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SIMULTANEOUS EXACT CONTROLLABILITY FOR MAXWELL EQUATIONS AND FOR A SECOND-ORDER HYPERBOLIC SYSTEM

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Abstract. We present a result on "simultaneous" exact controllability for two models that describe two hyperbolic dynamics. One is the system of Maxwell equations and the other a vector-wave equation with a pressure term. We obtain the main result using modified multipliers in order to generate a necessary observability estimate which allow us to use the Hilbert Uniqueness Method (HUM) introduced by Lions.

1. Introduction

We consider two models which describe two distinct hyperbolic dynamics: One of them is the system of Maxwell equations and the other is a vector wave equation with a pressure term. In the solution (and vector-valued) variables $\{E, H, u\}$ satisfy

$$\left.\begin{aligned} \mathcal{E}E_t - \operatorname{curl} H = 0 \\ \mu H_t + \operatorname{curl} E = 0 \\ \operatorname{div} E = 0, \quad \operatorname{div} H = 0 \end{aligned}\right\} \text{ in } \Omega \times (0,T) \tag{1.1}$$

$$E(x,0) = E_0(x), \quad H(x,0) = H_0(x) \quad \text{in } \Omega \tag{1.2}$$

$$\eta \times E = Q(x,t) \quad \text{on } \partial\Omega \times (0,T) \tag{1.3}$$

and

$$\left.\begin{aligned} \rho u_{tt} - \alpha \Delta u + \operatorname{grad} p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \text{ in } \Omega \times (0,T) \tag{1.4}$$

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x) \text{ in } \Omega \tag{1.5}$$

$$u = P(x,t) \quad \text{ on } \partial\Omega \times (0,T). \tag{1.6}$$

In (1.1), $E = (E_1, E_2, E_3)$ and $H = (H_1, H_2, H_3)$ denote the electric and magnetic field respectively and $\mathcal{E}$ and μ are positive constants representing the permittivity and magnetic permeability respectively.

In (1.4), ρ denotes the scalar density which we will assume to be a positive constant and $p = p(x,t)$ is the pressure term (an scalar function). Also, α is a positive constant depending on the elastic properties of the material. In (1.3),

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$\eta = \eta(x)$ denotes the unit normal vector at $x \in \partial\Omega$ pointing the exterior of Ω. Finally, grad, curl, Δ, div and $\times$ denote the usual gradient, rotational, Laplace operator, divergent and vector product in $\mathbb{R}^3$.

Generally speaking the problem of exact controllability (for either problem (1.1)–(1.3) or (1.4)–(1.6)) can be state as follows: Given a time $T > 0$ and any initial data and desired terminal data, to find a corresponding control F driving the system to the terminal data at time T. One of the most usefull methods to solve such problems of controllability is the Hilbert Uniqueness Method (HUM) introduced by J.L. Lions in the middle 80's and is based on the construction of appropriate Hilbert space structures on the space of initial data. These Hilbert structures are connected with uniqueness properties.

Several authors considered the problem of exact controllability either for problem (1.1)–(1.3) (see for instance Russell [16], Lagnese [9], Eller and Masters [2], Eller [3], Kapitonov [5], Weck [17], Phung [14]) or problem (1.4)–(1.6) (see for instance Lions [12] and Rocha dos Santos [15]).

Clearly, the above results provide the existence of controls $Q(x,t)$ and $P(x,t)$ which are not necessarily related one to the other. In the middle 80's, Russell [16] and Lions [10] raised the question if it is possible to solve the exact controllability problem for two evolution models using only one control. They named this problem "simultaneous" exact controllability. In the absence of dissipative effects, due to technical difficulties "simultaneous" exact controllability were only treated for one system with two different boundary conditions (see for instance, Lions [10], Kapitonov [6], Kapitonov, Raupp [7] and Kapitonov, Perla Menzala [8]).

Now, let us formulate the "simultaneous" exact controllability for systems (1.1)–(1.3) and (1.4)–(1.6): Given initial the states (f_1, f_2, g_1, g_2) and the terminal data $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ in suitable function spaces we ask if it is possible to find one vector-valued function $P = P(x,t)$ such that the solution $\{E, H, u\}$ de (1.1)–(1.6) with Q in terms of P satisfies at terminal time T

$$(E(T), H(T), u(T), u_t(T)) = (\varphi_1, \varphi_2, \psi_1, \psi_2).$$

The main purpose of this work is to prove that this is indeed the case, P serving as a control function for (1.4)–(1.6) while the vector-valued function

$$Q = \mu\eta \times (\eta \times P_t)$$

is a control function for (1.1)–(1.3).

Let us briefly describe the sections in this paper: In Section 2 we briefly indicate the function space where the solutions of systems (1.1)–(1.3) and (1.4)–(1.6) will be considered. Then, we use modified multipliers in order to obtain a boundary observability inequality valid for system (1.1)–(1.6) with P and Q identically equal to zero (see inequalities (2.17), (2.18)). However, the right hand side of this inequality is not suitable to apply the steps of the Hilbert Uniqueness Method. That is why we assume a suitable (numerically) relation between important parameters of (1.1)–(1.6). The observability inequality is obtain assuming that the region is substar-shaped. In Section 3, the "simultaneous" exact controllability is studied by means of the Hilbert Uniqueness Method (HUM) introduced by Lions (see [10, 11]).

Systems (1.1)–(1.3) and (1.4)–(1.6) are not directly coupled to each other. A more interesting problem would be to study the case when they are coupled say with coupling terms $-\gamma \operatorname{curl} E$ and $\gamma \operatorname{curl} u_t$ (with $\gamma > 0$) for system (1.1)–(1.3) and (1.4)–(1.6) respectively. As far as we know this remains an open problem. Our techniques seem to work when the coefficients $\mathcal{E}, \mu, \rho$ and α are functions of x which are smooth and they and their partial derivatives of first order are bounded below by strictly positive constants.

