Electronic Journal of Differential Equations, Vol. 2010(2010), No. 28, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

COMPACTNESS RESULTS FOR GINZBURG-LANDAU TYPE FUNCTIONALS WITH GENERAL POTENTIALS

MATTHIAS KURZKE

Abstract. We study compactness and Γ-convergence for Ginzburg-Landau type functionals. We only assume that the potential is continuous and positive definite close to one circular well, but allow large zero sets inside the well. We show that the relaxation of the assumptions does not change the results to leading order unless the energy is very large.

1. Introduction

We study the family of functionals

$$E_\varepsilon(u) = \frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{\varepsilon^2}P(u) \tag{1.1}$$

for a smooth bounded domain $U \subset \mathbb{R}^2$ and $u \in H^1(U;\mathbb{C})$, where P is a nonnegative function with $P = 0$ on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

In the case where $P(u) = P_{gl}(u) = \frac{1}{2}(1-|u|^2)^2$, this functional is the Ginzburg-Landau functional

$$G_\varepsilon(u) = \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2, \tag{1.2}$$

which has been widely studied since the groundbreaking work of Bethuel, Brezis and Hélein [4]. They considered minimizers and solutions of the Euler-Lagrange equations under Dirichlet boundary condition and were able to show convergence results and detailed energy asymptotics. Their methods relied on Rellich-Pohožaev type identities and thus on the PDE.

Later, Sandier [13] and Jerrard [6] were able to show the essential lower bounds using more direct comparison arguments and without PDE arguments. This approach was refined and later used by Jerrard and Soner [7, 8] to show compactness for the Jacobians and a Γ-convergence result for the energies. Independently, Alberti-Baldo-Orlandi [1, 2] obtained a proof in the general case of maps from $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$ and an energy related to the k-Dirichlet energy. In the context of the magnetic Ginzburg-Landau functional, a detailed presentation can be found in the monograph by Sandier-Serfaty [14].

2000 *Mathematics Subject Classification.* 35J50, 35B25.
Key words and phrases. Gamma-convergence; compactness for Jacobians;
Ginzburg-Landau functional.

Submitted September 22, 2009. Published February 18, 2010.
Supported by SFB 611.

The PDE approach of Bethuel-Brezis-Hélein [4] has been generalized to potentials P that vanish near S^1 to infinite order by Hadiji and Shafrir [5]. The main new feature is that certain potentials with sufficiently slow growth allow for a vortex energy that is not $\pi|\log\varepsilon| + O(1)$, but instead $\pi|\log\varepsilon| - o(|\log\varepsilon|)$.

Potentials given by distance functions to homeomorphic images of S^1 instead of S^1 itself were also studied, see the work of Shafrir [15] and André and Shafrir [3].

In this article, we investigate a different type of generalization: we allow the potential to vanish on a larger set, and in particular to have zeroes inside the unit ball. One simple such potential is

$$P_{csh}(u) = |u|^2(1-|u|^2)^2. \tag{1.3}$$

The study of this potential has applications in the theory of Chern-Simons-Higgs vortices. In Kurzke-Spirn [9], the analysis of (1.1) with this potential is the basis for a Γ-convergence analysis of the static Chern-Simons-Higgs functional,

$$G^{\varepsilon,\mu}_{csh}(u,A) = \frac{1}{2}\int_U |(\nabla - iA)u|^2 + \frac{\mu^2}{4}\frac{|\operatorname{curl} - h_{ex}|^2}{|u|^2} + \frac{1}{\varepsilon^2}|u|^2(1-|u|^2)^2, \tag{1.4}$$

which contains an interaction with a magnetic vector potential $A : U \to \mathbb{R}^2$ and an external magnetic field h_{ex}. The particular potential (1.3) has also been studied using the methods devised by Bethuel-Brezis-Hélein [4] for P_{gl}. Lassoued-Lefter [10] showed a convergence result for minimizers in bounded domains, and Ma [12] studied quantization properties for solutions of the corresponding Euler-Lagrange equations in $\mathbb{R}^2$. Both of these works rely heavily on the Pohožaev identity and on the equations, requiring some differentiability of the potential.

In contrast to these works, in the present article we only require the potential P to be continuous and nonnegative in $\{|z| \le 1\}$ and comparable to the Ginzburg-Landau potential near S^1. We do not require any further smoothness or symmetry assumptions, and allow the potential to vanish on large subsets of the unit ball. To ensure the presence of vortices, we study situations where $|u| = 1$ on the boundary (for example, this can be ensured by a Dirichlet boundary condition), although this assumption can be weakened.

