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EXISTENCE OF SOLUTIONS FOR AN EIGENVALUE PROBLEM WITH WEIGHT

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Abstract. In this work we study the existence of solutions for the nonlinear eigenvalue problem with p-biharmonic $\Delta_p^2 u = \lambda m(x)|u|^{p-2}u$ in a smooth bounded domain under Neumann boundary conditions.

1. Introduction

Let us consider the nonlinear eigenvalue problem

$$\begin{gathered} \Delta_p^2 u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p-2}\Delta u) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$; $1 < p < +\infty$; λ is a real parameter and m is a weight function in $L^r(\Omega)$ where $r = r(N,p)$ satisfying the conditions

$$\begin{gathered} r > N/2p \quad \text{if } N/p \geq 2 \\ r = 1 \quad \text{if } N/p < 2 \end{gathered} \tag{1.2}$$

We assume in addition that $\text{meas}(\Omega^+) \neq 0$, where $\Omega^+ = \{x \in \Omega / m(x) > 0\}$. Δ_p^2 is the p-biharmonic operator defined by $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$. For $p = 2$, $\Delta^2 = \Delta.\Delta$ is the iterated Laplacian which have been studied by many authors. For example, Gupta and Kwong [6] studied the existence of and L^p-estimates for the solutions of certain Biharmonic boundary value problems which arises in the study of static equilibrium of an elastic body.

In recent years, many papers including the p-Biharmonic operator ($p \neq 2$) have appeared (see [2, 4, 5, 8, 9]). In one dimensional case, Benedikt [2] studied the problem (1.1) under Dirichlet and Neumann boundary conditions. He proved that the spectrum consists on a sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ where λ_n is simple for $n > 0$ while $0 = \lambda_0$ is not and that any eigenfunction associated with λ_n, $n > 0$, has precisely $(n+1)$ zeros. In [5], El khalil, Kellati and Touzani showed that the spectrum of the problem (1.1) under Dirichlet boundary conditions contains at least one non decreasing sequence of eigenvalues $(\lambda_n)_n$, $\lambda_n \to +\infty$. We would like also mention the works in [4, 8, 9] where the authors studied various problems with p-biharmonic with Navier boundary conditions.

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The main goal of this paper is to show the existence of solutions for problem (1.1). For this end, we introduce the space

$$X = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\}.$$

We consider the functionals G and F defined on X by

$$G(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx; \quad F(u) = \frac{1}{p}\int_\Omega m(x)|u|^p dx$$

Let

$$\Gamma_n = \{K \subset M : K \text{ is compact symmetric and } \gamma(K) \geq n\}$$

where

$$M = \{u \in X : \int_\Omega m(x)|u|^p dx = 1\}.$$

and $\gamma(K)$ is the genus of K defined by

$$\gamma(K) = \begin{cases} \inf\{m : \exists h \in C^0(K; \mathbb{R}^m \setminus \{0\}), h(-u) = h(u)\} \\ \infty, \quad \text{if } \{\dots\} = \emptyset\,. \end{cases}$$

In particular, if $0 \in K$, $\gamma(\emptyset) = 0$ by definition.

Our main results are stated in the following theorems.

Theorem 1.1. *Problem* (1.1) *has at least one non decreasing sequence of nonnegative eigenvalues* $(\lambda_n)_n$ *defined as*

$$\lambda_n = \inf_{K \in \Gamma_n} \sup_{u \in K} pG(u), \tag{1.3}$$

and satisfying $\lambda_n \to +\infty$, *as* $n \to +\infty$.

Theorem 1.2. *The first eigenvalue* λ_1 *is*

$$\lambda_1 = \inf\{\|\Delta u\|_p^p : u \in X \text{ and } \int_\Omega m(x)|u|^p dx = 1\}, \tag{1.4}$$

and satisfies the following two properties:

(i) *If* $\int_\Omega m(x)dx \geq 0$ *then* $\lambda_1 = 0$.
(ii) *If* $\int_\Omega m(x)dx < 0$ *then* $\lambda_1 > 0$ *is the first nonnegative eigenvalue of* (1.1). *Moreover,* u_1 *is an eigenfunction associated to* λ_1 *if and only if*

$$G(u_1) - \lambda_1 F(u_1) = 0 = \inf_{u \in (X \setminus \{0\})} (G(u) - \lambda_1 F(u)).$$

The proofs of our main results are based on the Ljusternick Schnirelmann theory.