We will use standard notations which can be found in Duvaut and Lions' book [1]. For example, $H^m(\Omega)$ and $H^r(\partial\Omega)$ will denote the Sobolev spaces of order m and r on Ω and $\partial\Omega$ respectively. Given a real-valued function g the notation $\int_{\partial\Omega} g d\Gamma$ means the surface integral of g over the surface $\partial\Omega$. If X is a vector space, then $[X]^m$ means $X \times X \times X \times \cdots \times X$ m-times. For any vector $v \in \mathbb{R}^3$, $|v|$ denotes the usual norm of v in $\mathbb{R}^3$.

2. A Boundary Observability Inequality

We consider suitable function spaces where the solutions $\{E, H, u, u_t\}$ of problem (1.1)–(1.6) (with P and Q identically equal to zero) will be considered. Let Ω be a bounded region of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and $\mathcal{E} > 0$, $\mu > 0$ as indicated in Section 1. We consider the Hilbert space $\mathcal{H}$ defined as follows

$$\mathcal{H} = [L^2(\Omega)]^3 \times [L^2(\Omega)]^3$$

with inner product given by

$$\langle v, w\rangle_{\mathcal{H}} = \int_\Omega [\mathcal{E} v_1 \cdot w_1 + \mu v_2 \cdot w_2] dx$$

for any $v = (v_1, v_2)$, $w = (w_1, w_2)$ in $\mathcal{H}$. Here the central dot $\cdot$ means the usual inner product in $\mathbb{R}^3$. We also consider the Hilbert space

$$H(\mathrm{curl}, \Omega) = \{\mathrm{w} \in [\mathrm{L}^2(\Omega)]^3; \mathrm{curl}\,\mathrm{w} \in [\mathrm{L}^2(\Omega)]^3\}$$

with inner product

$$\langle v_1, v_2\rangle_{H(\mathrm{curl},\Omega)} = \int_\Omega [v_1 \cdot v_2 + \mathrm{curl}\, v_1 \cdot \mathrm{curl}\, v_2] dx$$

for any $v_1, v_2 \in H(\mathrm{curl}, \Omega)$. It is well known (see [1]) that the map $Z \mapsto \eta \times Z|_{\partial\Omega}$ from $[C_0^1(\overline{\Omega})]^3$ into $[C^1(\partial\Omega)]^3$ extends by continuity to a continuous linear map from $H(\mathrm{curl}, \Omega)$ into $[H^{-1/2}(\partial\Omega)]^3$. This result allow us to consider the space

$$H_0(\mathrm{curl}, \Omega) = \{\mathrm{w} \in \mathrm{H}(\mathrm{curl}, \Omega), \quad \eta \times \mathrm{w} = 0 \text{ on } \partial\Omega\}.$$

Here $\eta = \eta(x)$ denotes the unit normal vector at $x \in \partial\Omega$ pointing the exterior of Ω. We define the operator $A \colon \mathcal{D}(A) \subseteq \mathcal{H} \mapsto \mathcal{H}$ with domain $\mathcal{D}(A)$ given by

$$\mathcal{D}(A) = H_0(\mathrm{curl}, \Omega) \times \mathrm{H}(\mathrm{curl}, \Omega)$$

and A is defined as follows

$$A(v_1, v_2) = (\mathcal{E}^{-1}\,\mathrm{curl}\, v_2, -\mu^{-1}\,\mathrm{curl}\, v_1)$$

The skew-selfadjointness of A can be easily verified. By Stone's Theorem, the operator A generates a one-parameter group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}$. Observe that in order $U(t)f$ to solve problem (1.1)–(1.3) with $Q \equiv 0$ and given $f \in \mathcal{D}(A)$ remains to prove that the components of $U(t)f$ are divergent free. In order to overcome this issue we consider $M = \{(\mathrm{grad}\,\varphi_1, \mathrm{grad}\,\varphi_2)$ with $\varphi_1, \varphi_2 \in C_0^\infty(\Omega)\}$ and $M_1 = M^\perp$. We observe that M is not closed in $\mathcal{H}$ but M_1 is closed in $\mathcal{H}$. In the distributional sense it is easy to prove that whenever $w = (w_1, w_2) \in M_1$ then $\mathrm{div}\, w_1 = 0$ and $\mathrm{div}\, w_2 = 0$. Furthermore, we claim that

$U(t)$ takes $M_1 \cap \mathcal{D}(A)$ into itself. Indeed, for any w in M and v in $M_1 \cap \mathcal{D}(A)$ we have

$$\frac{d}{dt}\langle U(t)v, w\rangle_{\mathcal{H}} = \langle AU(t)v, w\rangle_{\mathcal{H}} = \langle U(t)v, A^* w\rangle_{\mathcal{H}} = 0 \quad \text{for any } t \in \mathbb{R}.$$

Thus

$$\langle U(t)v, w\rangle_{\mathcal{H}} = \text{constant} \quad \text{for any } t \in \mathbb{R}.$$

Therefore, taking $t = 0$ we have $\langle v, w\rangle = 0$. Consequently $U(t)v \in M_1 \cap \mathcal{D}(A)$ for any $t \in \mathbb{R}$ which proves our claim. Observe that for any element $v = (v_1, v_2)$ belonging to $M_1 \cap \mathcal{D}(A)$ and $w = (0, \operatorname{grad}\varphi_2)$ with $\varphi_2 \in H^2(\Omega)$ we have

$$0 = \langle v, w\rangle_{\mathcal{H}} = \int_\Omega \mu v_2 \cdot \operatorname{grad}\varphi_2 dx = \mu \int_{\partial\Omega} \varphi_2 v_2 \cdot \eta d\Gamma.$$

Since $\varphi_2 \in H^2(\Omega)$ is arbitrary, the above identify say that

$$\eta \cdot v_2 = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

We conclude that problem (1.1)–(1.3) with $Q \equiv 0$ has a generator A_1 which applies $M_1 \cap \mathcal{D}(A)$ into $M_1 \cap \mathcal{D}(A)$ and for any $w = (w_1, w_2) \in M_1 \cap \mathcal{D}(A)$ the relation $\eta \cdot w_2 = 0$ on $\partial\Omega$ holds.

