Electronic Journal of Differential Equations, Vol. 2010(2010), No. 18, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

TRANSPORT OPERATOR ON PHASE SPACES WITH FINITE TIME OF SOJOURN PROPERTY

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Abstract. In this article, the transport operator with general boundary conditions is discussed. According to a smallness hypothesis on the boundary operator and to finite time of sojourn property of phase spaces, we prove that the transport operator generates a strongly continuous semigroup and we give its upper bound.

1. Introduction

This article concerns the transport equation

$$\frac{\partial f}{\partial t}(x,v) = -v \cdot \nabla_x f(x,v), \quad (x,v) \in \Omega \tag{1.1}$$

where, $\Omega = X \times V$ with $X \subset \mathbb{R}^n$ $(n \geqslant 1)$ is a bounded open subset with smooth boundary ∂X and $d\mu$ is a Radon measure on $\mathbb{R}^n$ with bounded support V. If we denote by Γ_- (resp. Γ_+) the incoming (resp. outgoing) part of the phase space boundary $\Gamma = \partial X \times V$, then the boundary condition is modelled as

$$f(t)\big|_{\Gamma_-} = K\big(f(t)\big|_{\Gamma_+}\big) \tag{1.2}$$

where, $f(t)\big|_{\Gamma_-}$ (resp. $f(t)\big|_{\Gamma_+}$) is the incoming (resp. outgoing) particle flux. The boundary operator K is bounded linear into suitable function spaces on Γ_- and Γ_+ (for more explanations see next Section). All known boundary conditions (vacuum, specular reflections, periodic, ...) are special examples of our general context.

If $\|K\| \leqslant 1$, it is well known, in the pioneer works [1, 9, 10], that the transport model (1.1)–(1.2) is governed by a strongly continuous semigroup of contractions. However, the case $\|K\| > 1$ has been rarely studied and some contributions are made in [2, 3, 5, 6]. There is another contribution made in [8] (see last Section).

The difficulty concerning the case $\|K\| > 1$ is closely related to the increasing number of, on one hand, the incoming particles whose the time of sojourn $\tau(x,v)$ may be arbitrary small and on the other hand, to the particles in X of which the time of sojourn $t(x,v)$ may be arbitrary big. In order to take into account such as particles, we intuitively have to set hypotheses on the geometry of (X,V) and on the boundary operators K. So, the first hypothesis concerns boundary operators K satisfying

2000 *Mathematics Subject Classification.* 47D06, 82D75.
Key words and phrases. Positivity; strongly continuous semigroups; transport operator.

Submitted November 29, 2009. Published January 27, 2010.

(H1) There exists $\varepsilon_0 > 0$ such that $\|K\chi_{\varepsilon_0}\|_{\mathcal{L}(L^1(\Gamma_+),L^1(\Gamma_-))} < 1$, where the characteristic operator $\chi_{\varepsilon_0} \in \mathcal{L}(L^1(\Gamma_+))$ is defined by

$$\chi_{\varepsilon_0}\psi(x,v) = \begin{cases} \psi(x,v) & \text{if } \tau(x,v) \leqslant \varepsilon_0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

The second hypothesis acts on the geometry of V in the following sense

(H2) $0 \notin V$

which intuitively leads us to set our definition

Definition 1.1. The phase space (X, V) has *finite time of sojourn property* if

$$T_{\max} := \sup_{(x,v)\in\Omega} t(x,v) < \infty. \tag{1.4}$$

Clearly, the hypothesis (H2) implies that velocities cannot vanish; therefore Definition 1.1 holds because of the boundedness of X. For instance, let the phase space $\big((0,1)\times(a,b)\big)$ $(a > 0)$ be related to a model of cell dynamic populations already studied in [7]. The phase space $\big((0,1)\times(a,b)\big)$ $(a > 0)$ fulfils the definition above because of $T_{\max} = \frac{1}{a} < \infty$.

In this paper, we discuss the case $\|K\| > 1$ and at this end, we suppose that the hypotheses (H1) and (H2) hold. So, we prove that the transport model (1.1)–(1.2) is governed by a strongly continuous semigroup and we give its upper bound. We end this paper by remarks and comments.