This article is organized as follows: In section 2, several technical lemmas and definitions are presented. In section 3, we prove firstly the existence of positive eigenvalues of perturbed problem and after, we give the proof of our main result by passing to the limit.

2. Preliminaries

Throughout this paper, we will adopt the following notation:
$X = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$,
$\|u\|_p = (\int_\Omega |u|^p dx)^{1/p}$ is the norm in $L^p(\Omega)$,
$\|u\|_{2,p} = (\|\Delta u\|_p^p + \|u\|_p^p)^{1/p}$ is the norm in $W^{2,p}(\Omega)$.
For a function $u \in W^{2,p}(\Omega)$: the normal derivative $\frac{\partial u}{\partial \nu} = (\nabla u_{|\Gamma}).\overrightarrow{\nu}$ is defined where $\nabla u_{|\Gamma} \in (L^p(\Gamma))^N$, $\frac{\partial u}{\partial \nu} \in L^p(\Gamma)$ and $\Gamma = \partial\Omega$. Thus, it's clear that X is a nonempty, well defined and closed subspace of $W^{2,p}(\Omega)$. However, it's easy to see that X is reflexive separable space with the induced norm of $W^{2,p}(\Omega)$ and uniformly convex.

By weak solution u of (1.1), we mean a functions in $X \setminus \{0\}$ which satisfies: for all $\varphi \in X$ and all $\lambda > 0$,

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi\, dx = \lambda \int_\Omega m(x)|u|^{p-2}u\varphi\, dx. \tag{2.1}$$

Proposition 2.1. *If $u \in X$ is a weak solution of* (1.1) *and $u \in C^4(\overline{\Omega})$ then u is a classical solution of* (1.1).

Proof. Let $u \in C^4(\overline{\Omega})$ be a weak solution of problem (1.1) then for every $\varphi \in X$, we have

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi\, dx = \lambda \int_\Omega m(x)|u|^{p-2}u\varphi\, dx. \tag{2.2}$$

By applying Green formula, we have

$$\int_\Omega \Delta(|\Delta u|^{p-2}\Delta u)\Delta\varphi\, dx = -\int_\Omega \nabla(|\Delta u|^{p-2}\Delta u).\nabla\varphi\, dx + \int_{\partial\Omega} \varphi.\frac{\partial}{\partial\nu}(|\Delta u|^{p-2}\Delta u)\, dx \tag{2.3}$$

and

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi\, dx = -\int_\Omega \nabla(|\Delta u|^{p-2}\Delta u).\nabla\varphi\, dx + \int_{\partial\Omega} |\Delta u|^{p-2}\Delta u.\frac{\partial\varphi}{\partial\nu}\, dx\,. \tag{2.4}$$

Then

$$\begin{aligned}\int_\Omega \Delta(|\Delta u|^{p-2}\Delta u)\Delta\varphi\, dx &= \int_\Omega |\Delta u|^{p-2}\Delta u\Delta\varphi\, dx - \int_{\partial\Omega} |\Delta u|^{p-2}\Delta u.\frac{\partial\varphi}{\partial\nu}\, dx \\ &\quad + \int_{\partial\Omega} \varphi.\frac{\partial}{\partial\nu}(|\Delta u|^{p-2}\Delta u)\, dx\end{aligned} \tag{2.5}$$

Then the result follows. □

We will use the following results proved by Szulkin [7].