Our method is based on a few elementary Modica-Mortola type arguments and the co-area formula. By an observation of G. Orlandi, these arguments lead to a bound on the standard Ginzburg-Landau functional, and it is then possible to apply the standard compactness results of [2, 7, 8] also to the generalized functional. The method is applicable to the functional studied in [9], and yields a new and much shorter and clearer proof of the central compactness statement there.

The Γ-convergence result turns out to be essentially the same as for the Ginzburg-Landau energy, in contrast to the weighted Ginzburg-Landau energy with vanishing weight as studied by Lefter-Radulescu [11], where $P(u)$ is replaced by $w(x)P(u)$ with a nonnegative function $w : U \to [0,\infty)$, and the behaviour depends on the growth of the weight function near its zeroes.

A difference between our result and Ginzburg-Landau theory can be seen at very high energies: our compactness theorem only holds for a smaller range of energies, and we even have an explicit example of failure for very large energies, see Remark 2.7.

Notation and assumptions. We will deal with sequences (u_ε) of functions $u_\varepsilon \in C^1(\overline{U};\mathbb{C})$, labeled by an arbitrary (but assumed fixed) sequence $\varepsilon = \varepsilon_k \to 0$. Subsequences will be taken from this fixed sequence. As we are dealing mostly with compactness issues, the regularity requirement is mostly technical.

We will generally assume $|u_\varepsilon| \le 1$, $|u_\varepsilon| = 1$ on ∂U, where U is a Lipschitz domain in $\mathbb{R}^2$. The assumption $|u_\varepsilon| \le 1$ is technical and could be replaced by suitable growth conditions on $P(z)$ for $|z| \ge 1$.

The assumption $|u_\varepsilon| = 1$ on ∂U is slightly more restrictive: it completely rules out vortices on the boundary and at first glance seems to make it impossible to use the results presented here for dynamical situations where vortices enter or leave through the boundary. However, under reasonable assumptions on P and U, it is possible to use the extension technique of Kurzke-Spirn [9] to show that any u_ε on U with $|u_\varepsilon| \to 1$ in $L^2(U)$ can be extended to $\tilde{u}_\varepsilon$ on $\tilde{U} \supset U$ with $|\tilde{u}_\varepsilon| = 1$ on $\partial\tilde{U}$, and then the compactness results of this article can be applied in $\tilde{U}$.

Our assumptions on the potential P are as follows.

- (A1) $P \in C^0(\{|z| \le 1\})$, $P \ge 0$, $P(S^1) = \{0\}$
- (A2) There is a $\kappa \in (0,1)$ and an $a > 0$ such that $P(z) \ge a(1-|z|^2)^2$ for $|z| \in [1-\kappa, 1]$.

We note that both P_{csh} and P_{gl} satisfy these assumptions.

For $a, b \in \mathbb{C}$ we write $(a,b) := \mathrm{Re}(\overline{a}b)$ for the scalar product. Identifying $\mathbb{R}^2$ and $\mathbb{C}$, we can write the Jacobian of u as $\frac{1}{2}\,\mathrm{curl}(iu, \nabla u)$.

Our main results are a compactness result for the Jacobians, Theorem 2.6, holding for energies $E_\varepsilon(u_\varepsilon) \ll \frac{1}{\varepsilon}$, and Γ-convergence results for the energy scalings $E_\varepsilon(u_\varepsilon) \approx |\log\varepsilon|$ and $E_\varepsilon(u_\varepsilon) \approx |\log\varepsilon|^2$, Theorems 3.1 and 3.2. Unlike in the Ginzburg-Landau case, the Jacobian bounds of Theorem 2.6 cannot be extended to energies in the range $\frac{1}{\varepsilon} \ll E_\varepsilon(u_\varepsilon) \ll \frac{1}{\varepsilon^2}$, see Remark 2.7.

