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DYNAMICAL PROBLEMS WITHOUT INITIAL CONDITIONS FOR ELLIPTIC-PARABOLIC EQUATIONS IN SPATIAL UNBOUNDED DOMAINS

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Abstract. We consider a problem without initial conditions for degenerate nonlinear evolution equations with nonlinear dynamical boundary condition in spatial unbounded domains. We obtain sufficient conditions for the well-posedness of this problem without any restrictions at infinity.

1. Introduction

For $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, with $n\geq 1$, consider Euclidean norm $|x| := (|x_1|^2+\dots+x_n|^2)^{1/2}$. Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ which is a C^1 manifold of dimension $n-1$. Let Γ_0 be the closure of an open set on $\partial\Omega$ (in particular, $\Gamma_0=\partial\Omega$ or Γ_0 is an empty set), $\Gamma_1 := \partial\Omega\setminus\Gamma_0$. Let $\nu=(\nu_1,\dots,\nu_n)$ be the unit vector of the outer normal to $\partial\Omega$. Let S be either $(-\infty,0]$, $(-\infty,+\infty)$ or $(0,1]$. Put $Q:=\Omega\times S$, $\Sigma_0:=\Gamma_0\times S$, $\Sigma_1:=\Gamma_1\times S$. We will use this notation hereafter. Also we will assume that all quantities in this article are real-valued.

Consider the problem:

Find a function $u:\overline{\Omega}\times\overline{S}\to\mathbb{R}$ such that

$$\frac{\partial}{\partial t}(b_1(x)u)-\sum_{i=1}^{n}\frac{d}{dx_i}a_i(x,t,u,\nabla u)+a_0(x,t,u,\nabla u)=f_1(x,t),\quad (x,t)\in Q, \tag{1.1}$$

$$u(y,t)=0,\quad (y,t)\in\Sigma_0, \tag{1.2}$$

$$\frac{\partial}{\partial t}(b_2(y)u)+\sum_{i=1}^{n}a_i(y,t,u,\nabla u)\nu_i(y)+c(y,t,u)=f_2(y,t),\quad (y,t)\in\Sigma_1, \tag{1.3}$$

and if $S=(0,1]$ then in addition

$$\begin{gathered} b_1(x)(u(x,t)-u_1(x))|_{t=0}=0,\quad x\in\Omega,\\ b_2(y)(u(y,t)-u_2(y))|_{t=0}=0,\quad y\in\Gamma_1, \end{gathered} \tag{1.4}$$

where a_i $(i=\overline{0,n})$, c,f_1,f_2,u_1,u_2, $b_1\geq 0$, $b_2\geq 0$ are given real-valued functions.

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We allow at least one of following two relations to hold: $b_1 = 0$ and $b_2 \neq 0$ on subsets of Ω, or Γ_1 is of nonzero measure. Moreover we assume that the space part of the differential expression in the left side of (1.1) is nonlinear elliptic. Thus the partial differential equation (1.1) is parabolic at those $x \in \Omega$ for which $b_1(x) > 0$ and elliptic where $b_1(x) = 0$. Note that boundary condition (1.3) is dynamical on the subset of Γ_1 where $b_2 > 0$ and of the second type on the other part of Γ_1.

Further we will call this problem: problem (1.1)–(1.4) if $S = (0, 1]$, and problem (1.1)–(1.3) provided that S is either $(-\infty, 0]$ or $(-\infty, +\infty)$.

In the case when $S = (0, 1]$ and Ω is bounded , (1.1)–(1.4) can be considered as the Cauchy problem for an implicit evolution equation of the form

$$\big(\mathcal{B}u(t)\big)' + \mathcal{A}\big(t, u(t)\big) = f(t), \quad t \in S, \tag{1.5}$$

where $\mathcal{A}(t, \cdot)$ and $\mathcal{B}$ are some operators (see, e.g., [19]). The well-posedness of this problem has been studied extensively by many authors [1, 9, 12, 13, 14, 19, 21, 22]. Note if $\mathcal{B}$ is linear and $\mathcal{A}$ is either linear or nonlinear, the monographs by Showalter [21, 22] give sufficient conditions for existence and uniqueness of solutions of the Cauchy problem for equation (1.5).

If $S = (0, 1]$ and Ω is unbounded then it is known that for linear and some quasilinear equations (1.1) the existence and uniqueness of solutions of (1.1)–(1.4) need the additional assumptions on the growth of the data-in and the behavior of a solution as $|x| \to +\infty$ [15, 18]. Though such assumptions are not necessary for some nonlinear equations [2, 8, 10].

In the case when S is either $(-\infty, 0]$ or $(-\infty, +\infty)$ the initial conditions (1.4) are missed and the problem (1.1)–(1.3) is called the problem without initial conditions for evolutional equations. These problems arise when describing different non-stationary processes in nature under hypothesis that we consider so distant the initial time that the initial condition practically has no influence on present time, while boundary conditions do affect it. Thus we can assume that either $t = 0$ or ∞ is the *final time*, while $t = -\infty$ is the initial time. Sufficiently full survey of results on the problem can be found in [6]. Here we recall those results which are close to our investigation.

First of all recall that when dealing with problems related to equations in the form (1.1) different approaches are needed subject to whether the domain Ω is either bounded or not. In first case the problem without initial conditions (1.1)–(1.3) may be written in the form (1.5) (recall that S is either $(-\infty, 0]$ or $(-\infty, +\infty)$ now). It is known that when $\mathcal{B}$ is linear and $\mathcal{A}$ is either linear or almost linear this problem is well-posed if in addition some restrictions on behaviour of the solution and growth of data-in as t goes to $-\infty$ are imposed [3, 6, 11, 16, 17]. Same results were obtained in [20] when $\mathcal{A}$ may be set-valued. But the papers [3, 5] and others imply that for some nonlinear operators $\mathcal{A}$ equation (1.5) (with linear $\mathcal{B}$) admits a unique solution without any restrictions on its behaviour at $-\infty$ and the growth of the right-hand side as $t \to -\infty$ (that is, in the class of locally integrable functions on S).

When Ω is unbounded, a solution of the problem (1.1) – (1.3) cannot always be identified with the solution of the abstract equation (1.5). In addition a question about additional conditions on behaviour of the solution both as $t \to -\infty$ and as $|x| \to +\infty$ can arise. The answer to this question can be both positive and negative. The former one is in case of linear and almost linear equations [4, 15] (in the case of equations in the forms (1.1) and (1.3) there are $b_1 = 1$, $b_2 = 0$ respectively). The letter may be only for some nonlinear evolution equations. The results of such kind were obtained in [7] for the equations in the form (1.1) provided $b_1 = 1$, and when $\Gamma_0 = \partial\Omega$ (that is, condition (1.3) is missed).

In this paper, we generalize the results of [7] for the problem (1.1)-(1.3) when possible $b_1 = 0$ on nonzero measure subset of Ω and $b_2 \neq 0$ on nonzero surface measure subset of Γ_1. We obtain sufficient conditions for existence and uniqueness of solutions of this problem without an additional assumptions on the behavior of the solutions and data-in both as $t \to -\infty$ and as $|x| \to +\infty$. We also establish the continuous dependence on data-in of solutions of this problem.

Our paper is organized as follows. In Section 2 we state a problem and formulate the main results. Section 3 is devoted to some auxiliary statements needed in the sequel. We prove our main results in Section 4.

2. Statement of the problem and main results

Let us assume hereafter that Ω is unbounded and S is the interval $(-\infty, 0]$. Suppose that $0 \in \Omega$ and, for every $R > 0$, Ω_R is the connected component of the set $\Omega \cap \{x : |x| < R\}$ containing 0. For arbitrary $R > 0$ denote $\Gamma_{0,R} := \overline{\partial\Omega_R \setminus \Gamma_1}$, $\Gamma_{1,R} := \partial\Omega_R \setminus \Gamma_{0,R}$; $S_R := (-R, 0]$, $Q_R := \Omega_R \times S_R$, $\Sigma_{1,R} := \Gamma_{1,R} \times S_R$.

Let $p > 2$, $q > 2$ be real numbers which remain invariable throughout the paper. Denote $p' := p/(p-1)$, $q' := q/(q-1)$.

Hereafter we use some linear locally convex spaces which are introduced here.

Let G is either a domain or a regular surface in $\mathbb{R}^k$ for either $k = n$ or $k = n+1$, and let $Bs(G)$ be the set consisting of bounded measurable subsets of G. For each $r \in [1, \infty]$ define

$$L_{r,\mathrm{loc}}(\overline{G}) := \{v(z), z \in G | v \in L_r(G') \text{ for all } G' \in Bs(G)\}.$$

It is obvious that $L_{r,\mathrm{loc}}(\overline{G}) = L_r(G)$ when G is a bounded set. Suppose that on the space $L_{r,\mathrm{loc}}(\overline{G})$ there are introduced the standard linear operations and the system of semi-norms $\{\|\cdot\|_{L_r(G')} | G' \in Bs(G)\}$. In particular, it means that the sequence $\{v_k\}_{k=1}^\infty$ converges to v in $L_{r,\mathrm{loc}}(\overline{G})$ provided the sequence $\{v_k|_{G'}\}_{k=1}^\infty$ converges to $v|_{G'}$ in $L_r(G')$ for every $G' \in Bs(G)$. (Hereinafter for the function g defined on G and a subset G' of the set G the notation $g|_{G'}$ means the restriction of g on G'.)

Define $L^{0,+}_{r,\mathrm{loc}}(\overline{G})$ be the subset of $L_{r,\mathrm{loc}}(\overline{G})$ consisting of nonnegative functions, and $L^{+}_{r,\mathrm{loc}}(\overline{G})$ be the subset of $L^{0,+}_{r,\mathrm{loc}}(\overline{G})$ whose each element g is a function such that $\operatorname{ess\,inf}_{z \in G'} g(z) > 0$ for all bounded $G' \subset G$.

Let

$$H^1_{\mathrm{loc}}(\overline{\Omega}) := \{v \in L_{2,\mathrm{loc}}(\overline{\Omega}) | v|_{\Omega_R} \in H^1(\Omega_R) \quad \text{for all} \quad R > 0\}$$

with the system of semi-norms $\{\|\cdot\|_{H^1(\Omega_R))} | R > 0\}$. (Hereinafter $H^1(\widetilde{\Omega}) := \{v \in L_2(\widetilde{\Omega}) | v_{x_i} \in L^2(\widetilde{\Omega}), i = \overline{1,n}\}$ is the Sobolev space with the norm $\|v\|_{H^1(\widetilde{\Omega})} := (\|v\|^2_{L_2(\widetilde{\Omega})} + \sum_{i=1}^n \|v_{x_i}\|^2_{L_2(\widetilde{\Omega})})^{1/2}$ for any domain $\widetilde{\Omega} \subset \mathbb{R}^n$.)

Define

$$L_{2,\mathrm{loc}}(S; H^1_{\mathrm{loc}}(\overline{\Omega})) := \{v : S \to H^1_{\mathrm{loc}}(\overline{\Omega}) | v \in L_2(S_R; H^1(\Omega_R)) \quad \text{for all } R > 0\}$$

with the system of semi-norms $\{\|\cdot\|_{L_2(S_R; H^1(\Omega_R))} | R > 0\}$. Put

$$C(S; L_{2,\mathrm{loc}}(\overline{\Omega})) := \{v : S \to L_{2,\mathrm{loc}}(\overline{\Omega}) | v \in C(S_R; L_2(\Omega_R)) \quad \text{for all } R > 0\}$$

with the system of semi-norms $\{\|\cdot\|_{C(S_R;L_2(\Omega_R))}|R>0\}$,

$$C(S;L_{2,\mathrm{loc}}(\overline{\Gamma_1})):=\{w:S\to L_{2,\mathrm{loc}}(\overline{\Gamma_1})|v\in C(S_R;L_2(\Gamma_{1,R}))\quad\text{for all } R>0\}$$

with the system of semi-norms $\{\|\cdot\|_{C(S_R;L_2(\Gamma_{1,R}))}|R>0\}$. Let

$$\mathbb{F}_{\mathrm{loc}}:=\{(f_1,f_2)|f_1\in L_{p',\mathrm{loc}}(\overline{Q}),f_2\in L_{q',\mathrm{loc}}(\overline{\Sigma_1})\}\equiv L_{p',\mathrm{loc}}(\overline{Q})\times L_{q',\mathrm{loc}}(\overline{\Sigma_1})$$

with the topology generated by the Cartesian product of topological spaces.

The notation $\mathbb{V}_{\mathrm{loc}}$ means the linear locally convex space obtained by the closure of the space $\{v\in C^1(\overline{\Omega}):\operatorname{supp}v\in Bs(\overline{\Omega}),\operatorname{dist}\{\operatorname{supp}v,\Gamma_0\}>0\}$ in the topology generated by the system of semi-norms $\{\|\cdot\|_R:=\|\cdot\|_{H^1(\Omega_R)}+\|\cdot\|_{L_p(\Omega_R)}+\|\cdot\|_{L_q(\Gamma_{1,R})}|R>0\}$. Note that $\|\cdot\|_R$ is the norm for the space $H^1(\Omega_R)\cap L_p(\Omega_R)\cap L_q(\Gamma_{1,R})$, where $R>0$.

Now remark that since $\partial\Omega\in C^1$ then for every element of $H^1_{\mathrm{loc}}(\overline{\Omega})$ there exists its (uniquely) defined trace on $\partial\Omega$, which is the element of $L_{2,\mathrm{loc}}(\overline{\partial\Omega})$ and for every smooth function on $\overline{\Omega}$ it coincides with restriction of this function on the $\partial\Omega$. Therefore, taking into account the definition of the family of semi-norms on $\mathbb{V}_{\mathrm{loc}}$ (in particular, it follows that $\mathbb{V}_{\mathrm{loc}}\subset H^1_{\mathrm{loc}}(\overline{\Omega})$), we can conclude the proper definiteness, linearity and continuity of the operator $\gamma:\mathbb{V}_{\mathrm{loc}}\to L_{q,\mathrm{loc}}(\overline{\Gamma_1})$ which is the restriction of standard trace operator on the space $H^1_{\mathrm{loc}}(\overline{\Omega})$ to $\mathbb{V}_{\mathrm{loc}}$. Put

$$\mathbb{V}_c:=\{v\in\mathbb{V}_{\mathrm{loc}}|\operatorname{supp}v\text{ is a bounded set}\}.$$

Let us agree for every linear locally convex space W and interval $I\subset\mathbb{R}$ to understand the $(I\to W)$ as the linear space that is the factorization of the linear space of mappings of the set I to W by such equivalence relation that two mappings are equivalent if their values coincide for almost every value of the argument.