Lemma 2.2 ([7]). *Let E be a real Banach space and A, B be symmetric subsets of $E \setminus \{0\}$ which are closed in E. Then*

1. *If there exists an odd continuous mapping $f : A \to B$, then $\gamma(A) \le \gamma(B)$*
2. *If $A \subset B$ then $\gamma(A) \le \gamma(B)$.*
3. *$\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.*
4. *If $\gamma(B) < +\infty$ then $\gamma(\overline{A - B} \ge \gamma(A) - \gamma(B)$.*
5. *If A is compact then $\gamma(A) < +\infty$ and there exists a neighborhood N of A, N is a symmetric subset of $E \setminus \{0\}$, closed in E such that $\gamma(N) = \gamma(A)$.*
6. *If N is a symmetric and bounded neighborhood of the origin in $\mathbb{R}^k$ and if A is homeomorphic to the boundary of N by an odd homeomorphism then $\gamma(A) = k$.*

(7) *If E_0 is a subspace of E of codimension k and if $\gamma(A) > k$ then $A \cap E_0 \neq \phi$.*

Theorem 2.3 ([7]). *Suppose that M is a closed symmetric C^1-submanifold of a real Banach space X and $0 \notin M$. Suppose that $f \in C^1(M, \mathbb{R})$ is even and bounded below. Define*

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} f(x),$$

where $\Gamma_j = \{K \subset M : K \text{ is compact symmetric and } \gamma(K) \geq j\}$. If $\Gamma_k \neq \phi$ for some $k \geq 1$ and if f satisfies the Palais Smale condition for all $c = c_j$, $j = 1, \dots, k$, then f has at least k distinct pairs of critical points.

3. Proofs of main results

Let us consider a perturbation of the principal problem (1.1) as follows

$$\begin{gathered} \Delta_p^2 u + \varepsilon |u|^{p-2} u = \lambda m(x) |u|^{p-2} u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p-2} \Delta u) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where ε is enough small ($0 < \varepsilon < 1$).

Theorem 3.1. *The problem (3.1) has at least one non decreasing sequence of nonnegative eigenvalues $(\lambda_{n,\varepsilon})_{n \in \mathbb{N}^*}$ given by*

$$\lambda_{n,\varepsilon} = \inf_{K \in \Gamma_n} \sup_{v \in K} (\|\Delta u\|_p^p + \varepsilon \|u\|_p^p), \tag{3.2}$$

and satisfying $\lambda_{n,\varepsilon} \to +\infty$ as $n \to +\infty$. Here $\mathbb{N}^*$ is the set of positive integers.

Let us consider the functionals $G_\varepsilon, F : X \to \mathbb{R}$ defined by:

$$\begin{gathered} G_\varepsilon(u) = \frac{1}{p} \|\Delta u\|_p^p + \frac{\varepsilon}{p} \|u\|_p^p, \\ F(u) = \frac{1}{p} \int_\Omega m(x) |u|^p dx \end{gathered} \tag{3.3}$$

G_ε and F are of class C^1 in X and for all $u \in X$

$$G'_\varepsilon(u) = \Delta_p^2 u + \varepsilon |u|^{p-2} u \quad \textit{and} \quad F'(u) = m|u|^{p-2} u \quad \textit{in} \quad X'$$

Since meas$(\Omega^+) \neq 0$ then $M \neq \phi$ moreover M is a C^1-manifold.

For the proof of theorem 3.1, we first need to show the following lemmas.

Lemma 3.2. *(i) F' is completely continuous in X.*
(ii) G'_ε satisfies the (S^+) condition that is if $(v_n)_n$ is a sequence in X such that

$$v_n \rightharpoonup v \quad \textit{and} \quad \limsup_{n \to +\infty} < G'_\varepsilon(v_n), v_n - v > \leq 0$$

then $v_n \to v$ strongly in X.