2. Setting of the problem

In this section we state preparatory Lemmas for the next Section. Let us consider the Banach space $L^1(\Omega)$ whose natural norm is

$$\|\varphi\|_1 = \int_\Omega |\varphi(x,v)|\,dx\,d\mu(v) \tag{2.1}$$

where, $\Omega = X \times V$. We set $n(x)$ the outer unit normal at $x \in \partial X$, where, the boundary ∂X is equipped with the Lebesgue measure $d\gamma$ and we denote

$$\Gamma_\pm = \{(x,v) \in \Gamma : \pm v\cdot n(x) > 0\},$$

where $\Gamma = \partial X \times V$. For each $(x,v) \in \Omega$, we set

$$t(x,v) = \inf\{t : t > 0,\ x - tv \notin X\},$$

the time of sojourn in X, and

$$\theta(x,v) = t(x,v) + t(x,-v),$$

the chord of sojourn. Similarly, if $(x,v) \in \Gamma_+$ we set

$$\tau(x,v) = \inf\{t : t > 0,\ x - tv \notin X\}.$$

Next, we introduce the partial Sobolev space

$$W^1(\Omega) = \{\varphi \in L^1(\Omega) : v\cdot\nabla_x\varphi \in L^1(\Omega),\ \theta^{-1}\varphi \in L^1(\Omega)\}$$

whose norm is

$$\|\varphi\|_{W^1(\Omega)} = \|v\cdot\nabla_x\varphi\|_1 + \|\theta^{-1}\varphi\|.$$

Finally, we consider the trace spaces $L^1(\Gamma_\pm)$ endowed with the norm

$$\|\varphi\|_{L^1(\Gamma_\pm)} = \int_{\Gamma_\pm} |\varphi(x,v)| d\xi$$

where, $d\xi = |v\cdot n(x)|d\gamma d\mu(v)$. In this context, we define the following trace mapping

$$\gamma_+\varphi = \varphi|_{\Gamma_+} \quad \text{and} \quad \gamma_-\varphi = \varphi|_{\Gamma_-}$$

for which we have our new result.

Lemma 2.1 ([3]). *The trace mappings*

$$\gamma_+ : W^1(\Omega) \to L^1(\Gamma_+) \quad \textit{and} \quad \gamma_- : W^1(\Omega) \to L^1(\Gamma_-)$$

are continuous, surjective and admit continuous lifting operators.

Let K be a bounded linear operator from $L^1(\Gamma_+)$ into $L^1(\Gamma_-)$. So, it is clear that above Lemma allows us to give a sense to the following transport operator

$$\begin{gathered} T_K\varphi = -v\cdot\nabla_x\varphi \quad \text{on the domain} \\ D(T_K) = \{\varphi \in W^1(\Omega) : \gamma_-\varphi = K\gamma_+\varphi\}. \end{gathered} \tag{2.2}$$

If the boundary operator satisfies $K = 0$, then the corresponding operator T_0 is defined as follows

$$\begin{gathered} T_0\varphi = -v\cdot\nabla_x\varphi \quad \text{on the domain} \\ D(T_K) = \{\varphi \in W^1(\Omega) : \gamma_-\varphi = 0\} \end{gathered} \tag{2.3}$$

has some properties summarized next.

Lemma 2.2. *We have*

1. *The operator T_0 generates, on $L^1(\Omega)$, a strongly continuous semigroup of contractions $(U_0(t))_{t\geqslant 0}$. Furthermore, $U_0(t)$ is a positive operator; i.e., $U_0(t)\varphi \geqslant 0$ for all $\varphi \in (L^1(\Omega))_+$.*
2. *Let $\lambda > 0$ be fixed. Then, for all $\varphi \in (L^1(\Omega))_+ - \{0\}$ we have $(\lambda - T_0)^{-1}\varphi \in (L^1(\Omega))_+ - \{0\}$ and $\gamma_+(\lambda - T_0)^{-1}\varphi \in (L^1(\Gamma_+))_+ - \{0\}$.*
3. *Let $\lambda > 0$. Then*

$$\|(\lambda - T_0)^{-1}g\|_1 \leqslant \frac{\|g\|_1}{\lambda}, \tag{2.4}$$

$$\|\theta^{--1}(\lambda - T_0)^{-1}g\|_1 \leqslant \|g\|_1 \tag{2.5}$$

for all $g \in L^1(\Omega)$.