Next, we consider problem (1.3)–(1.6) with $P \equiv 0$. We can use Galerkin's method to find u and p (defined up to a constant). We choose the spaces

$$V = \{\varphi \in [C_0^\infty(\Omega)]^3, \operatorname{div}\varphi = 0 \text{ in } \Omega\}.$$

Let Y be the closure of V with respect to the norm of $[H_0^1(\Omega)]^3$ and

$$W = Y \cap [H^2(\Omega)]^3.$$

Considering $u_0 \in W$, $u_1 \in Y$ we obtain by using the Galerkin method a unique solution of problem (1.3)–(1.6) such that $u \in C([0, +\infty); V) \cap C^1([0, +\infty); H)$ where H denotes the closure of V with respect to the norm of $[L^2(\Omega)]^3$. We can also obtain more regularity. For example, if $u_0 \in V \cap [H^4(\Omega)]^3$ and $u_1 \in W$ then the solution $u \in C([0, \infty); W) \cap C^1([0, +\infty); V)$ with $p \in H^2(\Omega)$.

Let us now concern ourselves with the simultaneous boundary observability problem. We use the multiplier method. They are conveniently modified in such a way that we can handle the extra boundary terms appearing in the identities. Let $h(x)$ be an auxiliary scalar smooth function on $\overline{\Omega}$, which we will choose later on. For problem (1.1) we consider the multipliers

$$M_1 = tE + \mu \operatorname{grad} h \times H,$$
$$M_2 = tH - \mathcal{E} \operatorname{grad} h \times E$$

Since $\{E, H\}$ solves (1.1)–(1.2) then we have the identity

$$\begin{aligned} 0 &= 2M_1 \cdot (\mathcal{E} E_t - \operatorname{curl} H) + 2M_2 \cdot (\mu H_t + \operatorname{curl} E) \\ &\quad + 2\mathcal{E}(\operatorname{grad} h \cdot E) \operatorname{div} E + 2\mu(\operatorname{grad} h \cdot H) \operatorname{div} H. \end{aligned} \tag{2.2}$$

We can rearrange the terms on the right hand of (2.2) to obtain

$$\frac{\partial A}{\partial t} = \operatorname{div}(\vec{B}) + D \tag{2.3}$$

where

$$A = A(x,t) = t(\mathcal{E}|E|^2 + \mu|H|^2) + 2\mathcal{E}\mu \operatorname{grad} h \cdot (H \times E) \tag{2.4}$$

$$\begin{aligned}\vec{B} = \vec{B}(x,t) = 2tH \times E + \operatorname{grad} h(\mathcal{E}|E|^2 + \mu|H|^2)\\ - 2\mathcal{E}E(E \cdot \operatorname{grad} h) - 2\mu H(H \cdot \operatorname{grad} h),\end{aligned} \tag{2.5}$$

$$D = 2\sum_{i,j=1}^{3} \frac{\partial^2 h}{\partial x_i \partial x_j}(\mathcal{E}E_iE_j + \mu H_iH_j) - (\Delta h - 1)(\mathcal{E}|E|^2 + \mu|H|^2), \tag{2.6}$$

where

$$|E|^2 = \sum_{j=1}^{3} E_j^2, \quad |H|^2 = \sum_{j=1}^{3} H_j^2.$$

Similarly, for problem (1.4) we consider the multipliers

$$M_3 = tu_t + (\operatorname{grad} h \cdot \operatorname{grad})u + u,$$
$$M_4 = tp\frac{\partial}{\partial t} + p(\operatorname{grad} h \cdot \operatorname{grad}) + p.$$

Here $\operatorname{grad} h \cdot \operatorname{grad} = \sum_{j=1}^{3} \frac{\partial h}{\partial x_j}\frac{\partial}{\partial x_j}$. Observe that M_4 is actually an operator. Since $\{u,p\}$ is a solution of (1.4) then we have the identity

$$0 = 2M_3 \cdot (\rho u_{tt} - \alpha\Delta u + \operatorname{grad} p) + 2M_4 \operatorname{div} u. \tag{2.7}$$

We can rearrange terms in identity (2.7) to obtain

$$\frac{\partial A_1}{\partial t} = \operatorname{div} \vec{G} + \operatorname{div} \vec{F} + D_1 \tag{2.8}$$

where

$$A_1 = t\Big(\rho|u_t|^2 + \alpha\sum_{i=1}^{3}\big|\frac{\partial u}{\partial x_i}\big|^2\Big) + 2\rho[u_t \cdot (\operatorname{grad} h \cdot \operatorname{grad})u + u],$$
$$\vec{G} = (G_1, G_2, G_3)$$

with

$$G_i = 2[tu_t + (\operatorname{grad} h \cdot \operatorname{grad})u + u] \cdot \alpha\frac{\partial u}{\partial x_i} + \frac{\partial h}{\partial x_i}\Big(\rho|u_t|^2 - \alpha\sum_{j=1}^{3}\big|\frac{\partial u}{\partial x_j}\big|^2\Big), \tag{2.9}$$
$$\vec{F} = -2p[tu_t + (\operatorname{grad} h \cdot \operatorname{grad})u + u]$$

and

$$\begin{aligned}D_1 = (3 - \Delta h)\rho|u_t|^2 + (\Delta h - 1)\alpha\sum_{j=1}^{3}\big|\frac{\partial u}{\partial x_j}\big|^2\\ - 2\sum_{i,q=1}^{3}\frac{\partial^2 h}{\partial x_i \partial x_q}\alpha\Big(\frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_q}\Big) + 2p\sum_{i,j=1}^{3}\frac{\partial^2 h}{\partial x_i \partial x_j}\frac{\partial u_j}{\partial x_i}.\end{aligned}$$

Remark 2.1. If we choose $h(x) = \frac{1}{2}|x - x_0|^2$ for some $x_0 \in \mathbb{R}^3$ then we can verify that D and D_1 in (2.3) and (2.8) are identically zero. Therefore in the case (2.3) and (2.8) represent a conservation law.