Proof. (i) Firstly, we verify that the functional F' is well defined for $m \in L^r(\Omega)$ with r satisfying the conditions (1.2). For all $u, v \in X$, by Hölder inequality, we obtain

$$\Big| \int_\Omega m|u|^{p-2} u.v dx \Big| \leq \begin{cases} \|m\|_r \|u\|_s^{p-1} \|v\|_{p_2^*} & \text{if } \frac{N}{p} > 2 \\ \|m\|_r \|u\|_p^{p-1} \|v\|_s & \text{if } \frac{N}{p} = 2 \\ \|m\|_1 \|u\|_\infty^{p-1} \|v\|_\infty & \text{if } \frac{N}{p} < 2 \end{cases}$$

where s is defined as follow, there exists s such that

$$\frac{p-1}{s} = 1 - \frac{1}{r} - \frac{1}{p_2^*} \quad \text{if } \frac{N}{p} > 2$$
$$s \geq p \quad \text{if } \frac{N}{p} = 2$$

and where $p_2^* = \frac{Np}{N-2p}$. By Sobolev's imbedding theorem (cf [1]) F' is well defined. Now, we show that F' is completely continuous. Let $(u_n) \subset X$ be a sequence such that $u_n \rightharpoonup u$ weakly in X. We have to show that

$$\sup_{v\in X,\, \|v\|_{2,p}\leq 1} \Big| \int_\Omega m[|u_n|^{p-2}u_n - |u|^{p-2}u]v\,dx\Big| \to 0, \quad \text{as } n \to +\infty.$$

We distinguish three cases: (i) $\frac{N}{p} > 2$ and $r > \frac{N}{2p}$; (ii) $\frac{N}{p} = 2$ and $r > 1$; (iii) $\frac{N}{p} < 2$ and $r = 1$.

In case (i), we know that for $\frac{N}{p} > 2$ and $r > \frac{N}{2p}$, there exists $s \in [1, p_2^*[$ such that for all $u, v \in X$,

$$|\int_\Omega m|u|^{p-2}u.vdx| \leq \|m\|_r \|u\|_s^{p-1} \|v\|_{p_2^*}.$$

Then

$$\begin{aligned}
&\sup_{v\in X(\Omega),\, \|v\|_{2,p}\leq 1} |\int_\Omega m[|u_n|^{p-2}u_n - |u|^{p-2}u]vdx| \\
&\leq \sup_{v\in X,\, \|v\|_{2,p}\leq 1} [\|m\|_r \||u_n|^{p-2}u_n - |u|^{p-2}u\|_{\frac{s}{p-1}} \|v\|_{p_2^*}] \\
&\leq c\|m\|_r \||u_n|^{p-2}u_n - |u|^{p-2}u\|_{\frac{s}{p-1}},
\end{aligned}$$

where c is the constant of Sobolev's imbedding [1]. The Nemytskii's operator $u \mapsto |u|^{p-2}u$ is continuous from $L^s(\Omega)$ into $L^{\frac{s}{p-1}}(\Omega)$, and $u_n \rightharpoonup u$ in $X \subset W^{2,p}(\Omega)$. However, $W^{2,p}(\Omega) \hookrightarrow L^s(\Omega)$ then $u_n \rightharpoonup u$ in $L^s(\Omega)$ from where we get

$$\big\||u_n|^{p-2}u_n - |u|^{p-2}u\big\|_{\frac{s}{p-1}} \to 0, \quad \text{as } n \to +\infty.$$

The cases (ii) and (iii) can be treated similarly. The proof of (i) is complete.

(ii) We show that G'_ε satisfies the (S^+) condition: Let $(u_n)_n$ be a sequence in X such that $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle G'_\varepsilon(u_n), u_n - u\rangle \leq 0$. On one hand, we have

$$\limsup_{n\to+\infty}\langle G'_\varepsilon(u_n), u_n - u\rangle = \limsup_{n\to+\infty}\langle G'_\varepsilon(u_n) - G'_\varepsilon(u), u_n - u\rangle$$