Proof. The items (1), (2) and (3) follow easily from

$$(\lambda - T_0)^{-1}g(x,v) = \int_0^{t(x,v)} e^{-\lambda s} g(x - sv, v)ds$$

where, $\lambda > 0$ and $g \in L^1(\Omega)$. □

Lemma 2.3. *Let A be the operator*

$$A\psi(x,v) = \psi(x - \tau(x,v)v, v). \tag{2.6}$$

Then A is a positive isometry from $L^1(\Gamma_-)$ to $L^1(\Gamma_+)$; i.e.,

$$\|A\psi\|_{L^1(\Gamma_+)} = \|\psi\|_{L^1(\Gamma_-)} \tag{2.7}$$

for all $\psi \in L^1(\Gamma_+)$.

Proof. Let $\psi \in L^1(\Gamma_+)$. As $u(x,v) = \psi(x - t(x,v)v, v)$ is the unique solution of the boundary value problem

$$\begin{gathered} v \cdot \nabla_x u = 0 \\ \gamma_- u = \psi\,. \end{gathered}$$

Then multiplying the first equation by $(\operatorname{sgn} u)$ and using

$$(\operatorname{sgn} u) v \cdot \nabla_x u = v \cdot \nabla_x (|u|), \tag{2.8}$$

we obtain $v \cdot \nabla_x(|u|) = 0$. Integrating this equation over Ω and using Green's identity, we obtain

$$\int_{\Gamma_+} |\gamma_+ u(x,v)| d\xi = \int_{\Gamma_-} |\gamma_- u(x,v)| d\xi;$$

therefore,

$$\int_{\Gamma_+} |A\psi(x,v)| d\xi = \int_{\Gamma_-} |\psi(x,v)| d\xi$$

whence (2.7). The positivity of A is obvious. □

3. Generation Theorem

In this section, we are only concerned with boundary operators whose norm satisfies $\|K\| > 1$. So, according to (H1)–(H2), we prove that the transport operator T_K given by (2.2) generates, on $L^1(\Omega)$, a strongly continuous semigroup. Before we state this main goal, we have to show the following lemmas.

Lemma 3.1. *Let K be a boundary operator with $\|K\| > 1$ and suppose that* (H1) *holds. Let K_λ $(\lambda \geqslant 0)$ be the operator*

$$K_\lambda \psi := K(\alpha_\lambda \psi), \tag{3.1}$$

where

$$\alpha_\lambda(x,v) = e^{-\lambda\tau(x,v)}.$$

Then K_λ is a bounded linear operator from $L^1(\Gamma_+)$ to $L^1(\Gamma_-)$. Furthermore, we have

$$\lambda > \omega_0 \Longrightarrow \|K_\lambda\| < 1, \tag{3.2}$$

$$\|K_{\omega_0}\| \leqslant 1, \tag{3.3}$$

where

$$\omega_0 = \frac{1}{\varepsilon_0} \ln \|K\|. \tag{3.4}$$

Moreover, if K is a positive operator, then K_λ is also a positive operator.

Proof. Let $\lambda \geqslant 0$. For all $\psi \in L^1(\Gamma_+)$ we obviously have $\chi_{\varepsilon_0}^2 = \chi_{\varepsilon_0}$ which implies

$$\alpha_\lambda \psi = \chi_{\varepsilon_0}^2(\alpha_\lambda\psi) + \overline{\chi}_{\varepsilon_0}(\alpha_\lambda\psi);$$

therefore,

$$K_\lambda\psi = K\chi_{\varepsilon_0}(\chi_{\varepsilon_0}\alpha_\lambda\psi) + K\overline{\chi}_{\varepsilon_0}(\alpha_\lambda\psi),$$

where the characteristic operator χ_{ε_0} is given by (1.3), and $\overline{\chi}_{\varepsilon_0}$ is the characteristic operator

$$\overline{\chi}_{\varepsilon_0}\psi(x,v) = \begin{cases} \psi(x,v) & \text{if } \tau(x,v) > \varepsilon_0 \\ 0 & \text{otherwise.} \end{cases}$$