We will also need the space

$$\begin{aligned}\mathbb{U}_{\mathrm{loc}}:=\{u\in(S\to\mathbb{V}_{\mathrm{loc}})|u\in L_{2,\mathrm{loc}}(S;H^1_{\mathrm{loc}}(\overline{\Omega}))\cap L_{p,\mathrm{loc}}(\overline{Q}),b_1^{1/2}u\in C(S;L_{2,\mathrm{loc}}(\overline{\Omega})),\\ \gamma u\in L_{q,\mathrm{loc}}(\overline{\Sigma_1}),b_2^{1/2}\gamma u\in C(S;L_{2,\mathrm{loc}}(\overline{\Gamma_1}))\}\end{aligned}$$

with the topology generated by the system of semi-norms

$$\begin{aligned}\big\{\|u\|_R^*=\|u\|_{L_2(S_R;H^1(\Omega_R))}+\|u\|_{L_p(Q_R)}+\sup_{t\in S_R}\|b_1^{1/2}(\cdot)u(\cdot,t)\|_{L_2(\Omega_R)}\\ +\|\gamma u\|_{L_q(\Sigma_{1,R})}+\sup_{t\in S_R}\|b_2^{1/2}(\cdot)\gamma u(\cdot,t)\|_{L_2(\Gamma_{1,R})}|R>0\big\}.\end{aligned}$$

Let $\mathbb{B}$ be the set of pairs $b=(b_1,b_2)$ of the functions satisfying the condition

- (B) $b_1\in L_{p^*,\mathrm{loc}}(\overline{\Omega})$, $b_1\geq 0$ on Ω; $b_2\in L_{q^*,\mathrm{loc}}(\overline{\Gamma_1})$, $b_2\geq 0$ on Γ_1, where $p^*=p/(p-2)$, $q^*=q/(q-2)$.

Consider the set whose any element is an array $(a_0,a_1,\dots,a_n)$ of $n+1$ real-valued functions satisfying the following conditions:

- (A1) for each $i\in\{0,1,\dots,n\}$ the function $Q\times\mathbb{R}\times\mathbb{R}^n\ni(x,t,s,\xi)\to a_i(x,t,s,\xi)$ is a Caratheodory; i.e., for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n\equiv\mathbb{R}^{1+n}$ the function $a_i(\cdot,\cdot,s,\xi):Q\to\mathbb{R}$ is Lebesgue measurable and for a.e. $(x,t)\in Q$ the function $a_i(x,t,\cdot,\cdot):\mathbb{R}^{1+n}\to\mathbb{R}$ is continuous;
- (A1') $a_i(x,t,0,0)=0$ for a.e. $(x,t)\in Q$ and all $i\in\{0,1,\dots,n\}$;
- (A2) for a.e. $(x,t)\in Q$ and every $(s,\xi)\in\mathbb{R}^{1+n}$,

$$|a_0(x,t,s,\xi)|\leq h_{0,1}(x,t)(|s|^{p-1}+|\xi|^{2/p'})+h_{0,2}(x,t)$$

where $h_{0,1} \in L^+_{\infty,\mathrm{loc}}(\overline{Q}), h_{0,2} \in L^{0,+}_{p',\mathrm{loc}}(\overline{Q})$;

(A3) for a.e. $(x,t) \in Q$ and every $(s,\xi),(r,\eta) \in \mathbb{R}^{1+n}$,
$$\sum_{i=1}^n |a_i(x,t,s,\xi) - a_i(x,t,r,\eta)| \le d_1(x,t)|\xi-\eta| + d_2(x,t)|s-r|,$$
where $d_1 \in L^+_{\infty,\mathrm{loc}}(\overline{Q}), d_2 \in L^{0,+}_{\infty,\mathrm{loc}}(\overline{Q})$ are arbitrary functions;

(A4) for a.e. $(x,t) \in Q$ and every $(s,\xi),(r,\eta) \in \mathbb{R}^{1+n}$,
$$\begin{aligned}&\sum_{j=1}^n (a_i(x,t,s,\xi) - a_i(x,t,r,\eta))(\xi_i - \eta_i) + (a_0(x,t,s,\xi) - a_0(x,t,r,\eta))(s-r)\\ &\ge \rho_1(x,t)|\xi-\eta|^2 + \rho_2(x,t)|s-r|^p,\end{aligned}$$
where $\rho_1, \rho_2 \in L^+_{\infty,\mathrm{loc}}(\overline{Q})$.

On this set, define an equivalence relation such that the element $(a_0, a_1, \dots, a_n)$ is equivalent to the element $(\widetilde{a}_0, \widetilde{a}_1, \dots, \widetilde{a}_n)$ if for every $i \in \{0, 1, \dots, n\}$ the equality $a_i(x,t,s,\xi) = \widetilde{a}_i(x,t,s,\xi)$ holds for every $(s,\xi) \in \mathbb{R}^{1+n}$ and a.e. $(x,t) \in Q$. We denote by $\mathbb{A}$ the quotient-space obtained by this equivalence relation. We will not distinguish the notations of the elements of the space $\mathbb{A}$ (that are the classes of equivalent functions arrays) and their representatives. On the set $\mathbb{A}$ introduce the notion of convergence in such a way that a sequence $\{(a_0^k, a_1^k, \dots, a_n^k)\}_{k=1}^\infty$ is convergent to $(a_0, a_1, \dots, a_n)$ in $\mathbb{A}$, provided
$$\begin{aligned}\lim_{k\to\infty} \operatorname*{ess\,sup}_{(x,t)\in Q'} \sup_{(s,\xi)\in\mathbb{R}^{1+n}} \Big[&\sum_{i=1}^n |a_i^k(x,t,s,\xi) - a_i(x,t,s,\xi)|/(1+|s|+|\xi|)\\ &+ |a_0^k(x,t,s,\xi) - a_0(x,t,s,\xi)|/(1+|s|^{p-1}+|\xi|^{2/p'})\Big] = 0\end{aligned} \tag{2.1}$$
for every bounded domain $Q' \subset Q$.

Remark 2.1. It is easy to show that if $a_0(x,t,s,\xi) = \bar{a}_0(x,t)|s|^{p-2}s$, $a_i(x,t,s,\xi) = \bar{a}_i(x,t)\xi_i$ $(i = \overline{1,n})$, where $\bar{a}_j \in L^+_{\infty,\mathrm{loc}}(\overline{Q})$ $(j = \overline{0,n})$, then the array $(a_0, a_1, \dots, a_n)$ is an element of $\mathbb{A}$. Also note that for $a_0^k(x,t,s,\xi) = \bar{a}_0^k(x,t)|s|^{p-2}s$, $a_i^k(x,t,s,\xi) = \bar{a}_i^k(x,t)\xi_i$ $(i = \overline{1,n})$, where $k \in \mathbb{N}$, $\bar{a}_j^k \in L^+_{\infty,\mathrm{loc}}(\overline{Q})$ $(j = \overline{0,n})$, the sequence $\{(a_0^k, a_1^k, \dots, a_n^k)\}_{k=1}^\infty$ is convergent to $(a_0, a_1, \dots, a_n)$ in $\mathbb{A}$ if and only if $\bar{a}_j^k \to \bar{a}_j$ in $L_{\infty,\mathrm{loc}}(\overline{Q})$ $(j = \overline{0,n})$.

Consider the set of real-valued functions $c(y,t,s)$, $(y,t,s) \in \Sigma_1 \times \mathbb{R}$, satisfying the conditions:

(C1) c is a Caratheodory function, that is for every $s \in \mathbb{R}$ the function $c(\cdot,\cdot,s) : \Sigma_1 \to \mathbb{R}$ is Lebesgue measurable and for a.e. (in the sense of surface measure) $(y,t) \in \Sigma_1$ the function $c(y,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous;

(C1') $c(y,t,0) = 0$ for a.e. $(y,t) \in \Sigma_1$;

(C2) for a.e. $(y,t) \in \Sigma_1$ and every $s \in \mathbb{R}$,
$$|c(y,t,s)| \le g_1(y,t)|s|^{q-1} + g_2(y,t),$$
where $g_1 \in L^+_{\infty,\mathrm{loc}}(\overline{\Sigma_1}), g_2 \in L^{0,+}_{q',\mathrm{loc}}(\overline{\Sigma_1})$;

(C3) for a.e. $(y,t) \in \Sigma_1$ and every $s,r \in \mathbb{R}$ the inequality
$$(c(y,t,s) - c(y,t,r))(s-r) \ge \rho_3(y,t)|s-r|^q$$
is satisfied, where $\rho_3 \in L^+_{\infty,\mathrm{loc}}(\overline{\Sigma}_1)$ is some function.

On this set, introduce an equivalence relation such that two functions c and $\widetilde{c}$ of the given set are equivalent if $c(y,t,s) = \widetilde{c}(y,t,s)$ for all $s \in \mathbb{R}$ and for a.e. $(x,t) \in \Sigma_1$. We denote by $\mathbb{C}$ the obtained quotient-set. Define the convergence notation of the sequences of the elements of the set $\mathbb{C}$ such that the sequence $\{c^k\}_{k=1}^{\infty}$ is convergent to c in $\mathbb{C}$ if

$$\lim_{k\to\infty} \operatorname{ess\,sup}_{(y,t)\in\Sigma'} \sup_{s\in\mathbb{R}} |c^k(y,t,s) - c(y,t,s)|/(1+|s|^{q-1}) = 0 \tag{2.2}$$

for every bounded subset $\Sigma' \subset \Sigma_1$.

Remark 2.2. It is easy to show that $c(x,t,s) = \bar{c}(x,t)|s|^{q-2}s$ is an element of $\mathbb{C}$, when $\bar{c} \in L^{+}_{\infty,\mathrm{loc}}(\overline{\Sigma}_1)$. Also note that for $c^k(x,t,s) = \bar{c}^k(x,t)|s|^{q-2}s$, where $k \in \mathbb{N}, \bar{c}^k \in L^{+}_{\infty,\mathrm{loc}}(\overline{Q})$, the sequence $\{c^k\}_{k=1}^{\infty}$ is convergent to c in $\mathbb{C}$ if and only if $\bar{c}^k \to \bar{c}$ in $L_{\infty,\mathrm{loc}}(\overline{\Sigma}_1)$.

Remark 2.3. Conditions (A1') and (C1') are not essential. Indeed, let any of them or both do not hold. Then it is sufficient to suppose that for every $i \in \{1,\dots,n\}$ the function $t \to a_i(\cdot,t,0,0)$ belongs to the space $L_{2,\mathrm{loc}}(S; H^1_{\mathrm{loc}}(\overline{\Omega}))$ and in equations (1.1) and (1.3) make the substitution $a_i(x,t,u,\nabla u)$ for $\tilde{a}_i(x,t,u,\nabla u) := a_i(x,t,u,\nabla u) - a_i(x,t,0,0) (i \in \{0,1,\dots,n\})$, $f_1(x,t)$ for $\tilde{f}_1(x,t) := f_1(x,t) - a_0(x,t,0,0) + \sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x,t,0,0)$, $c(y,t,u)$ for $\tilde{c}(y,t,u) := c(y,t,u) - c(y,t,0)$, $f_2(y,t)$ for $\tilde{f}_2(y,t) := f_2(y,t) - c(y,t,0) - \sum_{i=1}^n a_i(y,t,0,0)\nu_i$. It is obvious that the functions $(\tilde{a}_0,\dots,\tilde{a}_n), \tilde{c}$, $(\tilde{f}_1,\tilde{f}_2)$ satisfy all above mentioned conditions for $(a_0,\dots,a_n), c$, (f_1,f_2) respectively.

Definition 2.4. Let $(b_1,b_2) \in \mathbb{B}$, $(a_0,a_1,\dots,a_n) \in \mathbb{A}, c \in \mathbb{C}, (f_1,f_2) \in \mathbb{F}_{\mathrm{loc}}$. We say that the function $u \in \mathbb{U}_{\mathrm{loc}}$ is generalized solution of the problem (1.1)–(1.3) if

$$\begin{aligned}
&\iint_Q \Big\{ \sum_{i=1}^n a_i(x,t,u,\nabla u)\psi_{x_i}\varphi + a_0(x,t,u,\nabla u)\psi\varphi - b_1(x)u\psi\varphi' \Big\}\,dx\,dt \\
&+ \iint_{\Sigma_1} \{c(y,t,\gamma u)\gamma\psi\varphi - b_2(y)\gamma u\gamma\psi\varphi'\}\,d\Gamma_y\,dt \\
&= \iint_Q f_1\psi\varphi\,dx\,dt + \iint_{\Sigma_1} f_2\gamma\psi\varphi\,d\Gamma_y\,dt
\end{aligned} \tag{2.3}$$

for every $\psi \in \mathbb{V}_c, \varphi \in C_0^1(-\infty,0)$.

Hereinafter denote by $C_0^1(I)$, where I is an interval of the number axis, the linear space of finite continuous-differentiable functions on I.