Remark 1.1. *For the questions studied in this article, we may assume without loss of generality that P is radial, more precisely, we can assume $P(u) = V(|u|)$ for some V with $V \in C^0([0,1])$ with $V \ge 0$, $V(1) = 0$ and $V(t) \ge a(1-t^2)^2$ for $t \in [1-\kappa, 1]$. To see this, set*

$$V^-(\rho) = \inf_{|u|=\rho} P(u) \quad \textit{and} \quad V^+(\rho) = \sup_{|u|=\rho} P(u). \tag{1.5}$$

It is clear that $\int_U V^-(|u|) \le \int_U P(u) \le \int_U V^+(|u|)$ and so compactness and lower bound results for functionals involving V^- carry over to P, while upper bound results for V^+ imply upper bound results for P. We note that by compactness, $V^- \le V^+ \le C(P)V^-$ and so any Γ-convergence results for radial functionals that are invariant under rescaling $V \mapsto \sigma V$ for $\sigma > 0$ automatically hold for non-radial P.

Acknowledgments. The author is grateful to Giandomenico Orlandi who suggested a significant improvement of a first draft of the article using Propositions 2.3 and 2.5. This greatly simplified the proofs.

The author enjoyed the support of DFG SFB 611.

2. Energy bounds and Jacobian compactness

Definition 2.1. For any compact set $K \subset \mathbb{R}^2$, we define its *radius* $\mathbf{r}(K)$ as

$$\mathbf{r}(K) = \inf\Big\{\sum_{i=1}^N r_i : K \subset \bigcup B_i, B_i \text{ a closed ball of radius } r_i\Big\} \tag{2.1}$$

Radius of a set and 1-dimensional Hausdorff measure $\mathcal{H}^1$ of its perimeter are related by the following inequality:

$$\mathbf{r}(K) \le \frac{1}{2}\mathcal{H}^1(\partial K). \tag{2.2}$$

For a proof, see [14, p. 71].

The radius also has the monotonicity property $\mathbf{r}(A) \le \mathbf{r}(B)$ for $A \subset B$.

Proposition 2.2. *There exists a constant $C = C(a) > 0$ such that for $u \in C^1(\overline{U}, \mathbb{C})$ with $|u| = 1$ on ∂U, $E_\varepsilon(|u|) \le M$ and for any $\delta < \kappa$ and any ε, we have the following covering estimate for $K_\delta := \{|u| \le 1 - \delta\}$:*

$$\mathbf{r}(K_\delta) \le C\frac{\varepsilon M}{\delta^2}. \tag{2.3}$$

In particular, the measure of $\{|u| \le 1 - \delta\}$ satisfies

$$|K_\delta| \le \tilde{C}\varepsilon^2 M^2 \tag{2.4}$$

for $\tilde{C} = \frac{4\pi C(a)}{\kappa^2}$.

Proof. Setting $\rho = |u|$, the energy bound

$$\frac{1}{2}\int_U |\nabla\rho|^2 + \frac{1}{\varepsilon^2}V(\rho) \le M$$

implies by Cauchy's inequality that

$$\int_U |\nabla\rho|\sqrt{V(\rho)} \le \varepsilon M.$$

Using the co-area formula, this shows

$$\int_{1-\kappa}^1 \sqrt{V(t)}\mathcal{H}^1(\{\rho = t\})dt \le \varepsilon M.$$

Using the assumption (A2) on the potential, we see that in particular

$$\int_{1-\delta}^{1-\frac{\delta}{2}} a(1-t^2)\mathcal{H}^1(\{\rho = t\}) \le \varepsilon M.$$

Using that $(1-t^2) \ge (1-(1-\frac{\delta}{2})^2) \ge \frac{\delta}{2}$ in $(1-\delta, 1-\frac{\delta}{2})$ and the mean value theorem, we obtain the existence of $t_0 \in (1-\delta, 1-\frac{\delta}{2})$ such that

$$\mathcal{H}^1(\{x \in U : \rho(x) = t_0\}) \le \frac{4\varepsilon M}{a\delta^2} \tag{2.5}$$

Using that $\{\rho \le t_0\}$ is compactly contained in U by the boundary condition $|u| = 1$, it follows that $\partial\{x \in U : \rho(x) \le t_0\} = \{x \in U : \rho(x) = t_0\}$). Hence we can use (2.2) and see that $\mathbf{r}(\{\rho \le t_0\}) \le \frac{4\varepsilon M}{a\delta^2}$. The monotonicity of $\mathbf{r}$ now yields (2.3).