This implies

$$\begin{aligned}
\|K_\lambda\psi\|_{L^1(\Gamma_-)} &\leqslant \|K\chi_{\varepsilon_0}(\chi_{\varepsilon_0}\alpha_\lambda\psi)\|_{L^1(\Gamma_-)} + \|K\overline{\chi}_{\varepsilon_0}(\alpha_\lambda\psi)\|_{L^1(\Gamma_-)}\\
&\leqslant \|K\chi_{\varepsilon_0}\|\|\chi_{\varepsilon_0}\alpha_\lambda\psi\|_{L^1(\Gamma_+)} + \|K\|\|\overline{\chi}_{\varepsilon_0}(\alpha_\lambda\psi)\|_{L^1(\Gamma_+)}\\
&\leqslant \|K\chi_{\varepsilon_0}\|\|\chi_{\varepsilon_0}\psi\|_{L^1(\Gamma_+)} + e^{-\lambda\varepsilon_0}\|K\|\|\overline{\chi}_{\varepsilon_0}\psi\|_{L^1(\Gamma_+)}\\
&\leq \max\{\|K\chi_{\varepsilon_0}\|,\ e^{-\lambda\varepsilon_0}\|K\|\}\{\|\chi_{\varepsilon_0}\psi\| + \|\overline{\chi}_{\varepsilon_0}\psi\|\}\\
&= \{\|K\chi_{\varepsilon_0}\|,\ e^{-\lambda\varepsilon_0}\|K\|\}\|\psi\|_{L^1(\Gamma_+)}
\end{aligned}$$

which leads to

$$\|K_\lambda\| \leqslant \max\{\|K\chi_{\varepsilon_0}\|,\ e^{-\lambda\varepsilon_0}\|K\|\}.$$

Now, the above relation clearly implies

$$\|K_\lambda\| < 1 \quad \text{if } \lambda > \omega_0$$

and $\|K_{\omega_0}\| \leqslant 1$. Finally, if K is a positive operator, the positivity of the operator K_λ is then obvious. The proof is now achieved. $\square$

Thanks Lemma above, the resolvent operator of (2.2) is given as follows.

Lemma 3.2. *Let K be a boundary operator whose satisfying $\|K\| > 1$, and suppose that* (H1) *holds. Then, for all $\lambda > \omega_0$, we have $\lambda \in \rho(T_K)$ and*

$$\begin{gathered}
(\lambda - T_K)^{-1}g(x,v) = (\lambda - T_0)^{-1}g(x,v)+\\
e^{-\lambda t(x,v)}(I - K_\lambda A)^{-1}K\gamma_-(\lambda - T_0)^{-1}g(x - t(x,v)v, v)
\end{gathered} \tag{3.5}$$

for almost all $(x,v) \in \Omega$ and for all $g \in L^1(\Omega)$, where, A is the operator given by (2.6)*. Furthermore, if K is a positive operator, $(\lambda - T_K)^{-1}$ is then a positive operator for all $\lambda > \omega_0$.*

Proof. Let $\lambda > \omega_0$. For all $g \in L^1(\Omega)$, the general solution of

$$\lambda\varphi = -v \cdot \nabla_x\varphi + g, \tag{3.6}$$

is given by

$$\varphi(x,v) = e^{-\lambda t(x,v)}\psi(x - t(x,v)v, v) + (\lambda - T_0)^{-1}g(x,v), \tag{3.7}$$

for almost all $(x,v) \in \Omega$, where T_0 is already studied in Lemma 2.2 and ψ is any function of $L^1(\Gamma_-)$. In the sequel, let us prove that $\varphi \in D(T_K)$.

Integrating (3.7) over Ω, a simple calculation together with (2.4) give us

$$\begin{aligned}
\|\varphi\|_1 &\leqslant \int_\Omega e^{-\lambda t(x,v)}|\psi(x - t(x,v)v, v)|\,dx\,d\mu(v) + \|(\lambda - T_0)^{-1}g\|_1\\
&\leqslant \frac{1}{\lambda}\|\psi\|_{L^1(\Gamma_-)} + \frac{\|g\|_1}{\lambda} < \infty
\end{aligned}$$

which implies, by (3.6), that

$$\|v \cdot \nabla_x\varphi\|_1 \leqslant \lambda\|\varphi\|_1 + \|g\|_1 < \infty.$$