For every $(b_1,b_2) \in \mathbb{B}$, ρ_1,ρ_2 in $L^{+}_{\infty,\mathrm{loc}}(\overline{Q})$, $\rho_3 \in L^{+}_{\infty,\mathrm{loc}}(\overline{\Sigma_1})$, $d_1,d_2 \in L^{0,+}_{\infty,\mathrm{loc}}(\overline{Q})$ put

$$\begin{aligned}
&\Psi(b_1,b_2,\rho_1,\rho_2,\rho_3,d_1,d_2;R) \\
&:= R^{-2p/(p-2)} \iint_{Q_R} \rho_1^{-p/(p-2)}\rho_2^{-2/(p-2)} d_1^{2p/(p-2)}\,dx\,dt \\
&\quad + R^{-p/(p-2)} \iint_{Q_R} \rho_2^{-2/(p-2)} b_1^{p/(p-2)}\,dx\,dt \\
&\quad + R^{-p/(p-2)} \iint_{Q_R} \rho_2^{-2/(p-2)} d_2^{p/(p-2)}\,dx\,dt \\
&\quad + R^{-q/(q-2)} \iint_{\Sigma_{1,R}} \rho_3^{-2/(q-2)} b_2^{q/(q-2)} d\Gamma_y\,dt, \quad R > 0.
\end{aligned} \tag{2.4}$$

Denote by $\mathbb{BAC}$ the subset of the Cartesian product $\mathbb{B}\times\mathbb{A}\times\mathbb{C}$ whose any element $((b_1,b_2),(a_0,a_1,\dots,a_n),c)$ such that $(a_0,a_1,\dots,a_n)$ satisfies conditions (A3), (A4), and c satisfies condition (C3) with $\rho_1,\rho_2,\rho_3,d_1,d_2$ satisfying

$$\Psi(b_1,b_2,\rho_1,\rho_2,\rho_3,d_1,d_2;R)\to 0 \quad \text{as } R\to+\infty. \tag{2.5}$$

Theorem 2.5. *Let $((b_1,b_2),(a_0,a_1,\dots,a_n),c)\in\mathbb{BAC}$, $(f_1,f_2)\in\mathbb{F}_{\rm loc}$. Then the problem* (1.1)–(1.3) *has a unique generalized solution. Moreover, for every R_0,R, $0<R_0<R$, this solution satisfies the estimation*

$$\begin{aligned}
&\sup_{t\in[-R_0,0]}\Big[\int_{\Omega_{R_0}} b_1(x)|u(x,t)|^2\,dx+\int_{\Gamma_{1,R_0}} b_2(y)|\gamma u(y,t)|^2 d\Gamma_y\Big]\\
&+\iint_{Q_{R_0}}\Big[\rho_1|\nabla u|^2+\rho_2|u|^p\Big]\,dx\,dt+\iint_{\Sigma_{1,R_0}}\rho_3|\gamma u|^q d\Gamma_y dt\\
&\le C\Big(R/(R-R_0)\Big)^{\sigma}\Big[\Psi(b_1,b_2,\rho_1,\rho_2,\rho_3,d_1,d_2;R)\\
&\quad+\iint_{Q_R}\rho_2^{-1/(p-1)}|f_1(x,t)|^{p'}\,dx\,dt+\iint_{\Sigma_{1,R}}\rho_3^{-1/(q-1)}|f_2(y,t)|^{q'}\,d\Gamma_y\,dt\Big],
\end{aligned}\tag{2.6}$$

where $\sigma:=\max\big\{p/(p-2),q/(q-2)\big\}+2p/(p-2)$, $C>0$ is a constant depending only on p,q.

In addition, for any sequences $\{(a_0^k,a_1^k,\dots,a_n^k)\},\{c^k\}$ and $\{(f_1^k,f_2^k)\}$ such that $((b_1,b_2),(a_0^k,a_1^k,\dots,a_n^k),c^k)\in\mathbb{BAC}$ and $(a_0^k,a_1^k,\dots,a_n^k)\to(a_0,a_1,\dots,a_n)$ in $\mathbb{A}$, $c^k\to c$ in $\mathbb{C}$, $(f_1^k,f_2^k)\to(f_1,f_2)$ in $\mathbb{F}_{\rm loc}$ as $k\to\infty$ we have $u^k\to u$ in $\mathbb{U}_{\rm loc}$ as $k\to\infty$, where for every $k\in\mathbb{N}$ u^k is a generalized solution of the problem differing from the problem (1.1)–(1.3) only by having functions $a_0^k,a_1^k,\dots,a_n^k,c^k,f_1^k,f_2^k$ instead of $a_0,a_1,\dots,a_n,c,f_1,f_2$ respectively.

Remark 2.6. If the functions $b_1,b_2,\rho_1,\rho_2,\rho_3,d_1,d_2$ are constant (positive), then condition (2.5) is equivalent to the condition

$$\operatorname{meas}\Omega_R\cdot R^{-2/(p-2)}+\operatorname{meas}\Gamma_{1,R}\cdot R^{-2/(q-2)}\to 0 \quad\text{as } R\to+\infty. \tag{2.7}$$

3. Auxiliary statements

Now we state some technical results needed later.

Lemma 3.1. *Let $R>0$, $\tau_1<\tau_2$ be any numbers, $(b_1,b_2)\in\mathbb{B}$. Suppose that a function $v\in((\tau_1,\tau_2)\to\mathbb{V}_{\rm loc})\cap L_2(\tau_1,\tau_2;H^1_{\rm loc}(\overline{\Omega}))\cap L_{p,{\rm loc}}\big(\overline{\Omega\times(\tau_1,\tau_2)}\big)$, $\gamma v\in L_{q,{\rm loc}}\big(\overline{\Gamma_1\times(\tau_1,\tau_2)}\big)$, and $g_0\in L_{p',{\rm loc}}\big(\overline{\Omega\times(\tau_1,\tau_2)}\big)$, $g_i\in L_{2,{\rm loc}}\big(\overline{\Omega\times(\tau_1,\tau_2)}\big)$, $(i=\overline{1,n})$, $h\in L_{q',{\rm loc}}\big(\overline{\Gamma_1\times(\tau_1,\tau_2)}\big)$ satisfy*

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{\Omega_R}\Big\{\sum_{i=1}^n g_i\psi_{x_i}\varphi+g_0\psi\varphi-b_1v\psi\varphi'\Big\}\,dx\,dt\\
&+\int_{\tau_1}^{\tau_2}\int_{\Gamma_{1,R}}\{h\gamma\psi\varphi-b_2\gamma v\gamma\psi\varphi'\}\,d\Gamma_y dt=0
\end{aligned}\tag{3.1}$$

for $\varphi\in C_0^1(\tau_1,\tau_2)$, $\psi\in\mathbb{V}_c$, $\operatorname{supp}\psi\subset\overline{\Omega_R}$. Then $b_1^{1/2}v\in C([\tau_1,\tau_2];L_2(\Omega_{R^*}))$, $b_2^{1/2}\gamma v\in C([\tau_1,\tau_2];L_2(\Gamma_{1,R^*}))$, for every $R^*\in(0,R)$. In addition, for every functions $\theta\in C^1([\tau_1,\tau_2])$, $w\in C^1(\overline{\Omega})$, $\operatorname{supp}w\subset\overline{\Omega_R}$, $w\ge 0$ and arbitrary numbers

t_1, t_2 such that $\tau_1 \le t_1 < t_2 \le \tau_2$, the equality holds

$$\begin{aligned} &\theta(t)\Big(\int_{\Omega_R} b_1(x)|v(x,t)|^2 w(x)dx + \int_{\Gamma_{1,R}} b_2(y)|\gamma v(y,t)|^2 w(y)d\Gamma_y\Big)\Big|_{t=t_1}^{t=t_2} \\ &- \int_{t_1}^{t_2}\Big(\int_{\Omega_R} b_1(x)|v(x,t)|^2 w(x)\,dx + \int_{\Gamma_{1,R}} b_2(y)|\gamma v(y,t)|^2 w(y)d\Gamma_y\Big)\theta'(t)\,dt \\ &+ 2\int_{t_1}^{t_2}\Big(\int_{\Omega_R}\Big\{\sum_{i=1}^n g_i(vw)_{x_i} + g_0 vw\Big\}\,dx + \int_{\Gamma_{1,R}} h\gamma vw\,d\Gamma_y\Big)\theta\,dt = 0. \end{aligned} \tag{3.2}$$

Proof. We assume without loss of generality that $\tau_1 = 0, \tau_2 = T$, where $T > 0$ is any number. We will use some ideas with proof of [22, Proposition 1.2, p. 106]. First of all construct the extension of the functions $\widehat{v}, \widehat{g}_i (i = \overline{0,n})$ for functions $v, g_i (i = \overline{0,n})$ respectively onto the cylinder $\Omega \times (-T, 2T)$ by putting for a.e. $x \in \Omega$,

$$\widehat{v}(x,t) := \begin{cases} v(x,-t), & -T < t < 0, \\ v(x,t), & 0 \le t \le T, \\ v(x,2T-t), & T < t < 2T, \end{cases} \quad \widehat{g}_i(x,t) := \begin{cases} -g_i(x,-t), & -T < t < 0, \\ g_i(x,t), & 0 \le t \le T, \\ -g_i(x,2T-t), & T < t < 2T. \end{cases}$$

Construct also the extension $\widehat{h}$ of the function h onto the surface $\Gamma_1 \times (-T, 2T)$:

$$\widehat{h}(y,t) := \begin{cases} -h(y,-t), & -T < t < 0, \\ h(y,t), & 0 \le t \le T, \\ -h(y,2T-t), & T < t < 2T \end{cases}$$

for a.e. $y \in \Gamma_1$. It is to verify that

$$\begin{aligned} &\int_{-T}^{2T}\int_{\Omega_R}\Big\{\sum_{i=1}^n \widehat{g}_i\psi_{x_i}\varphi + \widehat{g}_0\psi\varphi - b_1\widehat{v}\psi\varphi'\Big\}\,dx\,dt \\ &+ \int_{-T}^{2T}\int_{\Gamma_{1,R}}\{\widehat{h}\gamma\psi\varphi - b_2\gamma\widehat{v}\gamma\psi\varphi'\}\,d\Gamma_y dt = 0 \end{aligned} \tag{3.3}$$

is fulfilled for every $\varphi \in C_0^1(-T, 2T)$, $\psi \in \mathbb{V}_c, \operatorname{supp}\psi \subset \overline{\Omega_R}$.

Indeed, it is easy to ascertain that (3.3) holds for every $\psi \in \mathbb{V}_{\rm loc}$ and $\varphi \in C_0^1(-T, 2T)$, provided $\operatorname{supp}\varphi \subset (-T, 0) \cup (0, T) \cup (T, 2T)$ (it is enough to make the corresponding substitution of the variable t in to identity (2.3)). It remains to consider the case when $\operatorname{supp}\varphi \cap \{0, T\} \ne \emptyset$. For the simplicity we will assume without loss of generality that $\operatorname{supp}\varphi \subset (-T, T)$. Then for every $m \in \mathbb{N}$ choose the function $\chi_m \in C^1(\mathbb{R})$ such that $|\chi_m(t)| \le 1$, $|\chi_m'(t)| \le 2m$ and $\chi_m(-t) = \chi_m(t)$ when $t \in \mathbb{R}$, $\chi_m(t) = 1$ as $t \in (-\infty, -2/m) \cup (2/m, +\infty)$ and $\chi_m(t) = 0$, when $t \in (-1/m, 1/m)$.

It is obvious that for every $t \in \mathbb{R}\backslash\{0\}$ we have $\chi_m(t) \to 1$ as $m \to +\infty$. The above mentioned yields that (3.3) is fulfilled provided $\psi \in \mathbb{V}_{\rm loc}$ and with φ instead of $\chi_m\varphi$, where $m\in\mathbb{N}$. After the simple transformations we obtain

$$\begin{aligned}
&\int_{-T}^{2T}\int_{\Omega_R}\Big\{\sum_{i=1}^{n}\widehat{g}_i\psi_{x_i}\varphi+\widehat{g}_0\psi\varphi-b_1\widehat{v}\psi\varphi'\Big\}\chi_m\,dx\,dt\\
&+\int_{-T}^{2T}\int_{\Gamma_{1,R}}\Big\{\widehat{h}\gamma\psi\varphi-b_2\gamma\widehat{v}\gamma\psi\varphi'\Big\}\chi_m d\Gamma_y\,dt\\
&-\int_{-2/m}^{2/m}\int_{\Omega}b_1\widehat{v}\psi\varphi\chi_m'\,dx\,dt-\int_{2/m}^{-2/m}\int_{\Gamma_{1,R}}b_2\gamma\widehat{v}\gamma\psi\varphi\chi_m' d\Gamma_y\,dt=0,
\end{aligned}\tag{3.4}$$

where $m\in\mathbb{N}$, $\varphi\in C_0^1(-T,2T)$, $\operatorname{supp}\varphi\subset(-T,T)$, $\psi\in\mathbb{V}_c,\operatorname{supp}\psi\subset\overline{\Omega_R}$.

Change the third and fourth terms of the left side part of (3.4). We obtain

$$\begin{aligned}
&\int_{-2/m}^{2/m}\int_{\Omega_R}b_1\widehat{v}\psi\varphi\chi_m'\,dx\,dt\\
&=\int_{1/m}^{2/m}\int_{\Omega_R}b_1(x)\widehat{v}(x,t)\psi(x)\varphi(t)\chi_m'(t)\,dx\,dt\\
&\quad+\int_{-2/m}^{-1/m}\int_{\Omega_R}b_1(x)\widehat{v}(x,t)\psi(x)\varphi(t)\chi_m'(t)\,dx\,dt\\
&=\int_{1/m}^{2/m}\int_{\Omega_R}b_1(x)v(x,t)\psi(x)\varphi(t)\chi_m'(t)\,dx\,dt\\
&\quad+\int_{1/m}^{2/m}\int_{\Omega_R}b_1(x)v(x,t)\psi(x)\varphi(-t)\chi_m'(-t)\,dx\,dt\\
&=\int_{1/m}^{2/m}\int_{\Omega_R}b_1(x)v(x,t)\psi(x)(\varphi(t)-\varphi(-t))\chi_m'(t)\,dx\,dt\\
&=2\int_{1/m}^{2/m}\int_{\Omega_R}t\varphi'(\xi(t))\chi_m'(t)b_1(x)v(x,t)\psi(x)\,dx\,dt,
\end{aligned}\tag{3.5}$$

where $\xi(t)$ is some number between $-t$ and t. Here we made the replacement of t by $-t$ in one of terms, used the definition of $\widehat{v}$ and the Lagrange Theorem of finite decrements: $\varphi(t)-\varphi(-t)=\varphi'(\xi(t))t$, $t>0$. Note that $|t\chi_m'(t)|\le(2/m)2m=4$ for every $t\in[1/m,2/m]$ and

$$\operatorname{meas}_{n+1}\{(x,t)|x\in\operatorname{supp}\psi,t\in(1/m,2/m)\}\to0\quad asm\to+\infty.\tag{3.6}$$

It is obvious that

$$|t\varphi'(\xi(t))\chi_m'(t)b_1(x)v(x,t)\psi(x)|\le K|b_1(x)v(x,t)\psi(x)|,\quad(x,t)\in\Omega\times(-T,2T),\tag{3.7}$$

where $K>0$ is some constant not depending on m. Since the right side of the inequality (3.7) belongs to $L_1(\operatorname{supp}\psi\times(-T,2T))$, thus the left side belongs to $L_1(\operatorname{supp}\psi\times(-T,2T))$.

