'$, so from (3.3), we have $\|x\|_\infty < d + (d/\delta')$.

If $2d/\int_0^T |x'(t)|dt \le \delta'$, by (3.16), we have

$$\begin{aligned}
&\Big(d+\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p}\\
&\le \Big(\frac{1}{2}\int_0^T |x'(t)|dt\Big)^{p}\Big(1+\frac{2(p+1)d}{\int_0^T |x'(t)|dt}\Big)\\
&\le \big(\frac{1}{2}\big)^{p}\Big(\int_0^T |x'(t)|dt\Big)^{p} + \big(\frac{1}{2}\big)^{p-1}(p+1)d\Big(\int_0^T |x'(t)|dt\Big)^{p-1}\\
&\le \big(\frac{1}{2}\big)^{p}T^{\frac{p}{q}}\|x'\|_p^{p} + \big(\frac{1}{2}\big)^{p-1}(p+1)dT^{\frac{p-1}{q}}\|x'\|_p^{p-1}.
\end{aligned} \tag{3.17}$$

Substituting (3.15) and (3.17) into (3.12) and using Hölder inequality, we obtain

$$\begin{aligned}
\|Ax'\|_p^p \le (1+|c|)[(&\frac{1}{2^{p-1}}(r_2+\varepsilon)T^p + \frac{1}{2^{p-1}}r_1T^{\frac{p}{q}})\|x'\|_p^p + a_0\|x'\|_p^{p-1}\\
&+ a_1\|x'\|_p^2 + a_2\|x'\|_p + 2\widetilde{g}_\rho Td],
\end{aligned} \tag{3.18}$$

where a_0, a_1 and a_2 are constants depending on T, r_1, k, r_2, d, p and c. Then from Lemma(2.2), we have

$$|1-|c||^p\|x'\|_p^p = |1-|c||^p\|A^{-1}Ax'\|_p^p \le \|Ax'\|_p^p\,.$$

So it follows from (3.18) that

$$\begin{aligned} |1-|c||^p\|x'\|_p^p \le (1+|c|)[(\frac{1}{2^{p-1}}T^{p-1}(r_1+T(r_2+\varepsilon)))\|x'\|_p^p + a_0\|x'\|_p^{p-1} \\ + a_1\|x'\|_p^2 + a_2\|x'\|_p + 2\widetilde{g}_\rho Td]. \end{aligned} \tag{3.19}$$

As $p>2$ and $\frac{1}{2^{p-1}}(1+|c|)T^{p-1}(r_1+Tr_2) < |1-|c||^p$, there exists a constant $R_3>0$ such that

$$\|x'\|_p \le R_3. \tag{3.20}$$

Which together with (3.3) implies that there is a positive number R_4 such that

$$\|x\|_\infty \le R_4. \tag{3.21}$$

From (3.1), we have

$$\begin{aligned} &\int_0^T |(\phi_p(Ax')(t))'|dt \\ &\le \int_0^T [|f(x(t))x'(t)| + |g(t,x(t-\tau(t))) + |e(t)|]dt \\ &\le kT^{1/q}\|x'\|_p + \int_0^T r_1|x|^{p-2}|x'(t)| + Tg_{R_4} + \int_0^T |e(t)|dt \\ &\le kT^{1/q}\|x'\|_p + r_1\|x\|_\infty^{p-2}T^{1/q}\|x'\|_p + Tg_{R_4} + \int_0^T |e(t)|dt \\ &\le kT^{1/q}R_3 + r_1R_4^{p-2}T^{1/q}R_3 + Tg_{R_4} + \int_0^T |e(t)|dt = R_5, \end{aligned} \tag{3.22}$$

where $g_{R_4} = \max_{|x|\le R_4, t\in[0,T]} |g(t,x(t-\tau(t)))|$. As $(Ax)(0)=(Ax)(T)$, there exists $t_0 \in]0,T[$ such that $(Ax')(t_0)=0$, while $\phi_p(0)=0$ we see $\phi_p(Ax')(t_0)=0$. Thus, for any $t\in[0,T]$, we have

$$|\phi_p(Ax')(t))| = |\int_{t_0}^t \phi_p(Ax')(s))ds| \le \int_0^T |(\phi_p(Ax')(s))'|dt \le R_5.$$

From which, it follows that

$$\|Ax'\|_\infty \le R_5^{q-1}. \tag{3.23}$$

From Lemma 2.1, we derive

$$\|x'\|_\infty = \|A^{-1}Ax'\|_\infty \le \frac{\|Ax'\|_\infty}{|1-|c||} \le \frac{R_5^{q-1}}{|1-|c||} = R_6. \tag{3.24}$$

Now, let $y(t)=(Ax)(t)$, we can see that (3.1) is equivalent to the equation

$$(\phi_p(y'(t)))' + \lambda f((A^{-1}y)(t))(A^{-1}y')(t) + \lambda g(t,(A^{-1}y)(t-\tau(t))) = \lambda e(t). \tag{3.25}$$

So, if y is an periodic solution of (3.25), then $x=A^{-1}y$ is T-periodic solution of (3.1).