To prove (2.4), we note that $K_\delta \subset \bigcup B_{r_i}(a_i)$ with $\sum r_i \le \mathbf{r}(K_\delta)$. Hence $|K_\delta| \le \pi\sum r_i^2 \le \pi(\sum r_i)^2 \le \pi(\mathbf{r}(K_\delta))^2$, and (2.4) follows. □

The following observation was communicated to the author by Giandomenico Orlandi:

Proposition 2.3. *For $\alpha \in (0,1)$ and $\varepsilon > 0$ sufficiently small there exists a $C = C(a) > 0$ such that the following holds: For every $u \in C^1(\overline{U}, \mathbb{C})$ with $|u| \le 1$ in U, $|u| = 1$ on ∂U that satisfies the energy bound $E_\varepsilon(|u|) = M \le \varepsilon^{-\alpha}$, there holds the bound*

$$\int_U (1-|u|^2)^2 \le C\varepsilon^{2-\alpha} M. \tag{2.6}$$

Proof. We split $U = \{|u| \le 1 - \frac{\kappa}{2}\} \cup \{|u| > 1 - \frac{\kappa}{2}\}$. Using (2.5) applied to $\delta = \frac{\kappa}{2}$ and (2.2),

$$\begin{aligned} \int_{\{|u|\le 1-\frac{\kappa}{2}\}} (1-|u|^2)^2 &\le \pi \mathbf{r}(\{|u| \le 1 - \frac{\kappa}{2}\})^2 \\ &\le C(a)(\varepsilon M)^2 \\ &\le \varepsilon^2 \varepsilon^{-\alpha} M \\ &= C\varepsilon^{2-\alpha} M \end{aligned}$$

since $M \le \varepsilon^{-\alpha}$. On the other hand, using (A2) we estimate

$$\begin{aligned} \int_{\{|u|> 1-\frac{\kappa}{2}\}} (1-|u|^2)^2 &\le \frac{1}{a} \int_{\{|u|> 1-\frac{\kappa}{2}\}} V(|u|) \\ &\le C(a)\varepsilon^2 E_\varepsilon(|u|) \\ &\le C\varepsilon^2 M \\ &\le C\varepsilon^{2-\alpha} M. \end{aligned}$$

Combining these estimates we obtain (2.6). □

Remark 2.4. *From the proof of Proposition 2.3 we see that the assumption $|u| \le 1$ in U could be replaced by assuming that (A2) also holds for $|z| > 1$.*

Proposition 2.5. *Let $u \in C^1(\overline{U}, \mathbb{C})$ with $E_\varepsilon(u) \le M$, $|u| \le 1$ in U, $|u| = 1$ on ∂U, where $M \le \varepsilon^{-\alpha}$ for some $\alpha \in (0,1)$. Then for $\eta = \varepsilon^{1-\frac{\alpha}{2}}$, the Ginzburg-Landau functional $G_\eta(u)$ as defined in (1.2) satisfies the bound*

$$G_\eta(u) \le CM \tag{2.7}$$

If (u_ε) is a sequence satisfying the assumptions above, then the following bounds hold for the sequence $\eta_\varepsilon = \varepsilon^{1-\frac{\alpha}{2}} \to 0$:

(i) *If $E_\varepsilon(u_\varepsilon) \le \varepsilon^{-\alpha}$ then $G_{\eta_\varepsilon}(u_\varepsilon) \le C\eta_\varepsilon^{-\frac{2\alpha}{2-\alpha}}$.*
(ii) *If $E_\varepsilon(u_\varepsilon) \le K|\log\varepsilon|$ then $G_{\eta_\varepsilon}(u_\varepsilon) \le CK\frac{2}{2-\alpha}|\log\eta_\varepsilon|$.*
(iii) *If $E_\varepsilon(u_\varepsilon) \le K|\log\varepsilon|^2$ then $G_{\eta_\varepsilon}(u_\varepsilon) \le CK\frac{4}{(2-\alpha)^2}|\log\eta_\varepsilon|^2$.*

Proof. From (2.6) we obtain

$$\eta^{-2}\int_U (1-|u|^2)^2 \le C\eta^{-2}\eta^2 M = CM,$$

and trivially

$$\frac{1}{2}\int_U |\nabla u|^2 \le E_\varepsilon(u) \le M.$$

The bounds (i)-(iii) follow from (2.7) by simply expressing the energy bounds in terms of η_ε. □

Theorem 2.6. *Let (u_ε) be a sequence in $C^1(\overline{U},\mathbb{C})$ with $|u_\varepsilon| = 1$ on ∂U, $|u_\varepsilon| \le 1$ in U and $E_\varepsilon(u_\varepsilon) \le M_\varepsilon$, where*

$$|\log\varepsilon| \le M_\varepsilon \le \varepsilon^{-\alpha}$$

for some $\alpha \in (0,1)$. Set

$$\mu_\varepsilon = \frac{|\log\varepsilon|}{M_\varepsilon}\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon).$$

Then the sequence of measures (μ_ε) is precompact in $(C_0^{0,\beta})^*$ for any $\beta \in (0,1)$.