Let $(E_0, H_0) \in M_1 \cap \mathcal{D}(A)$ and (E, H) be the corresponding solution of problem (1.1)–(1.3) with $Q \equiv 0$. Integration in $\Omega \times (0,T)$ of identity (2.3) give us

$$\begin{aligned} &T\int_\Omega \{\mathcal{E}|E|^2 + \mu|H|^2\}dx + 2\mathcal{E}\mu \int_\Omega \operatorname{grad} h \cdot (H \times E)dx\Big|_{t=0}^{t=T} \\ &= \int_0^T \int_{\partial\Omega} \beta(E,H,h)d\Gamma dt + \int_0^T \int_\Omega D\, dxds \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \beta(E,H,h) &= 2t\eta\cdot(H\times E) + \frac{\partial h}{\partial \eta}(\mathcal{E}|E|^2 + \mu|H|^2) \\ &\quad - 2\mathcal{E}(E\cdot\eta)(E\cdot \operatorname{grad} h) - 2\mu(H\cdot\eta)(H\cdot\operatorname{grad} h). \end{aligned} \tag{2.11}$$

and D is given as in (2.6). Using the boundary conditions, (1.3) (with $Q \equiv 0$) and the vector identities on $\partial\Omega$ we get

$$\eta\cdot(H\times E) = -(\eta\times E)\cdot H = 0, \quad |E|^2 = (E\cdot\eta)^2 + |E\times\eta|^2 = (E\cdot\eta)^2$$
$$E = \eta\times(\eta\times E) + \eta(E\cdot\eta) = \eta(E\cdot\eta), \quad |H|^2 = (H\cdot\eta)^2 + |H\times\eta|^2 = |H\times\eta|^2$$

because $|\eta| = 1$ and (2.1) holds for H. Thus we obtain from (2.11)

$$\beta(E,H,h) = \frac{\partial h}{\partial\eta}\{\mu|H\times\eta|^2 - \mathcal{E}(E\cdot\eta)^2\}. \tag{2.12}$$

Now, we want to get appropriate estimates for the term $\int_0^T\int_\Omega D\, dxdt$ in (2.10). Let us choose a convenient function $h(x)$. We consider the elliptic problem

$$\Delta\Phi = 1 \quad \text{in } \Omega$$
$$\frac{\partial\Phi}{\partial\eta} = \frac{\operatorname{vol}(\Omega)}{\operatorname{area}(\partial\Omega)} \quad \text{on } \partial\Omega$$

which admits a solution $\Phi \in C^2(\Omega)\cap C^1(\overline{\Omega})$. Here area$(\partial\Omega)$ means the surface area of $S = \partial\Omega$. Let $0 < \delta < 1$ and $x_0 \in \mathbb{R}^3$ we define

$$h(x) = \delta\Phi(x) + \frac{1}{2}|x - x_0|^2. \tag{2.13}$$

substitution of such $h(x)$ into (2.6) give us

$$D = 2\delta\sum_{i,j=1}^3 \frac{\partial^2\Phi}{\partial x_i\partial x_j}(\mathcal{E}E_iE_j + \mu H_iH_j) - \delta(\mathcal{E}|E|^2 + \mu|H|^2).$$

Let $C_1 = C_1(\Phi)$ be the constant given by

$$C_1 = \max_{x\in\overline{\Omega},\, i,j=1,2,3} \Big|\frac{\partial^2\Phi(x)}{\partial x_i\partial x_j}\Big|.$$

We can easily verified that $C_1 \geq 1/3$ and

$$|D| \leq \delta C_2(\mathcal{E}|E|^2 + \mu|H|^2) \tag{2.14}$$

where $C_2 = 6C_1 - 1 > 0$ and $0 < \delta < 1$. Since the quantity $\frac{1}{2}\int_\Omega\{\mathcal{E}|E|^2 + \mu|H|^2\}dx$ is constant for any $t \in \mathbb{R}$ for the solution (E,H) of (1.1)–(1.3) (with $Q \equiv 0$) then, from (2.14) it follows that

$$\int_0^T\int_\Omega D\, dxdt \leq \delta C_2 T\int_\Omega\{\mathcal{E}|E|^2 + \mu|H|^2\}dx. \tag{2.15}$$