Multiplying (3.7) by θ^{-1} and integrating it over Ω, a simple calculation together with (2.5) lead to

$$\|\theta^{-1}\varphi\|_1 \leqslant \|\psi\|_{L^1(\Gamma_-)} + \|g\|_1 < \infty;$$

therefore, $\varphi \in W^1(\Omega)$. Next, φ satisfies $\gamma_-\varphi = K\gamma_-\varphi$ if and only if ψ satisfies

$$\psi = K_\lambda A\psi + K\gamma_-(\lambda - T_0)^{-1}g. \tag{3.8}$$

By (2.7) and (3.2) we obtain

$$\|K_\lambda A\| \leqslant \|K_\lambda\| \, \|A\| < 1; \tag{3.9}$$

therefore, (3.8) admits the unique solution

$$\psi = (I - K_\lambda A)^{-1} K \gamma_- (\lambda - T_0)^{-1} g$$

which we put in (3.7) to obtain (3.5). In order to achieve the proof, it suffices to show the positivity of the operator $(\lambda - T_K)^{-1}$.

Let $g \in (L^1(\Omega))_+$ and note that the positivity of the operator K implies that of the operator K_λ. By (3.5) and the second item of Lemma 2.2 we obtain

$$(\lambda - T_K)^{-1} g(x,v) \geqslant e^{-\lambda t(x,v)} (I - K_\lambda A)^{-1} K \gamma_+ (\lambda - T_0)^{-1} g(x - t(x,v)v, v)$$

for almost all $(x,v) \in \Omega$. Thanks to (3.9) we have

$$(I - K_\lambda A)^{-1} K = (\sum_{n=0}^{\infty}; (K_\lambda A)^n) K \geqslant IK = K$$

therefore,

$$(\lambda - T_K)^{-1} g(x,v) \geqslant e^{-\lambda t(x,v)} K \gamma_+ (\lambda - T_0)^{-1} g(x - t(x,v)v, v)$$

for almost all $(x,v) \in \Omega$. Finally, the positivity of K and the second item of Lemma 2.2 clearly imply the positivity of $(\lambda - T_K)^{-1} g$. The proof is now achieved. $\square$

Now, we are ready to state the main result of this work.

Theorem 3.3. *Let K be a boundary operator with $\|K\| > 1$, and suppose that* (H1)–(H2) *hold. Then, the transport operator T_K given by* (2.2) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(U_K(t))_{t \geqslant 0}$ satisfying*

$$\|U_K(t) g\|_1 \leqslant e^{\omega_0 (T_{\max} + t)} \|g\|_1 \quad t \geqslant 0, \tag{3.10}$$

for all $g \in L^1(\Omega)$, where, $T_{\max}$ and ω_0 are given by (1.4) *and* (3.4)*. Furthermore, if K is a positive operator, $(U_K(t))_{t \geqslant 0}$ is positive too.*

Proof. First, let us define on $L^1(\Omega)$ the norm

$$|||g|||_1 = \int_\Omega |g(x,v)| h(x,v) \, dx \, d\mu(v) \tag{3.11}$$

where, $h(x,v) = e^{\omega_0 t(x,v)}$. By (H2), (1.4) holds; therefore, the norms (2.1) and (3.11) are equivalent because

$$\|g\|_1 \leqslant |||g|||_1 \leqslant e^{\omega_0 T_{\max}} \|g\|_1 \tag{3.12}$$

for all $g \in L^1(\Omega)$.

Next, let $\lambda > \omega_0$ and $g \in L^1(\Omega)$. Thanks to Lemma 3.2 we obtain that

$$\varphi = (\lambda - T_K)^{-1} g \in D(T_K) \tag{3.13}$$

is the unique solution of $\lambda\varphi = T_K \varphi + g$. Therefore φ satisfies

$$\lambda\varphi = -v \cdot \nabla_x \varphi + g, \tag{3.14}$$

$$\gamma_- \varphi = K \gamma_- \varphi. \tag{3.15}$$

Multiplying (3.14) by $(\operatorname{sgn} \varphi) h$ and integrating it over Ω,

$$\begin{aligned} \lambda |||\varphi|||_1 &= -\int_\Omega v \cdot \nabla_x (|\varphi|) h(x,v) \, dx \, d\mu(v) + \int_\Omega ((\operatorname{sgn} \varphi) h g)(x,v) \, dx \, d\mu(v) \\ &:= I + J. \end{aligned} \tag{3.16}$$