On the other hand,

$$\begin{aligned}
&\langle G'_\varepsilon(u_n) - G'_\varepsilon(u), u_n - u\rangle \\
&= \|\Delta u_n\|_p^p + \|\Delta u\|_p^p - \int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta u dx - \int_\Omega |\Delta u|^{p-2}\Delta u \Delta u_n dx \\
&\quad + \|u_n\|_p^p + \|u\|_p^p - \varepsilon\int_\Omega |u_n|^{p-2}u_n.udx - \varepsilon\int_\Omega |u|^{p-2}u.u_n dx \\
&\geq (\|\Delta u_n\|_p^{p-1} - \|\Delta u\|_p^{p-1})(\|\Delta u_n\|_p - \|\Delta u\|_p) \\
&\quad + \varepsilon(\|u_n\|_p^{p-1} - \|u\|_p^{p-1})(\|u_n\|_p - \|u\|_p) \geq 0.
\end{aligned}$$

Then $\|u_n\|_p \to \|u\|_p$ and $\|\Delta u_n\|_p \to \|\Delta u\|_p$. This completes the proof. $\square$

Lemma 3.3. *(i) G'_ε is of class C^1 on M, even and bounded below.*
(ii) For all $n \in \mathbb{N}^*$, $\Gamma_n \neq \phi$.
(iii) G_ε satisfies the Palais Smale condition on M.

Proof. (i) It is easy to see that (i) is satisfied.
(ii) Since $\operatorname{meas}(\Omega^+) \neq 0$, there exists $u_1, u_2, \dots, u_n \in X$ such that $\operatorname{supp} u_i \cap \operatorname{supp} u_j = \phi$ if $i \neq j$ and $\int_\Omega m|u_i|^p dx = 1$ for every $i \in \{1, 2, \dots, n\}$.

Let $F_n = \operatorname{span}\{u_1, u_2, \dots, u_n\}$. F_n is a vectorial subspace, $\dim F_n = n$ and for all $n \in F_n$, there exists $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ such that $u = \sum_{i=1}^n \alpha_i u_i$. Thus $F(u) = \sum_{i=1}^n |\alpha_i|^p F(u_i) = \frac{1}{p} \sum_{i=1}^n |\alpha_i|^p$. It follows that the map $u \mapsto (pF(u))^{1/p}$ defines a norm on F_n. Consequently, there exists a constant $c > 0$ such that

$$c\|u\|_{2,p} \leq (pF(u))^{1/p} \leq \frac{1}{c}\|u\|_{2,p}.$$

Set $B = F_n \cap \{u \in X/(pF(u))^{1/p} = 1\}$. B is the unit sphere of F_n, B is closed, compact and symmetric then the genus of B, $\gamma(B) = n$. Therefore, $B \in \Gamma_n$ and the result holds.

(iii) G_ε satisfies the Palais Smale condition on M. Indeed, let $(u_n)_n \subset M$ such that $(G_\varepsilon(u_n))_n$ is bounded and $G'_\varepsilon(u_n) \to 0$. We show that $(u_n)_n$ has a subsequence which converges strongly. It is clear that G_ε is coercive then $(u_n)_n$ is bounded. For a subsequence still denoted by $(u_n)_n$, we have $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$.

Since G'_ε is of (S^+) type then it suffices to show that $\limsup_{n\to+\infty} < G'_\varepsilon(u_n), u_n - u > \leq 0$. Set $t_n = \frac{\langle G'_\varepsilon(u_n), u_n\rangle}{\langle F'(u_n), u_n\rangle}$ then $\alpha_n \to 0, n \to +\infty$ where $\alpha_n = G'_\varepsilon(u_n) - t_n F'(u_n)$, hence $\beta_n = \langle \alpha_n, u\rangle \to 0$. On the other hand,

$$\begin{aligned}
\langle G'_\varepsilon(u_n), u_n - u\rangle &= \langle G'_\varepsilon(u_n), u_n\rangle - \langle G'_\varepsilon(u_n), u\rangle \\
&= pG_\varepsilon(u_n) - \beta_n - t_n\langle F'(u_n), u\rangle \\
&= pG_\varepsilon(u_n)(1 - \langle F'(u_n), u\rangle) - \beta_n
\end{aligned}$$