From (3.5) by (3.6) and (3.7) we deduce

$$\int_{-2/m}^{2/m}\int_{\Omega_R}b_1\widehat{v}\psi\varphi\chi_m'\,dx\,dt\to0\quad\text{as }m\to\infty.\tag{3.8}$$

Arguing the same way, we derive

$$\int_{-2/m}^{2/m}\int_{\Gamma_{1,R}} b_2\gamma\widehat{v}\gamma\psi\varphi\chi_m'\,d\Gamma_y\,dt\to 0\quad\text{as } m\to\infty. \tag{3.9}$$

Passing to the limit in (3.4) as $m\to+\infty$, taking into account (3.8), (3.9) and the Lebesgue Theorem of boundary transition under the integral sign. It is obvious that as a result we obtain (3.3), which is required.

Let $\{\omega_\rho|\rho>0\}$ be the mollifier kernels, that is $\omega_\rho\in C^\infty(\mathbb{R})$, ω_ρ is an even function, $\operatorname{supp}\omega_\rho\subset[-\rho,\rho]$, $\int_{\mathbb{R}}\omega_\rho(s)\,ds=1$ for every $\rho>0$. Choose a number $k_0\in\mathbb{N}$ such that $1/k_0<T/2$, and for every $k\ge k_0$ put

$$\widehat{v}_k(x,\tau):=(\widehat{v}*\omega_{1/k})(x,\tau)\equiv\int_{\mathbb{R}}\widehat{v}(x,t)\omega_{1/k}(t-\tau)\,dt,\quad (x,\tau)\in\Omega\times(-T/2,3T/2),$$

$$\widehat{g}_{i,k}(x,\tau):=(\widehat{g}_i*\omega_{1/k})(x,\tau)\equiv\int_{\mathbb{R}}\widehat{g}_i(x,t)\omega_{1/k}(t-\tau)\,dt,(x,\tau)\in\Omega\times(-T/2,3T/2),\quad i\in\{0,\dots,n\},$$

$$\widehat{h}(y,\tau):=(\widehat{h}*\omega_{1/k})(y,\tau)\equiv\int_{\mathbb{R}}\widehat{h}(y,t)\omega_{1/k}(t-\tau)\,dt,\quad (y,\tau)\in\Gamma_1\times(-T/2,3T/2).$$

It is easy to ascertain the fact

$$(\gamma\widehat{v}*\omega_{1/k})(y,\tau)=\gamma\widehat{v}_k(y,\tau),\quad (y,\tau)\in\Gamma_1\times(-T/2,3T/2).$$

From well-known facts of homogenization theory we conclude

$$\begin{gathered}
\widehat{v}_k\underset{k\to\infty}{\longrightarrow}\widehat{v}\quad\text{in } L_{p,\mathrm{loc}}\big(\overline{\Omega\times(-T/2,3T/2)}\big)\cap L_2(-T/2,3T/2;H^1_{\mathrm{loc}}(\overline{\Omega})),\\
\gamma\widehat{v}_k\underset{k\to\infty}{\longrightarrow}\gamma\widehat{v}\quad\text{in } L_{q,\mathrm{loc}}\big(\overline{\Gamma_1\times(-T/2,3T/2)}\big),\\
\widehat{g}_{i,k}\underset{k\to\infty}{\longrightarrow}\widehat{g}_i\quad\text{in}\quad L_{2,\mathrm{loc}}\big(\overline{\Omega\times(-T/2,3T/2)}\big)\quad(i=\overline{1,n}),\\
\widehat{g}_{0,k}\underset{k\to\infty}{\longrightarrow}\widehat{g}_0\quad\text{in}\quad L_{p',\mathrm{loc}}\big(\overline{\Omega\times(-T/2,3T/2)}\big),\\
\widehat{h}_k\underset{k\to\infty}{\longrightarrow}\widehat{h}\quad\text{in}\quad L_{q',\mathrm{loc}}\big(\overline{\Gamma_1\times(-T/2,3T/2)}\big).
\end{gathered}\tag{3.10}$$

In (3.3) set $\varphi(t)=\omega_{1/k}(t-\tau)$, $t\in(-T,2T)$, where $\tau\in[-T/2,T]$, $k\ge k_0$ are some numbers. After simple transformations we obtain

$$\begin{aligned}
&\int_{\Omega_R}\big\{b_1(x)\frac{d}{d\tau}\widehat{v}_k(x,\tau)\psi(x)+\sum_{i=1}^n\widehat{g}_{i,k}(x,\tau)\psi_{x_i}(x)+\widehat{g}_{0,k}(x,\tau)\psi(x)\big\}dx\\
&+\int_{\Gamma_{1,R}}\big\{b_2(y)\frac{d}{d\tau}\gamma\widehat{v}_k(y,\tau)\gamma\psi(y)+\widehat{h}_k(y,\tau)\gamma\psi(y)\big\}d\Gamma_y=0
\end{aligned}\tag{3.11}$$

for every $\tau\in[-T/2,T]$, $\psi\in\mathbb{V}_c,\operatorname{supp}\psi\subset\overline{\Omega_R}$.

Let k, l be arbitrary natural numbers bigger than k_0. Subtracting from (3.11) the same equality with $k=l$ and putting $\widehat{v}_{kl}:=\widehat{v}_k-\widehat{v}_l$, $\widehat{g}_{i,kl}:=\widehat{g}_{i,k}-\widehat{g}_{i,l}$ $(i=\overline{0,n})$, $\widehat{h}_{kl}:=\widehat{h}_k-\widehat{h}_l$, we deduce

$$\begin{aligned}
&\int_{\Omega_R}\big\{b_1(x)\frac{d}{d\tau}\widehat{v}_{kl}(x,\tau)\psi(x)+\sum_{i=1}^n\widehat{g}_{i,kl}(x,\tau)\psi_{x_i}(x)+\widehat{g}_{0,kl}(x,\tau)\psi(x)\big\}dx\\
&+\int_{\Gamma_{1,R}}\big\{b_2(y)\frac{d}{d\tau}\gamma\widehat{v}_{kl}(y,\tau)\gamma\psi(y)+\widehat{h}_{kl}(y,\tau)\gamma\psi(y)\big\}d\Gamma_y=0
\end{aligned}\tag{3.12}$$

for every $\psi \in \mathbb{V}_c$, $\operatorname{supp}\psi \subset \overline{\Omega_R}$, $\tau \in [-T/2, T]$, $k, l \geq k_0$.

Let $w \in C^1(\overline{\Omega})$ be any function such that $\operatorname{supp} w \subset \overline{\Omega_R}$, $w \geq 0$, and let $\theta \in C^1(\mathbb{R})$ be an arbitrary function. In (3.12) let for every $\tau \in [-T/2, T]$ $\psi(x) = \widehat{v}_{kl}(x,\tau)w(x)\theta(\tau)$, $x \in \Omega$. Integrate the obtained equality over τ between t_1 and t_2 ($-T/2 \leq t_1 < t_2 \leq T$), keeping in mind that for every $\tau \in [-T/2, T]$

$$\begin{gathered}
\frac{d}{d\tau}\widehat{v}_{kl}(x,\tau)\cdot\widehat{v}_{kl}(x,\tau)\cdot\theta(\tau) = \frac{1}{2}\frac{d}{d\tau}\big(|\widehat{v}_{kl}(x,\tau)|^2\theta(\tau)\big) - \frac{1}{2}|\widehat{v}_{kl}(x,\tau)|^2\frac{d}{d\tau}\theta(\tau),\\
\frac{d}{d\tau}\gamma\widehat{v}_{kl}(x,\tau)\cdot\gamma\widehat{v}_{kl}(x,\tau)\cdot\theta(\tau) = \frac{1}{2}\frac{d}{d\tau}\big(|\gamma\widehat{v}_{kl}(x,\tau)|^2\theta(\tau)\big) - \frac{1}{2}|\gamma\widehat{v}_{kl}(x,\tau)|^2\frac{d}{d\tau}\theta(\tau).
\end{gathered}$$

As a result,

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_R}\big(b_1(x)|\widehat{v}_{kl}(x,\tau)|^2 w(x)\theta(\tau)\big)\Big|_{\tau=t_1}^{\tau=t_2}dx\\
&+\frac{1}{2}\int_{\Gamma_{1,R}}\big(b_2(y)|\gamma\widehat{v}_{kl}(y,\tau)|^2 w(y)\theta(\tau)\big)\Big|_{\tau=t_1}^{\tau=t_2}d\Gamma_y\\
&-\frac{1}{2}\int_{t_1}^{t_2}\Big(\int_{\Omega_R}b_1(x)|\widehat{v}_{kl}(x,\tau)|^2 w(x)dx + \int_{\Gamma_{1,R}}b_2(y)|\gamma\widehat{v}_{kl}(y,\tau)|^2 w(y)d\Gamma_y\Big)\theta'\,d\tau\\
&+\int_{t_1}^{t_2}\Big(\int_{\Omega_R}\Big\{\sum_{i=1}^n \widehat{g}_{i,kl}(x,\tau)(\widehat{v}_{kl}(x,\tau)w(x))_{x_i} + \widehat{g}_{0,kl}(x,\tau)\widehat{v}_{kl}(x,\tau)w(x)\Big\}\,dx\\
&+\int_{\Gamma_{1,R}}\widehat{h}_{kl}(y,\tau)\gamma\widehat{v}_{kl}(y,\tau)w(y)\,d\Gamma_y\Big)\theta(\tau)\,d\tau = 0.
\end{aligned} \tag{3.13}$$

Now, we impose additional conditions on the function θ:

$$\begin{gathered}
0 \leq \theta(\tau) \leq 1 \quad \text{when } \tau \in \mathbb{R}, \quad \theta(\tau) = 0 \quad \text{when } \tau \leq -T/2,\\
\theta(\tau) = 1 \quad \text{when } \tau \geq 0, \quad |\theta'(\tau)| \leq 4/T \quad \text{when } \tau \in [-T/2, 0].
\end{gathered}$$

Then from (3.13), having chosen $t_1 = -T/2$ and t_2 be any number from the interval $[0, T]$, we derive

$$\begin{aligned}
&\max_{\tau\in[0,T]}\Big(\int_{\Omega_R}b_1(x)|\widehat{v}_{kl}(x,\tau)|^2 w(x)dx + \int_{\Gamma_{1,R}}b_2(y)|\gamma\widehat{v}_{kl}(y,\tau)|^2 w(y)d\Gamma_y\Big)\\
&\leq \frac{4}{T}\int_{-T/2}^0\Big(\int_{\Omega_R}b_1(x)|\widehat{v}_{kl}(x,\tau)|^2 w(x)dx + \int_{\Gamma_{1,R}}b_2(y)|\gamma\widehat{v}_{kl}(y,\tau)|^2 w(y)d\Gamma_y\Big)d\tau\\
&\quad + 2\int_{-T/2}^T\Big(\int_{\Omega_R}\Big\{\sum_{i=1}^n|\widehat{g}_{i,kl}(x,\tau)||(\widehat{v}_{kl}(x,\tau)w(x))_{x_i}|\\
&\quad + |\widehat{g}_{0,kl}(x,\tau)||\widehat{v}_{kl}(x,\tau)|w(x)\Big\}dx + \int_{\Gamma_{1,R}}|\widehat{h}_{kl}(y,\tau)||\gamma\widehat{v}_{kl}(y,\tau)|w(y)d\Gamma_y\Big)d\tau.
\end{aligned}$$

From above inequality by (3.10) this implies that as $k, l \to +\infty$,

$$\begin{gathered}
(wb_1)^{1/2}\widehat{v}_{k,l} \to 0 \quad \text{in} \quad C([0,T]; L_2(\Omega_R)),\\
(wb_2)^{1/2}\gamma\widehat{v}_{k,l} \to 0 \quad \text{in} \quad C([0,T]; L_2(\Gamma_{1,R})).
\end{gathered}$$

This means that the sequences $\{(wb_1)^{1/2}\widehat{v}_k\}_{k=1}^{\infty}$, $\{(wb_2)^{1/2}\gamma\widehat{v}_k\}_{k=1}^{\infty}$ are fundamental in the spaces $C([0,T];L_2(\Omega_R))$, $C([0,T];L_2(\Gamma_{1,R}))$ respectively and

$$\begin{gathered}(wb_1)^{1/2}\widehat{v}_k \underset{k\to+\infty}{\longrightarrow} (wb_1)^{1/2}\widehat{v} \quad \text{in } C([0,T];L_2(\Omega_R)),\\ (wb_2)^{1/2}\gamma\widehat{v}_k \underset{k\to+\infty}{\longrightarrow} (wb_2)^{1/2}\gamma\widehat{v} \quad \text{in } C([0,T];L_2(\Gamma_{1,R})).\end{gathered} \tag{3.14}$$

Thus we conclude that

$$b_1^{1/2}v \in C([0,T];L_2(\Omega_{R^*})), \quad b_2^{1/2}\gamma v \in C([0,T];L_2(\Gamma_{1,R^*}))$$

for every $R^* \in (0,R)$.