Now, we want to estimate the term $2\mathcal{E}\mu\int_\Omega \operatorname{grad} h\cdot(H\times E)dx\Big|_{t=0}^{t=T}$ in (2.10). Let $C_3>0$ given by

$$C_3=\max_{x\in\overline{\Omega}}\{|\operatorname{grad}\Phi(x)|+|x-x_0|\}$$

Since $|H\times E|\le 2|E||H|$ and h as in (2.13) we deduce

$$\begin{aligned}2\int_\Omega \mathcal{E}\mu\operatorname{grad} h\cdot(H\times E)dx &\le 4(1+\delta)C_3\sqrt{\mathcal{E}\mu}\int_\Omega\sqrt{\mathcal{E}\mu}|E||H|dx\\ &\le 4(1+\delta)C_3\sqrt{\mathcal{E}\mu}\int_\Omega(\mathcal{E}|E|^2+\mu|H|^2)dx\end{aligned}\tag{2.16}$$

for any $0\le t\le T$. Thus, from identity (2.10) and inequalities (2.12), (2.15) and (2.16) we get

$$\begin{aligned}&(1-\delta C_2)(T-T_0)\int_\Omega\{\mathcal{E}|E|^2+\mu|H|^2\}dx\\ &\le\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\{\mu|H\times\eta|^2-\mathcal{E}(E\cdot\eta)^2\}d\Gamma\end{aligned}\tag{2.17}$$

where $T_0=4(1+\delta)C_3\sqrt{\mathcal{E}\mu}\Big/1-\delta C_2$. Similarly, using identity (2.8) and assuming some geometric condition on the region Ω (say for instance, "substar" like [4, 8]) we can prove that the solution of problem (1.4)–(1.6) (with $P\equiv 0$) satisfies the inequality

$$\begin{aligned}&(1-\delta\tilde{C}_2)(T-\tilde{T}_0)\int_\Omega\Big\{\rho|u_t|^2+\alpha\sum_{i=1}^3\big|\frac{\partial u}{\partial x_i}\big|^2\Big\}dx\\ &\le\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\alpha\big|\eta\times\frac{\partial u}{\partial\eta}\big|^2d\Gamma dt\end{aligned}\tag{2.18}$$

for some $\tilde{C}_2>0$ and $\tilde{T}_0>0$.

Remark 2.2. By choosing $\delta>0$ sufficiently small and adding inequalities (2.17) and (2.18) we would obtain a boundary observability provided $\frac{\partial h}{\partial\eta}\ge 0$ on $\partial\Omega$. However, in order to apply the techniques to use the HUM would not help that much.

We want to prove the following observability inequality.

Theorem 2.3. *Let $\{E,H,u,u_t\}$ be the solution of (1.1)–(1.6) with $P=Q=0$ on $\partial\Omega$. Suppose there exist $\delta_1>0$ and $x_0\in\mathbb{R}^3$ ($\delta_1<\min\{C_2^{-1},\tilde{C}_2^{-1}\}$) where C_2 and $\tilde{C}_2$ are as in (2.17) and (2.18) such that*

$$\delta_1\frac{\operatorname{vol}(\Omega)}{\operatorname{area}(\partial\Omega)}+(x-x_0)\cdot\eta>0\quad\text{for all } x\in\partial\Omega\tag{2.19}$$

and the parameters in (1.1) and (1.4) satisfy (numerically) the relation $\rho=\mathcal{E}\mu\alpha$. Then, there exist constants C_5,C_6 and a $T_1>0$ such that

$$\begin{aligned}&(2-\delta_1C_6)(T-T_1)\int_\Omega\Big\{\mathcal{E}|E|^3+\mu|H|^2+\rho|u_t|^2+\alpha\sum_{i=1}^3\big|\frac{\partial u}{\partial x_i}\big|^2\Big\}\\ &\le\int_0^T\int_{\partial\Omega}\Big\{\frac{C_5}{2}\big|\mu H+\alpha\frac{\partial u}{\partial\eta}\big|^2-\frac{\partial h}{\partial\eta}\mathcal{E}(E\cdot\eta)^2\Big\}d\Gamma dt\end{aligned}$$

Proof. In (2.13) we choose $\delta = \delta_1$. Observe that (2.19) tell us that $\frac{\partial h}{\partial \eta} > 0$ for all $x \in \partial\Omega$. we can easily verify the identity

$$\begin{aligned}
&\mu H \cdot (\rho u_{tt} - \alpha \Delta u + \operatorname{grad} p) + \rho \mathcal{E}^{-1} \operatorname{curl} u \cdot (\mathcal{E} E_t - \operatorname{curl} H) \\
&+ \rho u_t \cdot (\mu H_t + \operatorname{curl} E) + (\mu p - \alpha\mu \operatorname{div} u) \operatorname{div} H \\
&+ (\rho \mathcal{E}^{-1} - \alpha\mu) \operatorname{curl} u \cdot \operatorname{curl} H \\
&= \frac{\partial}{\partial t} A_2 - \operatorname{div} \vec{B}_2.
\end{aligned} \tag{2.20}$$

where

$$A_2 = \rho u_t \cdot \mathcal{E} H + \rho \operatorname{curl} u \cdot E,$$
$$\vec{B}_2 = \rho u_t \times E + \alpha\mu(\operatorname{div} u) H + \alpha\mu H \times \operatorname{curl} u - \mu p H.$$

From (2.20) it follows that $\frac{\partial}{\partial t} A_2 = \operatorname{div} B_2$. Integration over $\Omega \times (0,T)$ give us

$$\int_\Omega \{\rho u_t \cdot \mathcal{E} H + \rho \operatorname{curl} u \cdot E\} dx \Big|_{t=0}^{t=T} = -\alpha\mu \int_0^T \int_{\partial\Omega} (H \times \eta) \cdot \operatorname{curl} u \, d\Gamma dt. \tag{2.21}$$

We use the identity

$$|\mu(H \times \eta) - \alpha \operatorname{curl} u|^2 = \mu^2 |H \times \eta|^2 - 2\alpha\mu (H \times \eta) \cdot \operatorname{curl} u + \alpha^2 |\operatorname{curl} u|^2.$$