Integrating by parts,

$$\begin{aligned} I &= -\int_\Omega v\cdot\nabla_x(|h\varphi|)(x,v)\,dx\,d\mu(v) + \omega_0\int_\Omega |(h\varphi)(x,v)|\,dx\,d\mu(v) \\ &= \int_{\Gamma_-} |\gamma_-(h\varphi)(x,v)|d\xi - \int_{\Gamma_+} |\gamma_+(h\varphi)(x,v)|d\xi + \omega_0|||\varphi|||_1 \\ &= \int_{\Gamma_-} |\gamma_-\varphi(x,v)|d\xi - \int_{\Gamma_+} |\gamma_+(h\varphi)(x,v)|d\xi + \omega_0|||\varphi|||_1. \end{aligned}$$

By (3.15) and the fact that $(\gamma_+ h)\alpha_{\omega_0} = 1$, we obtain

$$\begin{aligned} I &= \int_{\Gamma_-} |K\gamma_+\varphi(x,v)|d\xi - \int_{\Gamma_+} |\gamma_+(h\varphi)(x,v)|d\xi + \omega_0|||\varphi|||_1 \\ &= \int_{\Gamma_-} |K(\alpha_{\omega_0}\gamma_+(h\varphi))(x,v)|d\xi - \int_{\Gamma_+} |\gamma_+(h\varphi)(x,v)|d\xi + \omega_0|||\varphi|||_1 \\ &= \int_{\Gamma_-} |K_{\omega_0}(\gamma_+(h\varphi))(x,v)|d\xi - \int_{\Gamma_+} |\gamma_+(h\varphi)(x,v)|d\xi + \omega_0|||\varphi|||_1 \\ &\leqslant (\|K_{\omega_0}\| - 1)\|\gamma_+(h\varphi)\|_{L^1(\Gamma_+)} + \omega_0|||\varphi|||_1; \end{aligned}$$

therefore

$$I \leqslant \omega_0|||\varphi|||_1 \tag{3.17}$$

because of (3.3). For the term J, we obviously have

$$J = \int_\Omega ((\operatorname{sgn}\varphi)hg)(x,v)\,dx\,d\mu(v) \leqslant |||g|||_1. \tag{3.18}$$

Putting now (3.13), (3.17) and (3.18) in (3.16) we obtain

$$|||(\lambda - T_K)^{-1}g|||_1 \leqslant \frac{|||g|||_1}{(\lambda-\omega_0)}.$$

Thanks to Hille-Yosida's theorem, the operator T_K generates on $L^1(\Omega)$ a strongly continuous semigroup $(U_K(t))_{t\geqslant 0}$ satisfying

$$|||U_K(t)g|||_1 \leqslant e^{t\omega_0}|||g|||_1 \quad t \geqslant 0 \tag{3.19}$$

for all $g \in L^1(\Omega)$. Now (3.10) follows from (3.12) and (3.19). In order to achieve the proof, it suffices to show the positivity of the semigroup $(U_K(t))_{t\geqslant 0}$.

Let $g \in (L^1(X\times V))_+$ and $t > 0$. Lemma 3.2 leads to

$$(\frac{n}{t} - T_K)^{-1}g \geqslant 0$$

for n large enough. Now, the exponential formula

$$U_K(t)g = \lim_{n\to\infty}[\frac{n}{t}(\frac{n}{t} - T_K)^{-1}]^n g \geqslant 0$$

achieves the proof. □

We finish this section by giving an example of a boundary operator K satisfying our hypothesis (H1).

Lemma 3.4. *Let $K \in \mathcal{L}(L^1(\Gamma_+), L^1(\Gamma_-))$ be a Maxwell boundary operator; i.e., $K = C + B$ where,*

$$C\psi(x,v) = \int_{\Gamma_+} k(x,v,x',v')\psi(x',v')|v'\cdot n(x')|d\gamma(x')d\mu(v') \quad (x,v)\in\Gamma_-$$

with $k \geqslant 0$ *and* $B \in \mathcal{L}(L^1(\Gamma_+), L^1(\Gamma_-))$ *is a given operator such that* $\|B\| < 1$. *If*

$$\limsup_{\varepsilon\to 0} \operatorname*{ess\,sup}_{\{\tau(y,v')\leqslant\varepsilon\}} \int_{\Gamma_-} k(x,v,y,v')|v\cdot n(x)|d\gamma(x)d\mu(v') < 1 - \|B\|,$$

then the hypothesis (H1) holds.