Now for every $\tau \in [0,T]$, in (3.11), put $\psi(x) = \widehat{v}_k(x,\tau)w(x)\theta(\tau)$, $x \in \Omega$, where $w \in C^1(\overline{\Omega})$, $\operatorname{supp} w \subset \overline{\Omega}_R$, $w \geq 0$, $\theta \in C^1([0,T])$, and integrate over τ between t_1 and t_2. The transformations similar to those we made above to obtain (3.13) yield

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_R}\big(b_1(x)|\widehat{v}_k(x,\tau)|^2w(x)\theta(\tau)\big)\Big|_{\tau=t_1}^{\tau=t_2}dx\\ &+\frac{1}{2}\int_{\Gamma_{1,R}}\big(b_2(y)|\gamma\widehat{v}_k(y,\tau)|^2w(y)\theta(\tau)\big)\Big|_{\tau=t_1}^{\tau=t_2}d\Gamma_y\\ &-\frac{1}{2}\int_{t_1}^{t_2}\Big(\int_{\Omega_R}b_1(x)|\widehat{v}_k(x,\tau)|^2w(x)dx+\int_{\Gamma_{1,R}}b_2(y)|\gamma\widehat{v}_k(y,\tau)|^2w(y)d\Gamma_y\Big)\theta'\,d\tau\\ &+\int_{t_1}^{t_2}\Big(\int_{\Omega_R}\Big\{\sum_{i=1}^n\widehat{g}_{i,k}(x,\tau)(\widehat{v}_k(x,\tau)w(x))_{x_i}+\widehat{g}_{0,k}(x,\tau)\widehat{v}_k(x,\tau)w(x)\Big\}\,dx\\ &+\int_{\Gamma_{1,R}}\widehat{h}_k(y,\tau)\gamma\widehat{v}_k(y,\tau)w(y)\,d\Gamma_y\Big)\theta(\tau)\,d\tau=0.\end{aligned}$$

Passing to the limit as $k\to+\infty$, by (3.10), (3.14) we obtain (3.2). $\square$

Lemma 3.2. *Let $(b_1,b_2)\in\mathbb{B}$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}$, $c\in\mathbb{C}$. Suppose that, for every $k\in\{1,2\}$, $u_k\in\mathbb{U}_{\rm loc}$, $(f_{1,k},f_{2,k})\in\mathbb{F}_{\rm loc}$, $\bar{f}_{i,k}\in L_{2,\rm loc}(\overline{Q})$ $(i=\overline{1,n})$ and*

$$\begin{aligned}&\iint_Q\Big\{\sum_{i=1}^n a_i(x,t,u_k,\nabla u_k)\psi_{x_i}\varphi+a_0(x,t,u_k,\nabla u_k)\psi\varphi-b_1(x)u_k\psi\varphi'\Big\}\,dx\,dt\\ &+\iint_{\Sigma_1}\{c(y,t,\gamma u_k)\gamma\psi\varphi-b_2(y)\gamma u_k\gamma\psi\varphi'\}\,d\Gamma_y dt\\ &=\iint_Q\Big\{\sum_{i=1}^n\bar{f}_{i,k}\psi_{x_i}\varphi+f_{1,k}\psi\varphi\Big\}\,dx\,dt+\iint_{\Sigma_1}f_{2,k}\gamma\psi\varphi\,d\Gamma_y dt\end{aligned}\tag{3.15}$$

holds for every $\psi\in\mathbb{V}_c, \operatorname{supp}\psi\subset\overline{\Omega_R}$, $\varphi\in C_0^1(-\infty,0)$, $\operatorname{supp}\varphi\subset S_R$, where $R\geq 1$ is some number. Then for arbitrary $R_0\in(0,R)$ we have

$$\begin{aligned}&\max_{t\in[-R_0,0]}\int_{\Omega_{R_0}}b_1(x)|u_1(x,t)-u_2(x,t)|^2dx\\ &+\max_{t\in[-R_0,0]}\int_{\Gamma_{1,R_0}}b_2(y)|\gamma u_1(y,t)-\gamma u_2(y,t)|^2d\Gamma_y\\ &+\iint_{Q_{R_0}}\big\{\rho_1|\nabla u_1-\nabla u_2|^2+\rho_2|u_1-u_2|^p\big\}\,dx\,dt+\iint_{\Sigma_{1,R_0}}\rho_3|\gamma u_1-\gamma u_2|^q d\Gamma_y dt\\ &\leq C\big(R/(R-R_0)\big)^\sigma\Big[\Psi(b_1,b_2,\rho_1,\rho_2,\rho_3,d_1,d_2;R)\end{aligned}$$

$$
\begin{aligned}
&+ \iint_{Q_R} \Big\{ \big(\rho_1^{-1} + n\rho_1 d_1^{-2}\big)\Big(\sum_{i=1}^n |\bar f_{i,1} - \bar f_{i,2}|^2 \Big) + \rho_2^{-1/(p-1)} |f_{1,1} - f_{1,2}|^{p'} \Big\}\, dx\, dt \\
&+ \iint_{\Sigma_{1,R}} \rho_3^{-1/(q-1)} |f_{2,1} - f_{2,2}|^{q'} d\Gamma_y dt \Big],
\end{aligned} \tag{3.16}
$$

where C, σ, Ψ are the same as in Theorem 2.5.

Proof. Introduce two "cutting" functions (see [2]):

$$
\zeta(x) = \begin{cases} (R^2 - |x|^2)/R, & |x| < R, \\ 0, & |x| \ge R, \end{cases} \quad \chi(t) = \begin{cases} t + R, & -R \le t \le 0, \\ 0, & t < -R. \end{cases}
$$

For given $\psi \in \mathbb{V}_c, \varphi \in C_0^1(-\infty, 0)$ such that $\operatorname{supp}\psi \subset \overline{\Omega_R}, \operatorname{supp}\varphi \subset S_R$, consider (3.15) when $k = 1$ and the same equality when $k = 2$. Subtract these equalities. Put

$$
\begin{gathered}
u_{12}(x,t) := u_1(x,t) - u_2(x,t), \quad f_{1,12}(x,t) := f_{1,1}(x,t) - f_{1,2}(x,t), \\
a_{i,12}(x,t) := a_i(x,t,u_1(x,t),\nabla u_1(x,t)) - a_i(x,t,u_1(x,t),\nabla u_1(x,t)), \\
\bar f_{i,12}(x,t) := \bar f_{i,1}(x,t) - \bar f_{i,2}(x,t), \quad (i = \overline{1,n}), \\
(x,t) \in Q, \quad i = \overline{0,n}, \\
\gamma u_{12}(y,t) := \gamma u_1(y,t) - \gamma u_2(y,t), \quad f_{2,12}(y,t) = f_{2,1}(y,t) - f_{2,2}(y,t), \\
c_{12}(y,t) := c(y,t,\gamma u_1(y,t)) - c(y,t,\gamma u_2(y,t)), \quad (y,t) \in \Sigma_1.
\end{gathered}
$$

Apply Lemma 3.1 to the obtained equality with $g_0 := a_{0,12} - f_{1,12}$, $g_i = a_{i,12} - \bar f_{i,12}$ $(i = \overline{1,n})$, $h = c_{12} - f_{2,12}$, $w = \zeta^s$, $\theta = \chi^r$, where $r = \max\big\{p/(p-2), q/(q-2)\big\}$, $s = 2p/(p-2)$, $t_1 = -R$, $t_2 = \tau \in (-R, 0]$. After simple transformation we obtain

$$
\begin{aligned}
&\eta^r(\tau)\Big(\int_{\Omega_R} b_1(x)|u_{12}(x,\tau)|^2 \zeta^s(x) dx + \int_{\Gamma_{1,R}} b_2(y) |\gamma u_{12}(y,\tau)|^2 \zeta^s(y) d\Gamma_y \Big) \\
&+ 2 \iint_{Q_R^\tau} \Big\{ \sum_{i=1}^n a_{i,12}(u_{12})_{x_i} + a_{0,12} u_{12} \Big\} \zeta^s \eta^r\, dx\, dt + 2 \iint_{\Sigma_{1,R}^\tau} b_{12} \gamma u_{12} \zeta^s \eta^r\, d\Gamma_y\, dt \\
&= r \iint_{Q_R^\tau} b_1 |u_{12}|^2 \zeta^s \eta^{r-1}\, dx\, dt + r \iint_{\Sigma_{1,R}^\tau} b_2 |\gamma u_{12}|^2 \zeta^s \eta^{r-1} d\Gamma_y\, dt \\
&\quad - 2s \iint_{Q_R^\tau} \Big(\sum_{i=1}^n a_{i,12} \zeta_{x_i} \Big) u_{12} \zeta^{s-1} \eta^r\, dx\, dt + 2 \iint_{Q_R^\tau} \sum_{i=1}^n \bar f_{i,12} (u_{12})_{x_i} \zeta^s \eta^r\, dx\, dt \\
&\quad + 2s \iint_{Q_R^\tau} \Big(\sum_{i=1}^n \bar f_{i,12} \zeta_{x_i} \Big) u_{12} \zeta^{s-1} \eta^r\, dx\, dt + 2 \iint_{Q_R^\tau} f_{1,12} u_{12} \zeta^s \chi^r\, dx\, dt \\
&\quad + 2 \iint_{\Sigma_{1,R}^\tau} f_{2,12} \gamma u_{12} \zeta^s \chi^r d\Gamma_y\, dt,
\end{aligned} \tag{3.17}
$$

where $Q_R^\tau := \Omega_R \times (-R, \tau), \Sigma_{1,R}^\tau := \Gamma_{1,R} \times (-R, \tau)$ when $\tau \in (-R, 0]$.

Making the appropriate estimation of the integrals of equality (3.17). From conditions (A4) and (B3) we have respectively

$$\begin{aligned} &\iint_{Q_R^\tau}\Big\{\sum_{i=1}^n a_{i,12}(u_{12})_{x_i}+a_{0,12}u_{12}\Big\}\zeta^s\eta^r\,dx\,dt \\ &\geq \iint_{Q_R^\tau}\Big\{\rho_1|\nabla u_{12}|^2+\rho_2|u_{12}|^p\Big\}\zeta^s\eta^r\,dx\,dt, \end{aligned} \tag{3.18}$$

$$\iint_{Q_R^\tau} b_{12}\gamma u_{12}\zeta^s\eta^r d\Gamma_y dt \geq \iint_{\sum_R^\tau}\rho_3|\gamma u_{12}|^q\zeta^s\eta^r d\Gamma_y dt. \tag{3.19}$$

Hereinafter we will use the Young inequality: For every $a\geq 0$, $b\geq 0$, $\varepsilon>0$, $\nu>1$, we have

$$ab\leq \varepsilon a^\nu + M(\nu,\varepsilon)b^{\nu'}, \tag{3.20}$$

where $1/\nu+1/\nu'=1$, $M(\nu,\varepsilon)>0$ is the constant depending only on ν and ε. Choose $\nu=p/2$ $a=\rho_2^{1/\nu}|u_{12}|^2\zeta^{s/\nu}\eta^{r/\nu}$, $b=\rho_2^{-1/\nu}b_1\zeta^{s/\nu'}\eta^{r/\nu'-1}$, $\varepsilon=\varepsilon_1>0$ $(\nu'=p/(p-2))$. By (3.20) we deduce

$$\begin{aligned} &\iint_{Q_R^\tau} b_1|u_{12}|^2\zeta^s\eta^{r-1}\,dx\,dt \\ &\leq \varepsilon_1\iint_{Q_R^\tau}\rho_2|u_{12}|^p\zeta^s\eta^r\,dx\,dt \\ &\quad + M(p/2,\varepsilon_1)\iint_{Q_R^\tau}\rho_2^{-2/(p-2)}b_1^{p/(p-2)}\zeta^s\eta^{r-p/(p-2)}\,dx\,dt, \end{aligned} \tag{3.21}$$

where $\varepsilon_1>0$ is an arbitrary number. In the same way we obtain the inequality

$$\begin{aligned} &\iint_{\sum_{1,R}^\tau} b_2|\gamma u_{12}|^2\zeta^s\eta^{r-1}d\Gamma_y dt \\ &\leq \varepsilon_2\iint_{\sum_{1,R}^\tau}\rho_3|\gamma u_{12}|^q\zeta^s\eta^r d\Gamma_y dt \\ &\quad + M(q/2,\varepsilon_2)\iint_{\sum_{1,R}^\tau}\rho_3^{-2/(q-2)}b_2^{q/(q-2)}\zeta^s\eta^{r-q/(q-2)}\,d\Gamma_y dt, \end{aligned} \tag{3.22}$$

where $\varepsilon_2>0$ is any number. On the basis of condition (A3), taking into account that $|\zeta_{x_i}|\leq 2$ $(i=\overline{1,n})$, we have

$$\begin{aligned} &\Big|\iint_{Q_R^\tau}\Big(\sum_{i=1}^n a_{i,12}\zeta_{x_i}\Big)u_{12}\zeta^{s-1}\eta^r\,dx\,dt\Big| \\ &\leq 2\iint_{Q_R^\tau}\Big(\sum_{i=1}^n|a_{i,12}|\Big)|u_{12}|\zeta^{s-1}\eta^r\,dx\,dt \\ &\leq 2\iint_{Q_R^\tau} d_1|\nabla u_{12}||u_{12}|\zeta^{s-1}\eta^r\,dx\,dt + 2\iint_{Q_R^r} d_2|u_{12}|^2\zeta^{s-1}\eta^r\,dx\,dt. \end{aligned} \tag{3.23}$$

Choose $\nu = 2$, $a = \rho_1^{1/2}|\nabla u_{12}|\zeta^{s/2}\eta^{r/2}$, $b = \rho_1^{-1/2}d_1|u_{12}|\zeta^{s/2-1}\eta^{r/2}$, $\varepsilon = \varepsilon_3 > 0$. From Young's inequality (3.20), we derive

$$\begin{aligned} &\iint_{Q_R^\tau} d_1|\nabla u_{12}||u_{12}|\zeta^{s-1}\eta^r \,dx\,dt \\ &\le \varepsilon_3 \iint_{Q_R^\tau} \rho_1|\nabla u_{12}|^2\zeta^s\eta^r \,dx\,dt + M(2,\varepsilon_3)\iint_{Q_R^\tau} \rho_1^{-1}d_1^2|u_{12}|^2\zeta^{s-2}\eta^r \,dx\,dt, \end{aligned} \tag{3.24}$$

where $\varepsilon_3 > 0$ is an arbitrary number.