Substitution into (2.21) give us

$$\begin{aligned}
&\int_\Omega \{\rho u_t \cdot \mathcal{E} H + \rho \operatorname{curl} u \cdot E\} dx \Big|_{t=0}^{t=T} \\
&= \int_0^T \int_{\partial\Omega} \Big\{ \frac{1}{2} |\mu(H \times \eta) - \alpha \operatorname{curl} u|^2 - \frac{1}{2} \mu^2 |H \times \eta|^2 - \frac{\alpha^2}{2} |\operatorname{curl} u|^2 \Big\} d\Gamma dt \\
&= \int_0^T \int_{\partial\Omega} \Big\{ \frac{1}{2} |\mu(H \times \eta) - \alpha \operatorname{curl} u|^2 - \frac{1}{2} \mu^2 |H \times \eta|^2 - \frac{\alpha^2}{2} \big| \frac{\partial u}{\partial \eta} \times \eta \big|^2 \Big\} d\Gamma dt
\end{aligned} \tag{2.22}$$

because $u = 0$ on $\partial\Omega \times (0,T)$ (and $u \in [H^2(\Omega) \cap H_0^1(\Omega)]^3$ then $\frac{\partial u_i}{\partial x_j} = \eta_j \frac{\partial u_i}{\partial \eta}$; therefore $\operatorname{curl} u = \eta \times \frac{\partial u}{\partial \eta}$ on $\partial\Omega \times (0;T)$. Observe that

$$|\operatorname{curl} u|^2 \le 2 \sum_{i,j=1}^3 \big(\frac{\partial u_i}{\partial x_j} \big)^2 = 2 \sum_{j=1}^3 \big| \frac{\partial u}{\partial x_j} \big|^2. \tag{2.23}$$

Using (2.23) we can also obtain the inequality

$$\begin{aligned}
&\Big| \int_\Omega \{\rho u_t \cdot \mathcal{E} H + \rho \operatorname{curl} u \cdot E\} dx \Big| \\
&\le C_4 \int_\Omega \Big\{ \rho |u_t|^2 + \alpha \sum_{j=1}^3 \big| \frac{\partial u}{\partial x_j} \big|^2 + \mathcal{E} |E|^2 + \mu |H|^2 \Big\} dx
\end{aligned} \tag{2.24}$$

where $C_4 = \max\{(\rho/\mu)^{1/2} \mathcal{E}, 2\rho/\sqrt{\mathcal{E}}\}$. Consider

$$C_5 = 2 \max \big\{ \big\| \frac{\partial h}{\partial \eta} \big\|_{L^\infty(\partial\Omega)} \alpha^{-1}, \big\| \frac{\partial h}{\partial \eta} \big\|_{L^\infty(\partial\Omega)} \mu^{-1} \big\}.$$

We multiply (2.22) by C_5 and obtain from (2.17), (2.18), (2.22) and (2.24)

$$
\begin{aligned}
&(1-\delta_1 C_6)(T-T_1)\int_\Omega \Big\{\mathcal{E}|E|^2+\mu|H|^2+\rho|u_t|^2+\alpha\sum_{i=1}^3\big|\frac{\partial u}{\partial x_i}\big|^2\Big\}dx\\
&\le \int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\Big\{\mu|H\times\eta|^2-\mathcal{E}(E\cdot\eta)^2+\alpha\big|\eta\times\frac{\partial u}{\partial\eta}\big|^2\Big\}\\
&\quad+\frac{C_5}{2}|\mu(H\times\eta)-\alpha\operatorname{curl}u|^2-\frac{C_5}{2}\mu^2|H\times\eta|^2-\frac{\alpha^2}{2}C_5\big|\frac{\partial u}{\partial\eta}\times\eta\big|^2 d\Gamma dt\\
&\le \int_0^T\int_{\partial\Omega}\Big\{\frac{C_5}{2}|\mu(H\times\eta)-\alpha\operatorname{curl}u|^2-\frac{\partial h}{\partial\eta}\mathcal{E}(E\cdot\eta)^2\Big\}d\Gamma dt
\end{aligned}
$$

where $C_6=C_2+\tilde{C}_2$ and

$$
T_1=\frac{(T_0+\tilde{T}_0)-\delta_1(C_2T_0+\tilde{C}_2\tilde{T}_0)}{2-\delta_1C_6}>0
$$

We claim that the term $|\mu(H\times\eta)-\alpha\operatorname{curl}u|$ on the right hand side of (2.19) equals to $|\alpha\frac{\partial u}{\partial\eta}+\mu H|$ for any $(x,t)\in\partial\Omega\times(0,T)$ q.t.p. In fact, using the boundary conditions we have

$$
|\mu(H\times\eta)-\alpha\operatorname{curl}u|=\big|\mu(H\times\eta)+\alpha\big(\frac{\partial u}{\partial\eta}\times\eta\big)\big|.
$$

Using the identity $|v\times\eta|^2+(v\cdot\eta)^2=|v|^2$ valid for any vector of $v\in\mathbb{R}^3$ we obtain

$$
\big|\big(\mu H+\alpha\frac{\partial u}{\partial\eta}\big)\times\eta\big|^2+\Big[\big(\mu H+\alpha\frac{\partial u}{\partial\eta}\big)\cdot\eta\Big]^2=\big|\alpha\frac{\partial u}{\partial\eta}+\mu H\big|^2
$$

because $H\cdot\eta=0$ and $\frac{\partial u}{\partial\eta}\cdot\eta=0$ on $\partial\Omega\times(0,T)$. This proves our claim and the conclusion of Theorem 2.3. □