Since $(G_\varepsilon(u_n))_n$ is bounded; i.e., $G_\varepsilon(u_n) \to c$ and $\beta_n \to 0$, it follows that

$$\limsup_{n\to+\infty} \langle G'_\varepsilon(u_n), u_n - u\rangle \leq pc \limsup_{n\to+\infty} (1 - \langle F'(u_n), u\rangle).$$

However,

$$1 - < F'(u_n), u > = < F'(u_n), u_n - u > \longrightarrow 0$$

then

$$\limsup_{n\to+\infty} \langle G'_\varepsilon(u_n), u_n - u\rangle \leq 0.$$

From where, we conclude that $(u_n)_n$ is convergent. The result then holds. $\square$

Proof of theorem 3.1. By Lemma 3.3 and theorem 2.3, we conclude that G_ε has n critical points $\lambda_{n,\varepsilon}$ given by

$$\lambda_{n,\varepsilon} = \inf_{K\in\Gamma_n} \sup_{v\in K} (pG_\varepsilon) \quad \forall n \in \mathbb{N}^*. \tag{3.4}$$

It is not difficult to verify that for all $n \in N^*$, $\lambda_{n,\varepsilon}$ is an eigenvalue of problem (3.1).

Now we prove that $\lambda_{n,\varepsilon} \to +\infty$. We proceed in the same way as in Szulkin [7]. Since X is separable, there exists a biorthogonal system $(e_n, e^*_m)_{n,m}$ such that

$e_n \in X$ and $e_m^* \in X'$. The e_n are linearly dense in X and the e_m^* are total for X'. For $k \in N^*$, set

$$F_k = \operatorname{span}\{e_1, \dots, e_k\}, \quad F_k^{\perp} = \operatorname{span}\{e_{k+1,e_{k+2},\dots}\}.$$

Then by assertion (7) of Lemma 2.2: for all $K \in \Gamma_k$, $K \cap F_k^{\perp} \neq \phi$. Thus

$$l_k := \inf_{K\in\Gamma_k} \sup_{u\in K\cap F_{k-1}^{\perp}} pG_\varepsilon(u) \to +\infty.$$

Indeed, if not, there exists $N > 0$ such that for every $k \in \mathbb{N}^*$, there exists $u_k \in F_{k-1}^{\perp}$ which verifies $pF(u_k) = 1$ and $l_k \leq pG_\varepsilon(u_k) \leq N$, this implies that $(u_k)_{k\geq 1}$ is bounded in X. For a subsequence still denoted $(u_k)_{k\geq 1}$, we can assume that $u_k \rightharpoonup u$ in X and $u_k \to u$ in $L^p(\Omega)$. However, for all $k > n$, $\langle e_n^*, e_k\rangle = 0$ then $u_k \to 0$. This contradicts the fact: $pF(u_k) = 1$ for all k. Since $\lambda_{k,\varepsilon} \geq l_k$, we obtain $\lambda_{n,\varepsilon} \to +\infty$. This achieves the proof. $\square$

In the following lemma, we show that when $\varepsilon \to 0$, $\lambda_{n,\varepsilon}$ converges to λ_n given by

$$\lambda_n = \inf_{K\in\Gamma_n} \sup_{u\in K} pG(u), \tag{3.5}$$

where $G(u) = \frac{1}{p}\|\Delta u\|_p^p$.

Lemma 3.4. *With the above notation,*

$$\lim_{\varepsilon\to 0} \lambda_{n,\varepsilon} = \lambda_n$$

Proof. Set $\varepsilon = 1/k$; $k \in \mathbb{N}^*$ and Let $\alpha > 0$ such that $\lambda_n < \alpha$. From the definition of λ_n, there exists $K = K(\alpha) \in \Gamma_n$ such that

$$\lambda_n \leq \sup_{u\in K} pG(u) < \alpha.$$

On the other hand,

$$\lambda_n \leq \lambda_{n,\varepsilon} \leq \sup_{u\in K} pG_\varepsilon(u) \leq \sup_{u\in K} pG(u) + \varepsilon \sup_{u\in K} \|u\|_p^p.$$

let $\varepsilon \to 0$ then there exists $N_\alpha > 0$ such that for all $k \geq N_\alpha$: $\sup_{u\in K} pG(u) + \varepsilon \sup_{u\in K} \|u\|_p^p < \alpha$. Thus for all $\alpha > 0$ there exists $N_\alpha > 0$ such that for all $k \geq N_\alpha$: $\lambda_n \leq \lambda_{n,\varepsilon} \leq \alpha$. This completes the proof. $\square$