Let $R_7 = 2(1+|c|)\max\{R_4,R_6,d\}$, $\Omega = \{y\in \mathcal{C}_T^1 : \|y\| < R_7\}$, we can see that (3.25) has no solution on $\partial\Omega$ for $\lambda\in(0,1)$. In fact, if $y=Ax$ is a solution (3.25) on $\partial\Omega$, then $\|y\| = R_7, \|y\|_\infty = R_7$ or $\|y'\|_\infty = R_7$. If $\|y\|_\infty = R_7$, then $\|x\|_\infty \ge \frac{\|y\|_\infty}{1+|c|} = 2\max\{R_4,R_6,d\} > R_4$, from (3.21) which is a contradiction.

Similarly, $\|y'\|_\infty = R_7$ is also impossible. If $y \in \partial\Omega \cap \mathbb{R}$, then y is a constant and $|y| = R_7$, $x = A^{-1}y = \frac{y}{1-c}$, $|x| \geq 2\max\{R_4, R_6, d\}$. Let

$$F(y) = \frac{1}{T}\int_0^T [e(t) - f((A^{-1}y)(t))(A^{-1}y')(t) - g(t, (A^{-1}y)(t - \tau(t)))].$$

Then $F(y) = \frac{1}{T}\int_0^T [e(t) - g(t, \frac{y}{1-c})]dt$ for $y \in \partial\Omega \cap \mathbb{R}$. From (A2), we know that $F(y) \neq 0$ on $\partial\Omega \cap \mathbb{R}$, so condition (ii) in Lemma 2.3 is satisfied. Define

$$H(y, \mu) = \mu(A^{-1}y) + (1 - \mu)F(y),$$

$y \in \partial\Omega \cap \mathbb{R}$, $\mu \in [0, 1]$. Then

$$(-A^{-1}y)H(y, \mu) = -\mu(A^{-1}y)^2 - (1-\mu)(A^{-1}y)\frac{1}{T}\int_0^T [e(t) - g(t, (A^{-1}y)(t - \tau(t)))]dt.$$

From (A2) we obtain $(A^{-1}y)H(y, \mu) > 0$. Thus $H(y, \mu)$ is a homotopic transformation and $\deg[F, \Omega \cap \mathbb{R}, 0] = \deg[A^{-1}y, \Omega \cap \mathbb{R}, 0] \neq 0$. So, for (3.25), all of conditions of Lemma 2.3 are satisfied. Applying Lemma 2.3, we conclude that

$$(\phi_p(y'(t)))' + f((A^{-1}y)(t))(A^{-1}y')(t) + g(t, (A^{-1}y)(t - \tau(t))) = e(t) \qquad (3.26)$$

has at least one T-periodic solution $\overline{y}$. Therefore, $\overline{x} = A^{-1}\overline{y}$ is an T-periodic solution of (1.4). $\square$

Similarly, we can prove the following Theorem.

Theorem 3.2. *Suppose that $p > 2$ and that there exist constants $r_1 \geq 0$, $r_2 \geq 0$, $d > 0$ and $k > 0$ such that*

- (A1) $|f(x)| \leq k + r_1|x|^{p-2}$ *for* $x \in \mathbb{R}$;
- (A2) $x[g(t, x) - e(t)] < 0$ *for* $|x| > d$ *and* $t \in \mathbb{R}$;
- (A3) $\lim_{x\to+\infty} \frac{|g(t,x)-e(t)|}{|x|^{p-1}} = r_2$.

then (1.4) *has at least one T-periodic solution if*

$$\frac{1}{2^{p-1}}(1 + |c|)T^{p-1}(r_1 + Tr_2) < |1 - |c||^p.$$

4. Example

In this section, we illustrate Theorem 3.1 with the following example. Consider the equation

$$(\phi_3(x(t) - 5x(t - \pi))')' + f(x(t))x'(t) + g(t, x(t - \sin(t))) = e^{\cos^2 t}, \qquad (4.1)$$

where $p = 3$, $c = 5$, $\sigma = 4$, $\mathrm{T} = 2\pi$, $\tau(t) = \sin t$, $e(t) = e^{\cos^2 t}$, $f(x) = 2 + \frac{\sqrt{|x|}}{\pi^2}$,

$$g(t, x) = \begin{cases} -xe^{\sin^2 t}, & x \geq 0 \\ \frac{x^2}{18\pi^2}, & x < 0. \end{cases}$$

Let $d = 3\pi\sqrt{2e}$, $r_1 = \frac{1}{\pi^2}$, $r_2 = \frac{1}{18\pi^2}$, $k = 4 + \frac{\max_{|x|\leq 1}|f(x)|}{\pi^2}$. We can easily check the condition (A1), (A2) and (A3) of Theorem 3.1 hold. Furthermore,

$$\frac{1}{2^{p-1}}(1 + |c|)T^{p-1}(r_1 + Tr_2) = 6 + \frac{2\pi}{3} < |1 - |c||^p = 64.$$

By Theorem 3.1, (4.1) has at least one 2π-periodic solution.

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