Corollary 2.4. *Let $\{E,H,u,u_t\}$ be the solution of* (1.1)-(1.6) *with zero boundary conditions and assume the conditions of Theorem 2.3. If the condition*

$$
\mu H+\alpha\frac{\partial u}{\partial\eta}=0\quad\text{on } \partial\Omega\times(0,T)
$$

holds, then for any $T>T_1$ we will have

$$
E(x,t)\equiv H(x,t)\equiv u(x,t)\equiv 0\quad\text{in } \Omega\times(0,T).
$$

3. Simultaneous exact controllability

Let $\{E,H,u,u_t\}$ be the solution of (1.1)–(1.6) with zero boundary conditions. In the function space of initial data (for strong solutions) we consider the Hilbert space $\mathcal{F}$ obtained by completing such space with respect to the norm

$$
\|(f,g)\|_{\mathcal{F}}=\Big(\int_0^T\int_{\partial\Omega}\big|\mu H+\alpha\frac{\partial u}{\partial\eta}\big|^2 d\Gamma dt\Big)^{1/2}
$$

for $T>T_1$ where $f=(f_1,f_2)$ and $g=(g_1,g_2)$ are the initial data of problems (1.1)–(1.3) and (1.4)–(1.6) respectively. From Corollary 2.4 it follows that $\|\cdot\|_{\mathcal{F}}$ is indeed a norm. Let us denote by $\|\cdot\|_K$ the energy norm

$$
\|(f,g)\|_K^2=\int_\Omega\Big\{\mathcal{E}|E|^2+\mu|H|^2+\rho|u_t|^2+\alpha\sum_{j=1}^3\big|\frac{\partial u}{\partial x_j}\big|^2\Big\}dx
$$

then, clearly we have $\mathcal{F} \subseteq K$ and $\|(f,g)\|_K \le C\|(f,g)\|_{\mathcal{F}}$ for some positive constant C. Let us consider the dual space of $\mathcal{F}$ with respect to K. We will denoted by $\mathcal{F}'$.

Definition. Given $R = R(x,t) \in [L^2(\partial\Omega \times (0,T))]^3$ and $(f_1, f_2, g_1, g_2) \in \mathcal{F}'$. We say that $\{E, H, u, u_t\}$ is a solution of the Maxwell/elasticity system if $\{E, H\}$ solves (1.1)–(1.2) with boundary condition.

$$\eta \times E = 0 = \mu\eta \times (\eta \times R) \quad \text{on } \partial\Omega \times (0,T) \tag{3.1}$$

and (u, u_t) solves (1.4)–(1.5) with boundary conditions $u_t = R$ on $\partial\Omega \times (0,T)$. Furthermore,

(a) $(E(\cdot,t), H(\cdot,t), u(\cdot,t), u_t(\cdot,t)) \in L^\infty(0,T;\mathcal{F}')$ and

(b)

$$\begin{aligned}
&\langle (E(\cdot,t), H(\cdot,t), u(\cdot,t), u_t(\cdot,t)), (\tilde{E}(\cdot,t), \tilde{H}(\cdot,t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t))\rangle_K \\
&= \langle (f_1, f_2, g_1, g_2), (\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2)\rangle_K \\
&\quad + \int_0^t \int_{\partial\Omega} R \cdot \Big(\mu\tilde{H}_0 + \alpha\frac{\partial\tilde{u}}{\partial\eta} - \tilde{p}\eta\Big) d\Gamma d\tau
\end{aligned} \tag{3.2}$$

holds for any $(\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2) \in \mathcal{F}$ and $t \in (0,T)$ where $(\tilde{E}, \tilde{H}, \tilde{u}, \tilde{u}_t)$ is a solution of (1.1)–(1.6) with zero boundary conditions. In (3.1)

$$\begin{aligned}
&\langle (f_1, f_2, g_1, g_2), (\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2)\rangle_K \\
&= \int_\Omega \Big\{\mathcal{E} f_1 \cdot \tilde{f}_1 + \mu f_2 \cdot \tilde{f}_2 + \alpha \sum_{i=1}^3 \frac{\partial g_1}{\partial x_i} \cdot \frac{\partial \tilde{g}_1}{\partial x_i} + \rho g_2 \cdot \tilde{g}_2\Big\} dx.
\end{aligned}$$

Here $\tilde{p}$ denotes the pressure term for the solution $(\tilde{u}, \tilde{u}_t)$ of (1.4)–(1.6) with zero boundary conditions.

Definition. Given $R = R(x,t) \in [L^2(\partial\Omega \times (0,T)]^3$ we say that $\{E, H, u, u_t\}$ is a solution of the Maxwell/elasticity system with zero initial data at time $t = T$ if $\{E, H\}$ solves (1.1)–(1.2) with boundary condition

$$\eta \times E = \mu\eta \times (\eta \times R) \quad \text{on } \partial\Omega \times (0,T)$$

and $\{u, u_t\}$ solves (1.4)–(1.5) with boundary condition

$$u_t = R \quad \text{on } \partial\Omega \times (0,T) \tag{3.3}$$

Furthermore,

(a) $(E(\cdot,t), H(\cdot,t), u(\cdot,t), u_t(\cdot,t)) \in L^\infty(0,T;\mathcal{F}')$ and

(b)

$$\begin{aligned}
&\langle (E(\cdot,t), H(\cdot,t), u(\cdot,t), u_t(\cdot,t)), (\tilde{E}(\cdot,t), \tilde{H}(\cdot,t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t))\rangle_K \\
&= -\int_t^T \int_{\partial\Omega} R \cdot \Big(\alpha\frac{\partial\tilde{u}}{\partial\eta} + \mu\tilde{H} - \tilde{p}\eta\Big) d\Gamma d\tau
\end{aligned} \tag{3.4}$$

holds for any $(\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2) \in \mathcal{F}$ and $t \in (0,T)$.

Due to the linearity and reversibility of system (1.1)-(1.6) it is clear that in order to solve the problem of exact controllability it is sufficient to prove that for any initial data $(f_1, f_2, g_1, g_2) \in \mathcal{F}'$ then the corresponding solution can be driven to the equilibrium state at time T.