To estimate the integral in the second term of the right side of (3.24) use again the Young's inequality (3.20), taking $\nu = p/2$, $a = \rho_2^{1/\nu}|u_{12}|^2\zeta^{s/\nu}\eta^{r/\nu}$, $b = \rho_1^{-1}\rho_2^{-1/\nu}d_1^2\zeta^{s/\nu'-2}\eta^{r/\nu'}$, $\varepsilon = \varepsilon_4 > 0(\nu' = p/(p-2))$. As a result we conclude

$$\begin{aligned} &\iint_{Q_R^\tau} \rho_1^{-1}d_1^2|u_{12}|^2\zeta^{s-2}\eta^r \,dx\,dt \\ &\le \varepsilon_4 \iint_{Q_R^\tau} \rho_2|u_{12}|^p\zeta^s\eta^r \,dx\,dt \\ &\quad + M(p/2,\varepsilon_4)\iint_{Q_R^\tau} \rho_1^{-p/(p-2)}\rho_2^{-2/(p-2)}d_1^{2p/(p-2)}\zeta^{s-2p/(p-2)}\eta^r \,dx\,dt, \end{aligned} \tag{3.25}$$

where $\varepsilon_4 > 0$ is an arbitrary number. In the same way, we obtain

$$\begin{aligned} &\iint_{Q_R^\tau} d_2|u_{12}|^2\zeta^{s-1}\eta^r \,dx\,dt \\ &\le \varepsilon_5 \iint_{Q_R^\tau} \rho_2|u_{12}|^p\zeta^s\eta^r \,dx\,dt \\ &\quad + M(p/2,\varepsilon_5)\iint_{Q_R^\tau} \rho_2^{-2/(p-2)}d_2^{p/(p-2)}\zeta^{s-p/(p-2)}\eta^r \,dx\,dt, \end{aligned} \tag{3.26}$$

where $\varepsilon_5 > 0$ is an arbitrary number. Using the Cauchy inequality we have

$$\begin{aligned} &\Big|\iint_{Q_R^\tau} \sum_{i=1}^n \bar f_{i,12}(u_{12})_{x_i}\zeta^s\eta^r \,dx\,dt\Big| \\ &\le \varepsilon_6 \iint_{Q_R^\tau} \rho_1|\nabla u_{12}|^2\zeta^s\eta^r \,dx\,dt + \frac{1}{4\varepsilon_6}\iint_{Q_R^\tau} \rho_1^{-1}\Big(\sum_{i=1}^n |\bar f_{i,12}|^2\Big)\zeta^s\eta^r \,dx\,dt, \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &\Big|\iint_{Q_R^\tau} \Big(\sum_{i=1}^n \bar f_{i,12}\zeta_{x_i}\Big)u_{12}\zeta^{s-1}\eta^r \,dx\,dt\Big| \\ &\le 2\iint_{Q_R^\tau} \Big(\sum_{i=1}^n |\bar f_{i,12}|\Big)|u_{12}|\zeta^{s-1}\eta^r \,dx\,dt \\ &\le \iint_{Q_R^\tau} \rho_1^{-1}d_1^2|u_{12}|^2\zeta^{s-2}\eta^r \,dx\,dt + n\iint_{Q_R^\tau} \rho_1 d_1^{-2}\Big(\sum_{i=1}^n |\bar f_{i,12}|^2\Big)\zeta^s\eta^r \,dx\,dt. \end{aligned} \tag{3.28}$$

Also on the basis of Young's inequality, we obtain

$$\begin{aligned}
&\Big|\iint_{Q_R^\tau} f_{1,12}u_{12}\zeta^s\eta^r\,dx\,dt\Big| \le \iint_{Q_R^\tau}|f_{1,12}||u_{12}|\zeta^s\eta^r\,dx\,dt\\
&\le \varepsilon_7\iint_{Q_R^\tau}\rho_2|u_{12}|^p\zeta^s\eta^r\,dx\,dt + M(p,\varepsilon_7)\iint_{Q_R^\tau}\rho_2^{-1/(p-1)}|f_{1,12}|^{p'}\zeta^s\eta^r\,dx\,dt,
\end{aligned}\tag{3.29}$$

and

$$\begin{aligned}
&\Big|\iint_{\Sigma_{1,R}^\tau} f_{2,12}\gamma u_{12}\zeta^s\eta^r d\Gamma_y dt\Big|\\
&\le \varepsilon_8\iint_{\Sigma_{1,R}^\tau}\rho_3|\gamma u_{12}|^q\zeta^s\eta^r d\Gamma_y dt + M(p,\varepsilon_7)\iint_{\Sigma_{1,R}^\tau}\rho_3^{-1/(q-1)}|f_{2,12}|^{q'}\zeta^s\eta^r d\Gamma_y dt,
\end{aligned}\tag{3.30}$$

where $\varepsilon_7>0$, $\varepsilon_8>0$ are arbitrary numbers. Then from (3.17), using (3.18), (3.19), (3.21)–(3.30) and taking $\varepsilon_1,\dots,\varepsilon_8$ be small enough, we deduce the estimate

$$\begin{aligned}
&\eta^r(\tau)\Big(\int_{\Omega_R} b_1(x)|u_{12}(x,\tau)|^2\zeta^s(x)dx + \int_{\Gamma_{1,R}} b_2(y)|\gamma u_{12}(y,\tau)|^2\zeta^s(y)d\Gamma_y\Big)\\
&+\iint_{Q_R^\tau}\big\{\rho_1|\nabla u_{12}|^2+\rho_2|u|^p\big\}\zeta^s\eta^r\,dx\,dt + \iint_{\Sigma_{1,R}^\tau}\rho_3|\gamma u_{12}|^q\zeta^s\eta^r d\Gamma_y dt\\
&\le C_1\Big(\iint_{Q_R^\tau}\rho_2^{-2/(p-2)}b_1^{p/(p-2)}\zeta^s\eta^{r-p/(p-2)}\,dx\,dt\\
&\quad+\iint_{\Sigma_{1,R}^\tau}\rho_3^{-2/(q-2)}b_2^{q/(q-2)}\zeta^s\eta^{r-q/(q-2)}d\Gamma_y dt\\
&\quad+\iint_{Q_R^\tau}\rho_1^{-p/(p-2)}\rho_2^{-2/(p-2)}d_1^{2p/(p-2)}\zeta^{s-2p/(p-2)}\eta^r\,dx\,dt\\
&\quad+\iint_{Q_R^\tau}\rho_2^{-2/(p-2)}d_2^{p/(p-2)}\zeta^{s-p/(p-2)}\eta^r\,dx\,dt\Big)\\
&\quad+C_2\Big(\iint_{Q_R^\tau}\rho_2^{-1/(p-1)}|f_{1,12}|^{p'}\zeta^s\eta^r\,dx\,dt\\
&\quad+\iint_{Q_R^\tau}\big(\rho_1^{-1}+n\rho_1 d_1^{-2}\big)\Big(\sum_{i=1}^n|\bar f_{i,12}|^2\Big)\zeta^s\eta^r\,dx\,dt\\
&\quad+\iint_{\Sigma_{1,R}^\tau}\rho_2^{-1/(q-1)}|f_{2,12}|^{q'}\zeta^s\eta^r\,d\Gamma_y dt\Big),
\end{aligned}\tag{3.31}$$

where $C_1>0, C_2>0$ are constants depending only on p,q. Let $R_0\in(0,R)$ be any number. Since $0\le\zeta(x)\le R$ for every $x\in\mathbb{R}^n$, $\zeta(x)\ge R-R_0$ when $|x|\le R_0$, and $0\le\eta(t)\le R$ for all $t\in\mathbb{R}$, $\eta(t)\ge R-R_0$ when $t\ge -R_0$, from (3.31) we obtain

$$\begin{aligned}
&\max_{\tau\in[-R_0,0]}\Big(\int_{\Omega_{R_0}} b_1(x)|u_{12}(x,\tau)|^2dx+\int_{\Gamma_{1,R_0}} b_2(y)|\gamma u_{12}(y,\tau)|^2d\Gamma_y\Big)\\
&+\iint_{Q_{R_0}}\big\{\rho_1|\nabla u_{12}|^2+\rho_2|u_{12}|^p\big\}\,dx\,dt+\iint_{\Sigma_{1,R_0}}\rho_3|\gamma u_{12}|^q d\Gamma_y dt\\
&\le (R/(R-R_0))^{s+r}\Big[C_3\Big(R^{-p/(p-2)}\iint_{Q_R}\rho_2^{-2/(p-2)}b_1^{p/(p-2)}\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ R^{-2p/(p-2)} \iint_{Q_R} \rho_1^{-p/(p-2)} \rho_2^{-2/(p-2)} d_1^{2p/(p-2)}\, dx\, dt \\
&+ R^{-p/(p-2)} \iint_{Q_R} \rho_2^{-2/(p-2)} d_2^{p/(p-2)}\, dx\, dt \\
&+ R^{-q/(q-2)} \iint_{\sum_{1,R}} \rho_3^{-2/(q-2)} b_2^{q/(q-2)} d\Gamma_y\, dt\Big) \\
&+ C_4\Big(\iint_{Q_R} \big(\rho_1^{-1} + n\rho_1 d_1^{-2}\big)\Big(\sum_{i=1}^{n} |\bar{f}_{i,12}|^2\Big)\, dx\, dt \\
&+ \iint_{Q_R} \rho_2^{-1/(p-1)} |f_{1,12}|^{p'}\, dx\, dt + \iint_{\sum_{1,R}} \rho_3^{-1/(q-1)} |f_{2,12}|^{q'}\, d\Gamma_y\, dt\Big)\Big],
\end{aligned}
$$

where $C_3 > 0,\ C_4 > 0$ are constants depending only on p, q. This yields (3.16). □

Remark 3.3. If in addition to the condition of Lemma 3.1 we assume that $\operatorname{supp} v \subset \overline{\Omega_R \times (\tau_1, \tau_2)}$, then the assertion of Lemma 3.1 is also true when $R_* = R$ and $w = 1$.

Corollary 3.4. *Let $(b_1, b_2) \in \mathbb{B}$, $(a_0, a_1, \dots, a_n) \in \mathbb{A}, c \in \mathbb{C}, (f_1, f_2) \in \mathbb{F}_{\rm loc}$. Suppose that for some $R > 0$ there exist constants $\alpha_j > 0, \beta_j > 0 (j = 1, 2), \alpha_3 \geq 0, \beta_3 \geq 0, \mu_1 > 0, \mu_2 \geq 0$ such that for a.e. $(x, t) \in Q_R$ and every $(s, \xi) \in \mathbb{R}^{1+n}$ we have*

$$\sum_{i=1}^{n} |a_i(x, t, s, \xi)| \leq \alpha_1 |\xi| + \alpha_2 |s| + \alpha_3, \tag{3.32}$$

$$\sum_{j=1}^{n} a_i(x, t, s, \xi)\xi_i + a_0(x, t, s, \xi)s \geq \beta_1 |\xi|^2 + \beta_2 |s|^p - \beta_3, \tag{3.33}$$

$$c(x, t, s)s \geq \mu_1 |s|^q - \mu_2, \tag{3.34}$$

Then for any generalized solution u of (1.1)–(1.3) and for every $R^* \in (0, R)$ the estimate

$$\|u\|_{L_2(S_{R^*}; H^1(\Omega_{R^*}))} + \|u\|_{L_p(Q_{R^*})} + \|\gamma u\|_{L_q(\Sigma_{1,R^*})} \leq C_5(R, R^*) \tag{3.35}$$

takes place, where $C_5(R, R^*) > 0$ is the constant depending only on $R, R^*, f_1|_{Q_R}$, $f_2|_{\Sigma_{1,R}}, \alpha_k, \beta_k (k = 1, 2, 3), \mu_j (j = 1, 2)$.

The statement of this Corollary can be obtained similarly as it made for (3.16).

4. Proof of main results

Proof of Theorem 2.5. **Step 1.** For every $k \in \mathbb{N}$ take the subdomain Ω^k of the domain Ω such that $\partial\Omega^k \in C^1$, $\Omega_k \subset \Omega^k, \Omega^k \subset \Omega^{k+1}$. Put $Q^k = \Omega^k \times S_k, \Gamma_0^k := \overline{\partial\Omega^k \setminus \Gamma_1}$, $\Gamma_1^k = \partial\Omega^k \setminus \Gamma_0^k$, $\Sigma_0^k := \Gamma_0^k \times S_k$, $\Sigma_1^k := \Gamma_1^k \times S_k$.