Proof. By Proposition 2.5, $G_{\eta_\varepsilon}(u_\varepsilon) \ll \eta^{-2}$. Hence we can apply Theorem 1.1 of [8], which yields the claim. □

Remark 2.7. *For the Ginzburg-Landau energy, the analog of the previous compactness result holds for energies up to $G_\varepsilon(u) \ll \varepsilon^{-2}$. In fact, we used that result in its full strength, as $E_\varepsilon(u_\varepsilon) \ll \varepsilon^{-1}$ just implies $G_\eta(u_\varepsilon) \ll \eta^{-2}$. That we cannot relax the assumption $E_\varepsilon(u_\varepsilon) \ll \varepsilon^{-1}$ is not just an artifact of the proof: for our more general potentials, the following example shows that compactness may fail for higher energies.*

Example 2.8. *Set $V(\rho) = (\rho^2-1)^2(\rho^2-\frac14)^2\chi_{[\frac12,1]}(\rho)$. Let $\frac1\varepsilon \ll d_\varepsilon$. Then there exist constants C_1, C_2 and a sequence of functions (u_ε), $u_\varepsilon \in C^1(\overline{B_1(0)},\mathbb{C})$ with $u_\varepsilon = 1$ on $\partial B_1(0)$, $E_\varepsilon(u_\varepsilon) \le C_1 d_\varepsilon$ and*

$$\|\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)\|_{(C_0^{0,1})^*} \ge C_2 E_\varepsilon(u_\varepsilon).$$

In particular, the measures (μ_ε) as in Theorem 2.6 will be unbounded.

Proof. Setting $d = d_\varepsilon$, we choose

$$u_\varepsilon(r,\theta) = \begin{cases} \frac12(4r)^d e^{id\theta} & 0 \le r < \frac14 \\ \frac12(2-4r)^d e^{id\theta} & \frac14 \le r < \frac12 \\ 0 & \frac12 \le r < 1-\varepsilon \\ 1 - \frac{1-r}{\varepsilon} & 1-\varepsilon \le r \le 1. \end{cases}$$

Then it is not hard to check that $|u_\varepsilon| \le \frac12$ on $B_{1/2}$ and $E_\varepsilon(u_\varepsilon) = Cd_\varepsilon + \frac{c}{\varepsilon} \le Cd_\varepsilon$. Testing with the $C^{0,1}$ function

$$\zeta(r,\theta) = \begin{cases} \frac18 & r < \frac18 \\ \frac14 - r & \frac18 \le r < \frac14 \\ 0 & r \ge \frac14 \end{cases}$$

shows the claim for the Jacobians. □

3. Gamma limits

In this section, we use the results of the previous section to obtain Γ-limit results in two scaling regimes: One is the smallest energy regime where vortices appear, namely $M \approx |\log\varepsilon|$. The second is the "natural" energy scaling $M \approx |\log\varepsilon|^2$, the only scaling where both the Jacobian and the current $(iu, \nabla u)$ make a significant contribution. A full discussion of possible regimes can be found in [8]. We note that results for Chern-Simons-Higgs type functionals (1.4) can be deduced as in [9], at least for certain parameter regimes.

Theorem 3.1 (Compactness and Γ-convergence in the $|\log\varepsilon|$ scaling)**.** *If (u_ε), $u_\varepsilon \in C^1(\overline{U};\mathbb{C})$ satisfies $|u_\varepsilon| \le 1$, $|u_\varepsilon| = 1$ on ∂U and $E_\varepsilon(u_\varepsilon) \le K|\log\varepsilon|$, then the sequence $\mu_\varepsilon = \frac{1}{2}\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)$ is precompact in $(C_0^{0,\beta})^*$, and any subsequential weak limit μ satisfies*

$$\mu = \pi \sum_{j=1}^{n} d_j \delta_{a_j} \tag{3.1}$$

for some $n \in \mathbb{N}$, $a_j \in U$ and $d_j \in \mathbb{Z}$. Moreover,

$$\|\mu\|_{\mathscr{M}} = \pi \sum_{j=1}^{n} |d_j| \le \liminf_{\varepsilon\to 0} \frac{1}{2|\log\varepsilon|}\int_U |\nabla u_\varepsilon|^2 \le \liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}E_\varepsilon(u_\varepsilon). \tag{3.2}$$