Theorem 3.1. *Under the assumptions of Theorem 2.3. If $T > T_1$, then for any initial data $(f_1, f_2, g_1, g_2) \in \mathcal{F}'$ of problems (1.1)–(1.2) and (1.4)–(1.5) there exist a control $P \in H^1(0,T;[L^2(\partial\Omega)]^3)$ such that $u = P$ on $\partial\Omega \times (0,T)$ and the corresponding solution satisfies*

$$(u, u_t)\big|_{t=T} = (0,0)$$

while the vector-valued function $Q = \mu\eta \times (\eta \times P_t)$ drives system (1.1)–(1.2) such that $\eta \times E = Q$ on $\partial\Omega \times (0,T)$ and the corresponding solution satisfies

$$(E, H)\big|_{t=T} = (0,0)$$

Proof. We use our previous discussion to apply the Hilbert Uniqueness Method (HUM). Let (h_1, h_2, q_1, q_2) be an (arbitrary) element of $\mathcal{F}$ and (φ, ψ, v, v_t) the solution of (1.1)–(1.6) with zero boundary conditions and initial data

$$(\varphi, \psi, v, v_t)\big|_{t=0} = (h_1, h_2, q_1, q_2). \tag{3.5}$$

Finally, let (E, H, u, u_t) be the solution of (1.1), (3.1), (1.4), (3.3) with zero initial data at $t = T > T_1$ where R is chosen to be

$$-R(x,t) = \mu\psi + \alpha\frac{\partial v}{\partial\eta} \quad \text{on } \partial\Omega \times (0,T). \tag{3.6}$$

We consider the map $M : \mathcal{F} \mapsto \mathcal{F}'$ given by

$$M(h_1, h_2, q_1, q_2) = (E, H, u, u_t)\big|_{t=0}.$$

Our objective is to show that M is an isomorphism from $\mathcal{F}$ onto $\mathcal{F}'$. From to (3.4) (with $t = 0$) and (3.6) it follows

$$\begin{aligned} &\langle M(h_1, h_2, q_1, q_2), (\tilde{f}_1, \tilde{f}_2, \tilde{q}_1, \tilde{q}_2)\rangle_K \\ &= \int_0^T \int_{\partial\Omega} \Big(\mu\psi + \alpha\frac{\partial v}{\partial\eta}\Big) \cdot \Big(\alpha\frac{\partial\tilde{u}}{\partial\eta} + \mu\tilde{H} - \tilde{p}\eta\Big) d\Gamma d\tau \end{aligned} \tag{3.7}$$

where $(\tilde{E}, \tilde{H}, \tilde{u}, \tilde{u}_t)$ is a solution of (1.1)–(1.6) with zero boundary conditions. Since we know that $\psi \cdot \eta = 0$ and $\frac{\partial v}{\partial\eta} \cdot \eta = 0$ on $\partial\Omega \times (0,T)$ because (2.1) and $v = 0$ on $\partial\Omega \times (0,T)$, then it follows that $\Big(\mu\psi + \alpha\frac{\partial v}{\partial\eta}\Big) \cdot \tilde{p}\eta = 0$ on $\partial\Omega \times (0,T)$. Therefore from (3.7) we deduce

$$\begin{aligned} &\langle M(h_1, h_2, q_1, q_2), (\tilde{f}_1, \tilde{f}_2, \tilde{q}_1, \tilde{q}_2)\rangle_K \\ &= \int_0^T \int_{\partial\Omega} \Big(\mu\psi + \alpha\frac{\partial v}{\partial\eta}\Big) \cdot \Big(\alpha\frac{\partial\tilde{u}}{\partial\eta} + \mu\tilde{H}\Big) d\Gamma d\tau \\ &= \langle (h_1, h_2, q_1, q_2), (\tilde{f}_1, \tilde{f}_2, \tilde{q}_1, \tilde{q}_2)\rangle_{\mathcal{F}} \end{aligned} \tag{3.8}$$

for any $(\tilde{f}_1, \tilde{f}_2, \tilde{q}_1, \tilde{q}_2) \in \mathcal{F}$. Clearly (3.8) implies that M is an isomorphism from $\mathcal{F}$ onto $\mathcal{F}'$. Now, we return to problems (1.1), (1.2), (3.1) and (1.4), (1.5), (3.3). Let $(f_1, f_2, g_1, g_2) \in \mathcal{F}'$. We set

$$(h_1, h_2, q_1, q_2) = M^{-1}(f_1, f_2, g_1, g_2), \quad R = -\Big(\mu\psi + \alpha\frac{\partial v}{\partial\eta}\Big)$$

where (φ, ψ, v, v_t) is the solution of (1.1), (1.6) with zero boundary conditions and initial data at $t = 0$ as in (3.5). From (3.4) with $t = T > T_1$ we deduce

$$\begin{aligned}
&\langle (E(T), H(T), u(T), u_t(T)), (\tilde{E}(T), \tilde{H}(T), \tilde{u}(T), \tilde{u}_t(T)) \rangle_K \\
&= \langle M(h_1, h_2, q_1, q_2), (\tilde{f}_1, \tilde{f}_2, \tilde{q}_1, \tilde{q}_2) \rangle_K \\
&\quad - \langle (h_1, h_2, q_1, q_2), (\tilde{f}_1, \tilde{f}_2, \tilde{q}_1, \tilde{q}_2) \rangle_{\mathcal{F}}.
\end{aligned} \tag{3.9}$$

Using (3.8), we conclude that the right hand side of (3.9) is equal to zero. This means that $(E(T), H(T), u(T), u_t(T))$ generates the zero functional on $\mathcal{F}$. Now, the conclusion of Theorem 3.1 is a consequence of the above discussion: Construct $R(x,t)$ as in (3.6) and let

$$P(x,t) = \int_0^t R(x,s)ds + g_1(x)$$

Obviously $P = u$ and $\eta \times E = \mu\eta \times (\eta \times P_t)$ by construction. $\square$

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