For every $k \in \mathbb{N}$, let $\mathbb{V}^k$ be the Banach space obtained by closure of the space $\{v \in C^1(\overline{\Omega^k}) : \operatorname{dist}\{\operatorname{supp} v, \Gamma_0^k\} > 0\}$ by the norm $\|v\|_{\mathbb{V}^k} := \|v\|_{H^1(\Omega^k)} + \|v\|_{L_p(\Omega^k)} + \|v\|_{L_q(\Gamma_1^k)}$. Note that for every $k \in \mathbb{N}$ the extensions by zero on Ω of functions from $\mathbb{V}^k$ generate the subspace of the space $\mathbb{V}_c \subset \mathbb{V}_{\rm loc}$. Thus we can consider the operator $\gamma^k : \mathbb{V}^k \to L_q(\Gamma_1^k)$ as the contraction of the operator $\gamma : \mathbb{V}_{\rm loc} \to L_{q,{\rm loc}}(\overline{\Gamma_1})$. So further we will write γ instead of γ^k. Define

$$\mathbb{U}^k := \{w \in (S_k \to \mathbb{V}^k) : w \in L_2(S_k; H^1(\Omega^k)) \cap L_p(Q^k),$$

$$b_1^{1/2}w \in C(\overline{S_k}; L_2(\Omega^k)), \gamma w \in L_q(\Sigma_1^k), b_2^{1/2}\gamma w \in C(\overline{S_k}; L_2(\Gamma_1^k))\}$$

be the Banach space with the norm

$$\begin{aligned}\|w\|_{\mathbb{U}^k} &:= \|w\|_{L^2(S_k;H^1(\Omega^k))} + \|w\|_{L_p(Q^k)} + \max_{t\in S^k}\|b_1^{1/2}(\cdot)w(\cdot,t)\|_{L_2(\Omega^k)} \\ &\quad + \|\gamma w\|_{L_q(\Sigma_1^k)} + \max_{t\in S^k}\|b_2^{1/2}(\cdot)\gamma w(\cdot,t)\|_{L_2(\Gamma_1^k)}.\end{aligned}$$

Consider the family of mixed problems

$$\begin{gathered}\frac{\partial}{\partial t}(b_1(x)u^k) - \sum_{i=1}^n \frac{d}{dx_i}a_i(x,t,u^k,\nabla u^k) + a_0(x,t,u^k,\nabla u^k) = f_1(x,t), \\ (x,t) \in Q_k,\end{gathered} \tag{4.1}$$

$$u^k(y,t) = 0, \quad (y,t) \in \Sigma_0^k, \tag{4.2}$$

and

$$\frac{\partial}{\partial t}(b_2(y)u^k) - \sum_{i=1}^n a_i(y,t,u^k,\nabla u^k)\nu_i(y) + c(y,t,u^k) = f_2(y,t), \quad (y,t) \in \Sigma_1^k, \tag{4.3}$$

$$b_1^{1/2}u^k|_{t=-k} = 0, \quad x \in \Omega^k, \quad b_2^{1/2}u^k|_{t=-k} = 0, \quad y \in \Gamma_1^k. \tag{4.4}$$

Definition 4.1. A function $u^k \in \mathbb{U}^k$ is called a generalized solution of the problem (4.1)-(4.4) (for arbitrary $k \in \mathbb{N}$) if it satisfies the initial data (4.4) and the integral equality

$$\begin{aligned}&\iint_{Q^k}\Big\{\sum_{i=1}^n a_i(x,t,u^k,\nabla u^k)\psi_{x_i}\varphi + a_0(x,t,u^k,\nabla u^k)\psi\varphi - b_1(x)u^k\psi\varphi'\Big\}\,dx\,dt \\ &+ \iint_{\Sigma_1^k}\Big\{c(y,t,\gamma u^k)\gamma\psi\varphi - b_2(y)\gamma u^k\gamma\psi\varphi'\Big\}\,d\Gamma_y\,dt \\ &= \iint_{Q^k} f_1^k\psi\varphi\,dx\,dt + \iint_{\Sigma_1^k} f_2^k\gamma\psi\varphi\,d\Gamma_y\,dt\end{aligned} \tag{4.5}$$

for every $\psi \in \mathbb{V}^k$, $\varphi \in C^1([-k,0])$, $\varphi(0) = 0$.

The existence and uniqueness of the generalized solution of (4.1) – (4.4) (for every $k \in \mathbb{N}$) can be easily proved using research technique from [22].

Step 2. For every $k \in \mathbb{N}$ extend u^k by zero to Q and keep the notation u^k to this extension. It easy to verify that $u^k \in \mathbb{U}_{\rm loc}$ for all $k \in \mathbb{N}$. Consider the sequence $\{u^k\}_{k=1}^\infty$ and show that it contains the subsequence converging to the generalized solution of problem (1.1) – (1.3) in some sense.

First we show that for every $R_0 > 0$ the sequences $\{u^k|_{Q_{R_0}}\}_{k=1}^\infty$, $\{\gamma u^k|_{\Sigma_{1,R_0}}\}_{k=1}^\infty$, $\{b_1^{1/2}u^k|_{Q_{R_0}}\}_{k=1}^\infty$, and $\{b_2^{1/2}\gamma u^k|_{\Sigma_{1,R_0}}\}_{k=1}^\infty$ are respectively fundamental in spaces $L_2(S_{R_0}; H^1(\Omega_{R_0})) \cap L_p(Q_{R_0})$, $L_q(\Sigma_{1,R_0})$, $C(\overline{S_{R_0}}; L_2(\Omega_{R_0}))$ and $C(\overline{S_{R_0}}; L_2(\Gamma_{1,R_0}))$. For this purpose use Lemma 3.2, choosing $R > 2R_0$ be any number, $u_1 = u^k$, $u_2 = u^l$ for arbitrary $k,l > R$. From its assertion and inequality

$$R/(R-R_0) = 1 + R_0/(R-R_0) \le 2 \quad \text{when } R \ge 2R_0 \tag{4.6}$$

we obtain

$$
\begin{aligned}
&\max_{t\in\overline{S_{R_0}}}\|b_1^{1/2}(\cdot)(u^k(\cdot,t)-u^l(\cdot,t))\|^2_{L_2(\Omega_{R_0})}\\
&\quad+\max_{t\in\overline{S_{R_0}}}\|b_2^{1/2}(\cdot)(\gamma u^k(\cdot,t)-\gamma u^l(\cdot,t))\|^2_{L_2(\Gamma_{1,R_0})}\\
&\quad+\|u^k-u^l\|^2_{L_2(S_{R_0};H^1(\Omega_{R_0}))}+\|u^k-u^l\|^p_{L_p(Q_{R_0})}+\|\gamma u^k-\gamma u^l\|^q_{L_q(\Sigma_{1,R_0})}\\
&\le C_6(R_0)\Psi(b_1,b_2,\rho_1,\rho_2,\rho_3,d_1,d_2;R),
\end{aligned}
\tag{4.7}
$$

where $C_6(R_0)>0$ is a constant depending on R_0, but not depending on R.

From (2.5) and 4.6 we conclude that the right side of (4.7) tends to zero as $R\to+\infty$. Thus for arbitrarily small value $\varepsilon>0$ there exists $k^*\in\mathbb{N}$ such that for every $k,l\ge k^*$ the left side of (4.7) is less then ε. It yields the fundamentality of the sequences $\{u^k|_{Q_{R_0}}\}$ in $L_2(S_{R_0};H^1(\Omega_{R_0}))\cap L_p(Q_{R_0})$, $\{\gamma u^k|_{\Sigma_{1,R_0}}\}$ in $L_q(\Sigma_{1,R_0})$, $\{b_1^{1/2}u^k|_{Q_{R_0}}\}$ in $C(\overline{S_{R_0}};L_2(\Omega_{R_0}))$, $\{b_2^{1/2}\gamma u^k|_{\Sigma_{1,R_0}}\}$ in $C(\overline{S_{R_0}};L_2(\Gamma_{1,R_0}))$. Since R_0 is an arbitrary number, the above stated yields the existence of functions $u\in(S\to\mathbb{V}_{\rm loc})\cap L_{2,{\rm loc}}(S;H^1_{\rm loc}(\overline{\Omega}))\cap L_{p,{\rm loc}}(\overline{Q})$, $\gamma u\in L_{q,{\rm loc}}(\overline{\Sigma_1})$, $\widehat{u}\in C(S;L_{2,{\rm loc}}(\overline{\Omega}))$, $\widehat{u}\in C(S;L_{2,{\rm loc}}(\overline{\Gamma_1}))$ such that

$$u^k\underset{k\to\infty}{\longrightarrow}u\quad\text{in } L_{2,{\rm loc}}(S;H^1_{\rm loc}(\overline{\Omega}))\cap L_{p,{\rm loc}}(\overline{Q}),\tag{4.8}$$

$$\gamma u^k\underset{k\to\infty}{\longrightarrow}\gamma u\quad\text{in } L_{q,{\rm loc}}(\overline{\Sigma_1}),\tag{4.9}$$

$$b_1^{1/2}u^k\underset{k\to\infty}{\longrightarrow}\widehat{u}\quad\text{in } C(S;L_{2,{\rm loc}}(\overline{\Omega})),\tag{4.10}$$

$$b_2^{1/2}\gamma u^k\underset{k\to\infty}{\longrightarrow}\widehat{u}\quad\text{in } C(S;L_{2,{\rm loc}}(\overline{\Gamma_1})).\tag{4.11}$$

It remains to show that

$$\widehat{u}=b_1^{1/2}u,\quad \widehat{u}=b_2^{1/2}\gamma u.\tag{4.12}$$

Indeed, from (4.8)–(4.11) it follows that there exists a subsequence $\{u^{k_j}\}_{j=1}^\infty$ such that

$$u^{k_j}\underset{j\to\infty}{\longrightarrow}u,\quad \partial u^{k_j}/\partial x_i\underset{j\to\infty}{\longrightarrow}\partial u/\partial x_i\quad\text{a.e. in } Q,\tag{4.13}$$

$$\gamma u^{k_j}\underset{j\to\infty}{\longrightarrow}\gamma u\quad\text{a.e. on}\quad\Sigma_1,\tag{4.14}$$

$$b_1^{1/2}u^{k_j}\underset{j\to\infty}{\longrightarrow}\widehat{u}\quad\text{a.e. in } Q,\tag{4.15}$$

$$b_2^{1/2}\gamma u^{k_j}\underset{j\to\infty}{\longrightarrow}\widehat{u}\quad\text{a.e. on}\quad\Sigma_1.\tag{4.16}$$

From this it easily follows (4.12). On the basis of (4.10)– (4.12) we conclude that $b_1^{1/2}u\in C(S;L_{2,{\rm loc}}(\overline{\Omega}))$, $b_2^{1/2}\gamma u\in C(S;L_{2,{\rm loc}}(\overline{\Gamma_1}))$, and therefore $u\in\mathbb{U}_{\rm loc}$.

Now show that u is generalized solution of (1.1)–(1.3). First of all note that under condition (A3) and (4.8), we have

$$a_i(\cdot,\cdot,u^k(\cdot,\cdot),\nabla u^k(\cdot,\cdot))\underset{k\to\infty}{\longrightarrow}a_i(\cdot,\cdot,u(\cdot,\cdot),\nabla u(\cdot,\cdot))\quad\text{in } L_{2,{\rm loc}}(\overline{Q}),\ i=\overline{1,n}.\tag{4.17}$$

Now prove the existence of a subsequence $\{u^{k_i}\}_{i=1}^\infty$ of the sequence $\{u^k\}$ such that for arbitrary fixed $R>0$

$$a_0(\cdot,\cdot,u^{k_i}(\cdot,\cdot),\nabla u^{k_i}(\cdot,\cdot))\underset{i\to\infty}{\longrightarrow}a_0(\cdot,\cdot,u(\cdot,\cdot),\nabla u(\cdot,\cdot))\quad\text{weakly in } L_{p'}(Q_R),\tag{4.18}$$

$$c(\cdot,\cdot,\gamma u^{k_i}(\cdot,\cdot))\underset{i\to\infty}{\longrightarrow}c(\cdot,\cdot,\gamma u(\cdot,\cdot))\quad\text{weakly in } L_{q'}(\Sigma_{1,R}).\tag{4.19}$$

Indeed, (4.8) and (4.9) yield the estimates

$$\|u^k\|_{L_2(S_R;H^1(\Omega_R))\cap L_p(Q_R)} \le C_7(R), \quad k\in\mathbb{N}, \tag{4.20}$$

$$\|\gamma u^k\|_{L_q(\Sigma_{1,R})} \le C_8(R), \quad k\in\mathbb{N}, \tag{4.21}$$

where $C_7(R), C_8(R) > 0$ are constants probably depending on R, but not depending on k.

By (A2) and (4.20) we have

$$\|a_0(\cdot,\cdot,u^k(\cdot,\cdot),\nabla u^k(\cdot,\cdot))\|_{L_{p'}(Q_R)} \le C_9(R), \quad k\in\mathbb{N}, \tag{4.22}$$

and on the basis of condition (B2) and (4.21), we have

$$\|c(\cdot,\cdot,\gamma u^k(\cdot,\cdot))\|_{L_{q'}(\Gamma_{1,R})} \le C_{10}(R), \quad k\in\mathbb{N}, \tag{4.23}$$

where $C_9(R), C_{10}(R) > 0$ are constants probably depending on R, but not depending on k. Using the [22, Proposition 3.4,p. 51], from $\mathbf{A}_1$, (4.13) and (4.22) we obtain (4.18), and from (B1), (4.14) and (4.23) we obtain (4.19).

Let $\psi\in\mathbb{V}_{\rm c}$, $\varphi\in C_0^1(-\infty,0)$ be arbitrary functions and l be a natural number such that $\operatorname{supp}\psi\subset\overline{\Omega^l}$, $\operatorname{supp}\varphi\subset(-l,0)$. Then for every $k>l$ ($k\in\mathbb{N}$) the equality (4.5) is fulfilled. In fact the integrals in this equality are taken over Q^l instead of Q^k and Σ_1^l instead of Σ_1^k. Put $k=k_i (i\in\mathbb{N})$ in (4.5) and pass to the limit as $i\to\infty$, taking into account (4.8),(4.9),(4.17)–(4.19). As a result we obtain (2.3); i.e., exactly what is needed.

To obtain (2.6) let us use Lemma 3.2. Let u be the generalized solution of the problem (1.1)–(1.3) with given $(b_1,b_2)\in\mathbb{B}$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}$, $c\in\mathbb{C}$, $(f_1,f_2)\in\mathbb{F}_{\rm loc}$. Note that on the basis of conditions $\mathbf{A_1'}, \mathbf{C_1'}$ the function $u=0$ is the solution of (1.1)–(1.3) with the same (b_1,b_2), $(a_0,a_1,\dots,a_n)$, b, but with $(f_1,f_2)=(0,0)$. Let $R\ge 0$ be an arbitrary number, $u_1=u$, $u_2=0$, $f_{1,1}=f_1$, $f_{2,1}=f_2$, $f_{1,2}=0$, $f_{2,2}=0$, $\overline{f}_{i,1}=\overline{f}_{i,2} (i=\overline{1,n})$. Then from Lemma 3.2 (see (3.16)) it is easy to get (2.6).