Conversely, for any μ of the form (3.1) there exists a sequence (u_ε), $u_\varepsilon \in C^1(\overline{U};\mathbb{C})$ such that

$$\|\mu\|_{\mathscr{M}} = \lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}E_\varepsilon(u_\varepsilon) \tag{3.3}$$

Proof. Let $\alpha \in (0,1)$. From Proposition 2.5, $G_\eta(u_\varepsilon) \le C|\log\varepsilon|$, where $\eta = \varepsilon^{1-\frac{\alpha}{2}}$. We have $|\log\eta| = (1-\frac{\alpha}{2})|\log\varepsilon|$ and so

$$G_\eta(u_\varepsilon) \le \frac{C}{1-\frac{\alpha}{2}}|\log\eta|.$$

In particular, we can apply the standard compactness theory for the Ginzburg-Landau functional in the form of Theorem 3.1 of [7], which yields the compactness and the structure statement on μ. Furthermore, it yields the bound as well as the following

$$\|\mu\|_{\mathscr{M}} \le \frac{1}{|\log\eta|}G_{\eta_\varepsilon}(u_\varepsilon). \tag{3.4}$$

By the observation made in the Remark following Theorem 1.1 of [2] (the bound in (3.4) does not depend on the shape of the potential), we can replace $(1-|u|^2)^2$ by $\sigma(1-|u|^2)^2$ and then let $\sigma \to 0$ and arrive at

$$\|\mu\|_{\mathscr{M}} \le \frac{1}{2|\log\eta_\varepsilon|}\int_U |\nabla u_\varepsilon|^2. \tag{3.5}$$

Finally, we let $\alpha \to 0$ so $\frac{|\log\eta_\varepsilon|}{|\log\varepsilon|} = 1 - \frac{\alpha}{2} \to 1$, and we obtain (3.2). The upper bound construction of [7] provides (3.3). Here our different potential does not substantially change the proof, cf. Remark 1.1: Since $P(u) \le K(P)(1-|u|^2)^2$ and so $\varepsilon^{-2}P(u) \le \frac{1}{2}\tilde{\varepsilon}^{-2}(1-|u|^2)^2$ for $\tilde{\varepsilon} = \sqrt{2K(P)}\varepsilon$ and $\frac{|\log\varepsilon|}{|\log\tilde{\varepsilon}|} \to 1$ for $\varepsilon \to 0$, the precise form of the potential is irrelevant. □

Theorem 3.2 (Compactness and Γ-convergence in the $|\log\varepsilon|^2$ scaling)**.** *Assume that (u_ε), $u_\varepsilon \in C^1(\overline{U};\mathbb{C})$ satisfies $|u_\varepsilon| \le 1$, $|u_\varepsilon| = 1$ on ∂U and $E_\varepsilon(u_\varepsilon) \le K|\log\varepsilon|^2$. Then a subsequence of $v_\varepsilon = \frac{1}{|\log\varepsilon|}(iu_\varepsilon, \nabla u_\varepsilon)$ converges weakly in L^2 to $v \in L^2(U)$. The measures $w_\varepsilon = \operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)$ converge subsequentially in $(C_0^{0,\beta})^*$ to $w = \operatorname{curl} v \in H^{-1}(U)$.*

The energy satisfies

$$\liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|^2}E_\varepsilon(u_\varepsilon) \ge \liminf_{\varepsilon\to 0}\frac{1}{2|\log\varepsilon|^2}\int_U |\nabla u_\varepsilon|^2 \ge \frac{1}{2}\int_U |v|^2 + \frac{1}{2}\|\operatorname{curl} v\|_{\mathscr{M}}. \tag{3.6}$$

Furthermore, for any $v \in L^2(U)$ such that $\operatorname{curl} v$ is a Radon measure, there exists a sequence (u_ε) such that v_ε and w_ε defined as above converge weakly in L^2 and in $(C^{0,\beta})^*$, respectively, and such that equality holds in (3.6).