Proof. It is clear that there exists $\varepsilon_0 > 0$ such that

$$\|C\chi_{\varepsilon_0}\| = \operatorname*{ess\,sup}_{\{\tau(y,v')\leqslant\varepsilon_0\}} \int_{\Gamma_-} k(x,v,y,v')|v\cdot n(x)|d\gamma(x)d\mu(v') < 1 - \|B\|;$$

therefore

$$\|K\chi_{\varepsilon_0}\| \leqslant \|C\chi_{\varepsilon_0}\| + \|B\chi_{\varepsilon_0}\| < 1 - \|B\| + \|B\| = 1.$$

The proof is achieved. $\square$

4. Remarks and Comments

As we pointed in the introduction, this section deals with some remarks and comments on [8], using our notation.

Remark 4.1. In [8, page 288, line 14], the authors claim that the traces $\psi|_{\Gamma_\pm}$ $(= \gamma_\pm\psi)$ of ψ in

$$\mathcal{W}^1(\Omega) = \big\{\varphi \in L^1(\Omega), v\cdot\nabla_x\varphi \in L^1(\Omega)\big\},$$

are well defined and belong to $L^1(\Gamma_\pm)$. According to our Lemma 2.1, this claim is incorrect.

Remark 4.2. Note that [8, Theorem 5.2] is incorrect. Indeed, the authors consider positive boundary operators K satisfying

($\overline{\mathrm{H1}}$)

$$\lim_{\varepsilon\to 0} \|K\chi_\varepsilon\|_{\mathcal{L}(L^1(\Gamma_+),L^1(\Gamma_-))} < 1$$

where the characteristic operator χ_{ε_0} is given by (1.3), and

($\overline{\mathrm{H2}}$)

$$\|K\psi\|_{L^1(\Gamma_-)} \geqslant \|\psi\|_{L^1(\Gamma_+)}$$

for all $\psi \in (L^1(\Gamma_+))_+$.

According to ($\overline{\mathrm{H1}}$)–($\overline{\mathrm{H2}}$), the authors claim that the operator T_K defined by

$$T_K\varphi = -v\cdot\nabla_x\varphi \quad \text{on the domain}$$

$$D(T_K) = \{\varphi \in \mathcal{W}^1(\Omega),\ \gamma_-\varphi = K\gamma_+\varphi\}$$

generates, on $L^1(\Omega)$, a strongly continuous semigroup.

However, by ($\overline{\mathrm{H1}}$), there exist $\varepsilon_0 > 0$ and

$$0 < \alpha < 1 \tag{4.1}$$

such that

$$\|K\chi_{\varepsilon_0}\|_{\mathcal{L}(L^1(\Gamma_+),L^1(\Gamma_-))} < \alpha \tag{4.2}$$

which implies that

$$\|K\chi_{\varepsilon_0}\psi\|_{L^1(\Gamma_-)} < \alpha\|\psi\|_{L^1(\Gamma_+)} \tag{4.3}$$

for all $\psi \in L^1(\Gamma_+)$. Next, let us consider $\psi \in (L^1(\Gamma_+))_+$ such that $\chi_{\varepsilon_0}\psi \neq 0$. Now, clearly the fact that $\chi_{\varepsilon_0}^2 = \chi_{\varepsilon_0}$ together with the hypothesis ($\overline{\mathrm{H}}_2$) and (4.3) lead us to

$$\|\chi_{\varepsilon_0}\psi\| = \|\chi_{\varepsilon_0}^2\psi\| \leqslant \|K(\chi_{\varepsilon_0}^2\psi)\| = \|K\chi_{\varepsilon_0}(\chi_{\varepsilon_0}\psi)\| < \alpha\|\chi_{\varepsilon_0}\psi\|$$

whence $1 < \alpha$ which contradicts (4.1). Therefore, there are no positive boundary operators K satisfying simultaneously ($\overline{\text{H1}}$) and ($\overline{\text{H2}}$).

Finally, note that (4.1) and (4.2) imply

$$\|K\chi_{\varepsilon_0}\|_{\mathcal{L}(L^1(\Gamma_+),L^1(\Gamma_-))} < 1,$$

and our hypothesis (H1) holds.

Acknowledgement. The author wants to thank the professors M. Babin and J. Pequiniot and all their colleagues and students of the Marais Poitevin's institute for their help. This research was supported by a grant from LMCM-RMI.

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