Step 3. Now prove the continuous dependence on data-in of generalized solutions of (1.1)–(1.3). Let $\{(a_0^k,a_1^k,\dots,a_n^k)\},\{c^k\}$ and $\{(f_1^k,f_2^k)\}$ be the sequences such that $((b_1,b_2),(a_0^k,a_1^k,\dots,a_n^k),c^k)\in\mathbb{B}\mathbb{A}\mathbb{C}$ and

$$(a_0^k,a_1^k,\dots,a_n^k)\underset{k\to\infty}{\longrightarrow}(a_0,a_1,\dots,a_n) \quad\text{in } \mathbb{A}, \quad c^k\underset{k\to\infty}{\longrightarrow} c \quad\text{in } \mathbb{C}, \tag{4.24}$$

$$(f_1^k,f_2^k)\underset{k\to\infty}{\longrightarrow}(f_1,f_2) \quad\text{in } \mathbb{F}_{\rm loc}. \tag{4.25}$$

Take any number $k\in\mathbb{N}$. Reformulate the integral identity, which define the function u^k as a generalized solution corresponding problem (similar to (2.3)), in the form

$$\begin{aligned}
&\iint_Q\Big\{\sum_{i=1}^n a_i(x,t,u^k,\nabla u^k)\psi_{x_i}\varphi + a_0(x,t,u^k,\nabla u^k)\psi\varphi - b_1(x)u^k\psi\varphi'\Big\}\,dx\,dt\\
&+\iint_{\Sigma_1}\{c(y,t,\gamma u^k)\gamma\psi\varphi - b_2(y)\gamma u^k\gamma\psi\varphi'\}\,d\Gamma_y dt\\
&=\iint_Q\Big\{\sum_{i=1}^n\big(a_i(x,t,u^k,\nabla u^k)-a_i^k(x,t,u^k,\nabla u^k)\big)\psi_{x_i}\varphi\\
&\quad+\big(a_0(x,t,u^k,\nabla u^k)-a_0^k(x,t,u^k,\nabla u^k)+f_1^k\big)\psi\varphi\Big\}\,dx\,dt
\end{aligned}$$

$$+\iint_{\Sigma_1}\{c(y,t,\gamma u^k)-c^k(y,t,\gamma u^k)+f_2^k\}\gamma\psi\varphi\,d\Gamma_y dt \tag{4.26}$$

for every $\psi\in\mathbb{V}_c,\varphi\in C_0^1(-\infty,0)$.

From (4.26) and (2.3) on the basis of Lemma 3.2, putting $u_1=u^k$, $u_2=u$ and $\bar f_{i,1}=a_i(\cdot,\cdot,u^k,\nabla u^k)-a_i^k(\cdot,\cdot,u^k,\nabla u^k)$ $(i=\overline{1,n})$, $f_{1,1}=a_0(\cdot,\cdot,u^k,\nabla u^k)-a_0^k(\cdot,\cdot,u^k,\nabla u^k)+f_1^k,$, $f_{2,1}=c(\cdot,\cdot,\gamma u^k)-c^k(\cdot,\cdot,\gamma u^k)+f_2^k$, $\bar f_{i,2}=0$ $(i=\overline{1,n})$, $f_{1,2}=f_1$, $f_{2,2}=f_2$ and choosing $R>0$, $R_0\in(0,R)$ to be arbitrary, we obtain

$$\begin{aligned}
&\max_{t\in[-R_0,0]}\int_{\Omega_{R_0}}b_1(x)|u^k(x,t)-u(x,t)|^2\,dx\\
&+\max_{t\in[-R_0,0]}\int_{\Gamma_{1,R_0}}b_2(y)|\gamma u^k(y,t)-\gamma u(y,t)|^2\,d\Gamma_y\\
&+\iint_{Q_{R_0}}\Big\{\rho_1|\nabla u^k-\nabla u|^2+\rho_2|u^k-u|^p\Big\}\,dx\,dt+\iint_{\Sigma_{1,R_0}}\rho_3|\gamma u^k-\gamma u|^q\,d\Gamma_y\,dt\\
&\le C_{11}(R/(R-R_0))^\sigma\Big[\Psi(b_1,b_2,\rho_1,\rho_2,\rho_3,d_1,d_2;R)\\
&\quad+\iint_{Q_R}\rho_2^{-1/(p-1)}|f_1^k-f_1|^{p'}\,dx\,dt+\iint_{\Sigma_{1,R}}(\rho_3)^{-1/(q-1)}|f_2^k-f_2|^{q'}\,d\Gamma_y\,dt\\
&\quad+\iint_{Q_R}\big(\rho_1^{-1}+n\rho_1d_1^{-2}\big)\sum_{i=1}^n|a_i(x,t,u^k,\nabla u^k)-a_i^k(x,t,u^k,\nabla u^k)|^2\,dx\,dt\\
&\quad+\iint_{Q_R}\rho_2^{-1/(p-1)}|a_0(x,t,u^k,\nabla u^k)-a_0^k(x,t,u^k,\nabla u^k)|^{p'}\,dx\,dt\\
&\quad+\iint_{\Sigma_{1,R}}\rho_3^{-1/(q-1)}|c(y,t,\gamma u^k)-c^k(x,t,\gamma u^k)|^{q'}\,dx\,dt\Big],
\end{aligned}\tag{4.27}$$

where σ,Ψ are the same as in the statement of the Theorem 2.5, $C_{11}>0$ is a constant depending only on p,q.

Let $\varepsilon>0$ be an arbitrary small number and $R_0>0$ be any number. By virtue of (2.5) we can take $R\ge 2R_0$ such that

$$\Psi(b_1,b_2,\rho_1,\rho_2,\rho_3,d_1,d_2;R)<\varepsilon/(5C_{11}2^\sigma)\,. \tag{4.28}$$

Fix such R. On the basis of (4.25) there exists $k_1\in\mathbb{N}$ such that

$$\begin{aligned}
&\iint_{Q_R}\rho_2^{-1/(p-1)}|f_1^k-f_1|^{p'}\,dx\,dt+\iint_{\Sigma_{1,R}}\rho_3^{-1/(q-1)}|f_2^k-f_2|^{q'}\,d\Gamma_y\,dt\\
&<\varepsilon/(5C_{11}2^\sigma)
\end{aligned}\tag{4.29}$$

for every $k\ge k_1$. Now we show the existence of constants $k_2\ge k_1$ $(k_2\in\mathbb{N})$, $C_{12}>0$ such that

$$\iint_{Q_R}\big(|u^k|^2+|u^k|^p+|\nabla u^k|^2\big)\,dx\,dt+\iint_{\Sigma_{1,R}}|\gamma u^k|^q\le C_{12} \tag{4.30}$$

for all $k\ge k_2$. To this effect we use Corollary 3.4. Let k be any natural number. Put $\bar\rho_j:=\operatorname{ess\,inf}_{(x,t)\in Q_{R+1}}\rho_j(x,t)>0$ $(j=1,2)$, $\bar\rho_3:=\operatorname{ess\,inf}_{(y,t)\in\Sigma_{1,R+1}}\rho_3(y,t)>0$, $\bar d_1:=\operatorname{ess\,sup}_{(x,t)\in Q_{R+1}}d_1(x,t)$, $\bar d_2:=\operatorname{esssup}_{(x,t)\in Q_{R+1}}d_2(x,t)$.

From conditions (A1'), (A3), (A4), (C1'), (C3) and simple consideration for a.e. $(x,t) \in Q_{R+1}$ it follows that

$$
\begin{aligned}
&\sum_{i=1}^n |a_i^k(x,t,s,\xi)| \\
&\le \sum_{i=1}^n |a_i(x,t,s,\xi)| + \sum_{i=1}^n |a_i^k(x,t,s,\xi) - a_i(x,t,s,\xi)| \\
&\le \bar{d}_1|\xi| + \bar{d}_2|s| \\
&\quad + \Big(\operatorname*{ess\,sup}_{(x,t)\in Q_{R+1}} \sup_{(s,\xi)\in\mathbb{R}^{1+n}} \sum_{i=1}^n \frac{|a_i^k(x,t,s,\xi) - a_i(x,t,s,\xi)|}{1+|s|+|\xi|}\Big)(1+|s|+|\xi|),
\end{aligned}
\tag{4.31}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^n a_i^k(x,t,s,\xi)\xi_i + a_0^k(x,t,s,\xi)s \\
&= \sum_{i=1}^n a_i(x,t,s,\xi)\xi_i + a_0(x,t,s,\xi)s \\
&\quad + \sum_{i=1}^n \big(a_i^k(x,t,s,\xi) - a_i(x,t,s,\xi)\big)\xi_i + \big(a_0^k(x,t,s,\xi) - a_0(x,t,s,\xi)\big)s \\
&\ge \rho_1(x,t)|\xi|^2 + \rho_2(x,t)|s|^p - \Big(\sum_{i=1}^n \big|a_i^k(x,t,s,\xi) - a_i(x,t,s,\xi)\big||\xi| \\
&\quad + \big|a_0^k(x,t,s,\xi) - a_0(x,t,s,\xi)\big||s|\Big) \\
&\ge \bar{\rho}_1|\xi|^2 + \bar{\rho}_2|s|^p \\
&\quad - \Big[\Big(\operatorname*{ess\,sup}_{(x,t)\in Q_{R+1}} \sup_{(s,\xi)\in\mathbb{R}^{1+n}} \sum_{i=1}^n \frac{|a_i^k(x,t,s,\xi) - a_i(x,t,s,\xi)|}{(1+|s|+|\xi|)}\Big)(|\xi| + |s||\xi| + |\xi|^2) \\
&\quad + \Big(\operatorname*{ess\,sup}_{(x,t)\in Q_{R+1}} \sup_{(s,\xi)\in\mathbb{R}^{1+n}} \frac{|a_0^k(x,t,s,\xi) - a_0(x,t,s,\xi)|}{(1+|s|^{p-1}+|\xi|^{2/p'})}\Big)(|s| + |s|^p + |s||\xi|^{2/p'})\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
c^k(y,t,s)s &= c(y,t,s)s - |c^k(y,t,s) - c(y,t,s)||s| \\
&\ge \bar{\rho}_3|s|^p - \Big(\operatorname*{ess\,sup}_{(x,t)\in\Sigma_{1,R+1}} \sup_{s\in\mathbb{R}} \frac{|c^k(y,t,s) - c(y,t,s)|}{1+|s|^{p-1}}\Big)\big(|s| + |s|^p\big).
\end{aligned}
\tag{4.32}
$$

Now note that on the basis of Young's inequalities we obtain

$$
\begin{gathered}
|\xi| \le |\xi|^2/2 + 1/2, \quad |s||\xi| \le |\xi|^2/2 + |s|^p/p + (p-2)/(2p), \\
|s| \le |s|^p/p + 1/p', \quad |s||\xi|^{2/p'} \le |\xi|^2/p' + |s|^p/p.
\end{gathered}
$$

From this, (4.31)-(4.32), and (4.24) it follows that there exists natural number $k_2 \ge k_1$ such that for each $k \ge k_2$ we obtain assumptions of Corollary 3.4 (in particular, (3.32)-(3.34)) with $a_0^k, a_1^k, \dots, a_n^k, c^k$ instead of $a_0, a_1, \dots, a_n, c$ respectively). Note that in the given case constants α_j, β_j, μ_l are independent of k. Taking into consideration (4.25) from the statement of Corollary 3.4 (4.30) follows.

We resume to estimate the terms in the right side of (4.27). After easy transformations we obtain the inequalities

$$
\begin{aligned}
&\iint_{Q_R} \big(\rho_1^{-1}+n\rho_1 d_1^{-2}\big)\sum_{i=1}^n |a_i(x,t,u^k,\nabla u^k)-a_i^k(x,t,u^k,\nabla u^k)|^2\,dx\,dt\\
&\le 3 \operatorname*{ess\,sup}_{(x,t)\in Q_R}\big(\rho_1^{-1}(x,t)+n\rho_1(x,t)d_1^{-2}(x,t)\big)\\
&\quad\times \operatorname*{ess\,sup}_{(x,t)\in Q_R}\sup_{(s,\xi)\in\mathbb{R}^{1+n}}\Big(\sum_{i=1}^n |a_i^k(x,t,s,\xi)-a_i(x,t,s,\xi)|/(1+|s|+|\xi|)\Big)^2\\
&\quad\times\iint_{Q_R}\big(1+|u^k|^2+|\nabla u^k|^2\big)\,dx\,dt,
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
&\iint_{Q_R}\rho_2^{-1/(p-1)}|a_0(x,t,u^k,\nabla u^k)-a_0^k(x,t,u^k,\nabla u^k)|^{p'}\,dx\,dt\\
&\le C_{13}(p)\operatorname*{ess\,sup}_{(x,t)\in Q_R}\rho_2^{-1/(p-1)}(x,t)\\
&\quad\times\operatorname*{ess\,sup}_{(x,t)\in Q_R}\sup_{(s,\xi)\in\mathbb{R}^{1+n}}\big[|a_0^k(x,t,s,\xi)-a_0(x,t,s,\xi)|/(1+|s|^{p-1}+|\xi|^{2/p'})\big]^{p'}\\
&\quad\times\iint_{Q_R}\big(1+|u^k|^p+|\nabla u^k|^2\big)\,dx\,dt,
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{\Sigma_{1,R}}\rho_3^{-1/(q-1)}|c(y,t,\gamma u^k)-c^k(x,t,\gamma u^k)|^{q'}\,dx\,dt\\
&\le C_{14}(p)\operatorname*{ess\,sup}_{(y,t)\in\Sigma_{1,R}}\rho_3^{-1/(q-1)}(y,t)\Big(\operatorname*{ess\,sup}_{(y,t)\in\Sigma_{1,R}}\sup_{s\in\mathbb{R}}|c^k(y,t,s)\\
&\quad - c(y,t,s)|/(1+|s|^{q-1})\Big)^{q'}\iint_{\Sigma_{1,R}}\big(1+|\gamma u^k|^q\big)\,d\Gamma_y\,dt.
\end{aligned}
\tag{4.34}
$$

From (4.30) on the basis of (4.24) it follows that there exists natural number $k_3\ge k_2$ such that the right side of each inequalities (4.33)-(4.34) is less than $\varepsilon/(52^\sigma C_{11})$ for all $k\ge k_3$. From this and (4.6), (4.28), (4.29) it follows that the right side of the inequality (4.27) with R and k_3 being chosen above is less then ε for every $k\ge k_3$. Therefore $u^k\longrightarrow_{k\to\infty} u$ in $\mathbb{U}_{\mathrm{loc}}$. $\square$

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