Proof. We note that $\|v_\varepsilon\|_{L^2}^2 \le \left\| \frac{v_\varepsilon}{|u_\varepsilon|} \right\|_{L^2}^2 \le K$. By Theorem 2.6, the w_ε are compact.

We again choose $\alpha \in (0,1)$ and set $\eta_\varepsilon = \varepsilon^{1-\frac{\alpha}{2}}$. By Proposition 2.5, the Ginzburg-Landau energy satisfies the bound

$$G_{\eta_\varepsilon}(u_\varepsilon) \le C|\log \eta_\varepsilon|^2. \tag{3.7}$$

Hence we can use Theorem 1.2 of [8] and obtain the compactness and compactness and structure results of the theorem. To prove (3.6), we note that the cited theorem implies

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log \eta_\varepsilon|^2} G_{\eta_\varepsilon}(u_\varepsilon) \ge \liminf_{\varepsilon\to 0} \frac{1}{2|\log \eta_\varepsilon|^2} \int_U |\nabla u_\varepsilon|^2 \ge \frac{1}{2}\int_U |v|^2 + \frac{1}{2}\|\operatorname{curl} v\|_{\mathscr{M}}, \tag{3.8}$$

where we have again used the observation of [2] to drop the potential term in the lower bound. Just as in the proof of the previous theorem, we may now let $\alpha \to 0$, which implies $\frac{|\log \eta_\varepsilon|^2}{|\log \varepsilon|^2} \to 1$, hence (3.6).

For the construction of a recovery sequence that shows equality, we note that the sequence constructed in Proposition 7.1 of [8] can be used without changes, keeping in mind the remarks made at the end of the proof of Theorem 3.1. □

References

[1] Alberti, G. Un risultato di convergenza variazionale per funzionali di tipo Ginzburg-Landau in dimensione qualunque. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 4 (2001), 289–310.

[2] Alberti, G., Baldo, S., and Orlandi, G. Variational convergence for functionals of Ginzburg-Landau type. *Indiana Univ. Math. J. 54*, 5 (2005), 1411–1472.

[3] André, N., and Shafrir, I. On a singular perturbation problem involving a "circular-well" potential. *Trans. Amer. Math. Soc. 359*, 10 (2007), 4729–4756 (electronic).

[4] Bethuel, F., Brezis, H., and Hélein, F. *Ginzburg-Landau vortices.* Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.

[5] Hadiji, R., and Shafrir, I. Minimization of a Ginzburg-Landau type energy with potential having a zero of infinite order. *Differential Integral Equations 19*, 10 (2006), 1157–1176.

[6] Jerrard, R. L. Lower bounds for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal. 30*, 4 (1999), 721–746 (electronic).

[7] Jerrard, R. L., and Soner, H. M. The Jacobian and the Ginzburg-Landau energy. *Calc. Var. Partial Differential Equations 14*, 2 (2002), 151–191.

[8] Jerrard, R. L., and Soner, H. M. Limiting behavior of the Ginzburg-Landau functional. *J. Funct. Anal. 192*, 2 (2002), 524–561.

[9] Kurzke, M., and Spirn, D. Gamma limit of the non-selfdual Chern-Simons-Higgs energy. *Journal of Functional Analysis 255* (2008), 535–588.

[10] Lassoued, L., and Lefter, C. On a variant of the Ginzburg-Landau energy. *NoDEA Nonlinear Differential Equations Appl. 5*, 1 (1998), 39–51.

[11] Lefter, C., and Rădulescu, V. Asymptotics for the minimizers of the Ginzburg-Landau energy with vanishing weight. *Adv. Math. Sci. Appl. 7*, 1 (1997), 261–273.

[12] Ma, L. Quantization effects for a variant of the Ginzburg-Landau type system. *Electron. J. Differential Equations* (2008), No. 135, 6.

[13] Sandier, E. Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal. 152*, 2 (1998), 379–403.

[14] Sandier, E., and Serfaty, S. *Vortices in the magnetic Ginzburg-Landau model.* Progress in Nonlinear Differential Equations and their Applications, 70. Birkhäuser Boston Inc., Boston, MA, 2007.

[15] Shafrir, I. On a class of singular perturbation problems. In *Stationary partial differential equations. Vol. I*, Handb. Differ. Equ. North-Holland, Amsterdam, 2004, pp. 297–383.

Matthias Kurzke
Institut für Angewandte Mathematik, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: kurzke@iam.uni-bonn.de