Proof of theorem 1.1. Let $k \in \mathbb{N}^*$ and set $\varepsilon = \frac{1}{k}$. There exists a sequence $(u_k)_{k\in\mathbb{N}^*}$ of eigenvalues associated with $\lambda_{n,k}, k \in \mathbb{N}^*$ such that $pG_k(u_k) = 1$ then $(u_k)_k$ is bounded in X. For a subsequence still denoted $(u_k)_k$, we can assume that $u_k \rightharpoonup u$ in X and $u_k \to u$ in $L^p(\Omega)$. Since the operator $G' + J : W^{2,p}(\Omega) \to (W^{2,p}(\Omega))'$ is of type (S^+) and is an homeomorphism then $u_k \to u$. However, $G'(u_k)+\frac{1}{k}|u_k|^{p-2}u_k = \lambda_{n,k}F'(u_k)$ and F' is strongly continuous on X , it follows that $G'(u)= \lambda_n F'(u)$ the result then hold. The assertion $\lambda_n \to +\infty$ can be proved in the same way as for $\lambda_{n,\varepsilon}$. $\square$

Remark 3.5. (i) The existence of solutions for nonlinear eigenvalue problems with weight holds under some conditions on F and G. For example, the coercivity of the functional G is of main importance to establish the desired results. In the cases where this condition is not satisfied, we often use a perturbation of the principal problem as above.

(ii)It is easy to see that λ_1 is defined as follows:

$$\lambda_1 = \inf\{\|\Delta u\|_p^p : u \in X \text{ and } \int_\Omega m(x)|u|^p dx = 1\}. \tag{3.6}$$

This can be deduced from the formula (3.5) . For the proof, one can see for example [5].

Proof of theorem 1.2. (i) We distinguish two cases:

Case 1: $\int_\Omega m(x)dx > 0$. In this case there exists a constant $c > 0$ such that $\int_\Omega mc^p dx = 1$. Thus $0 \le \lambda_1 \le \|\Delta c\|_p^p = 0$ then $\lambda_1 = 0$.

Case 2: $\int_\Omega m(x)dx = 0$. Let us consider the functional $\Phi : W^{2,p}(\Omega) \to \mathbb{R}$ defined as $\Phi(u) = \|\Delta u\|_p^p - \lambda_1 \int_\Omega m(x)|u|^p dx$. Φ is weakly lower semi continuous, positive and of class C^1. Moreover, $u_0 \equiv 1$ is a minimum of Φ then $\Phi'(u_0) = 0$, i.e $\Delta_p^2 u_0 = \lambda_1 m(x) = 0$. Or meas$(\Omega^+) \neq 0$ then $\lambda_1 = 0$.

(ii) If $\int_\Omega m(x)dx < 0$ then $\lambda_1 > 0$. Indeed, there exists a sequence $(u_n)_n \subset X$ such that

$$\|\Delta u_n\|_p^p \to \lambda_1 \quad \text{as } n \to +\infty \quad \text{and} \quad \int_\Omega m(x)|u_n|^p dx = 1. \tag{3.7}$$

$(u_n)_n$ is bounded. Indeed, if not, set $v_n = \frac{u_n}{\|u_n\|_{2,p}}$. It's clear that $(v_n)_n$ is bounded in X then for a subsequence still denoted $(v_n)_n$, v_n converges weakly to a limit v in X and strongly to v in $L^p(\Omega)$ and we have

$$\|v\|_{2,p} \le \liminf_{n\to+\infty}[(\int_\Omega |\Delta v_n|^p dx + \int_\Omega |v_n|^p dx)^{1/p}].$$

Or

$$\int_\Omega |\Delta v_n|^p dx = \frac{\int_\Omega |\Delta u_n|^p dx}{\|u_n\|_{2,p}^p} \to 0 \quad \text{as } n \to +\infty,$$

then

$$\|v\|_{2,p} \le \liminf_{n\to+\infty}(\int_\Omega |v_n|^p dx)^{1/p}.$$

i.e.,

$$\|\Delta v\|_p^p + \|v\|_p^p \le \|v\|_p^p$$

i.e., $\|\Delta v\|_p^p = 0$ thus $\Delta v = 0$.

By applying the Green formula we have

$$\int_\Omega v\Delta v dx + \int_\Omega \nabla v \nabla v dx = \int_{\partial\Omega} v.\frac{\partial v}{\partial \nu} d\sigma$$

where ν is the outre normal derivative. However, $v \in X$ then $\frac{\partial v}{\partial \nu} = 0$ in $\partial\Omega$ then it follows that

$$\int_\Omega \nabla v \nabla v dx = \int_\Omega |\nabla v|^2 dx = 0,$$

i.e., $v = c \neq 0$ is constant. On the other hand, we have

$$\int_\Omega m|v_n|^p dx = \frac{\int_\Omega m|u_n|^p dx}{\|u_n\|_{2,p}^p} = \frac{1}{\|u_n\|_{2,p}^p} \to 0, \quad \text{as } n \to +\infty$$

and

$$\int_\Omega m|v_n|^p dx \to \int_\Omega m|v|^p dx = 0,$$

then, since v is constant it follows that $\int_\Omega m dx = 0$ which is impossible. Then $(u_n)_n$ is bounded in X. For a subsequence still denoted by $(u_n)_n$, $u_n \to u$ in $L^p(\Omega)$ and $u_n \rightharpoonup u$ in X. By passing to the limit in (3.7), we obtain

$$\lambda_1 = \|\Delta u\|_p^p \quad \text{and} \quad \int_\Omega m|u|^p dx = 1.$$

We remark that $\Delta u \neq 0$. If not, we obtain $u = c$ is a constant and $\int_\Omega m(x)dx > 0$ which is impossible. Then $\lambda_1 > 0$.

Let us now show that $\lambda_1 > 0$ is the first eigenvalue associated to the problem (1.1) and that u_1 is an eigenfunction associated to λ_1 if and only if

$$G(u_1) - \lambda_1 F(u_1) = 0 = \inf_{u \in X \setminus \{0\}} (G(u) - \lambda_1 F(u)).$$

Indeed, Let $n, m \in \mathbb{N}^*$ such that $n \leq m$ then $\Gamma_m \subset \Gamma_n$. Since

$$\lambda_m = \inf_{K \in \Gamma_m} \sup_{u \in K} pG(u),$$

it follows that $\lambda_m \geq \lambda_n$. Thus

$$0 < \lambda_1 \leq \lambda_2 \leq \dots . \leq \lambda_n.$$

Let u_1 be an eigenfunction associated to λ_1. Without loss of generality, we can assume that $u_1 \in M$, then the infimum is achieved at u_1; i.e., $\lambda_1 = \inf_{u \in M} pG(u) = pG(u_1)$; i.e., $\lambda_1 F(u_1) = G(u_1)$. Hence

$$G(u_1) - \lambda_1 F(u_1) = 0 = \inf_{u \in X \setminus \{0\}} (G(u) - \lambda_1 F(u)).$$

Suppose now that there exists $\lambda \in]0, \lambda_1[$ with λ is an eigenvalue of problem (1.1) and let v be an eigenfunction associated to λ then

$$G(u_1) - \lambda_1 F(u_1) = 0 \leq G(v) - \lambda_1 F(v) < G(v) - \lambda F(v) = 0$$

which is impossible. Thus λ_1 is the first eigenvalue associated to problem (1.1). $\square$

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