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SOLVABILITY OF A NONLINEAR THIRD-ORDER THREE-POINT GENERAL EIGENVALUE PROBLEM ON TIME SCALES

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Abstract. We study the existence of eigenvalue intervals for the third-order nonlinear three-point boundary value problem on time scales satisfying general boundary conditions. Values of a parameter are determined for which the boundary value problem has a positive solution by utilizing a fixed point theorem on a cone in a Banach space.

1. Introduction

The study of obtaining optimal eigenvalue intervals for the existence of positive solutions to boundary value problems(BVPs) on time scales has gained prominence and is a rapidly growing field, since it arises in many applications. By a time scale we mean a nonempty closed subset of $\mathbb{R}$. For an excellent introduction to the overall area of dynamic equations on time scales, we refer to the text book by Bohner and Peterson [5].

In this paper, we focus on determining the eigenvalue intervals for which there exists a positive solution to the third order boundary value problem on time scales

$$y^{\Delta^3}(t) + \lambda f(t, y(t), y^{\Delta}(t), y^{\Delta^2}(t)) = 0, \quad t \in [t_1, \sigma^3(t_3)] \tag{1.1}$$

satisfying the general three point boundary conditions

$$\begin{gathered} \alpha_{11}y(t_1) + \alpha_{12}y^{\Delta}(t_1) + \alpha_{13}y^{\Delta^2}(t_1) = 0 \\ \alpha_{21}y(t_2) + \alpha_{22}y^{\Delta}(t_2) + \alpha_{23}y^{\Delta^2}(t_2) = 0 \\ \alpha_{31}y(\sigma^3(t_3)) + \alpha_{32}y^{\Delta}(\sigma^2(t_3)) + \alpha_{33}y^{\Delta^2}(\sigma(t_3)) = 0 \end{gathered} \tag{1.2}$$

where $t_1 < t_2 < \sigma^3(t_3)$ and α_{ij}, for $i, j = 1, 2, 3$ are real constants. The BVPs of this form arise in the modelling of nonlinear diffusion via nonlinear sources, thermal ignition of gases, and in chemical concentrations in biological problems. In these applied settings, only positive solutions are meaningful.

Optimal eigenvalue intervals were obtained for the existence of positive solutions of boundary value problems for ordinary differential equations, as well as for finite

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difference equations using the Krasnosel'skii fixed point theorem [20] on a cone. A few papers along these lines are Agarwal, Bohner and Wang [2], Anderson and Davis [4], Davis, Eloe and Henderson [9], Davis, Henderson, Prasad and Yin [10, 11], Eloe and Henderson [12], Erbe and Tang [16], Henderson and Wang [18], Jiang and Liu [19], Prasad and Murali [21]. Recently, Prasad and Rao [22] extended these results to third order general three point boundary value problem.

In order to unify the results on differential equations and difference equations, the theory of dynamical equations on time scales is being developed. It has a great potential in nonlinear analysis and its applications in the modeling of physical and biological systems. Some papers on boundary value problems on time scales are Chyan and Henderson [6], Chyan, Davis, Henderson and Yin [7], DaCunha, Davis and Singh [8] and Erbe and Peterson [13, 14, 15]. This paper generalizes many papers in the literature. By choosing different values to the constants in the boundary conditions we get various three point BVPs.

For simplicity we make the following notation: $\beta_i = \alpha_{i1}t_i + \alpha_{i2}$, $\gamma_i = \alpha_{i1}t_i^2 + \alpha_{i2}(t_i + \sigma(t_i)) + 2\alpha_{i3}$, for $i = 1, 2$, $\beta_3 = \alpha_{31}\sigma^3(t_3) + \alpha_{32}$ and $\gamma_3 = \alpha_{31}(\sigma^3(t_3))^2 + \alpha_{32}(\sigma^2(t_3) + \sigma^3(t_3)) + 2\alpha_{33}$. We define

$$m_{ij} = \frac{\alpha_{i1}\gamma_j - \alpha_{j1}\gamma_i}{2(\alpha_{i1}\beta_j - \alpha_{j1}\beta_i)}; \quad M_{ij} = \frac{\beta_i\gamma_j - \beta_j\gamma_i}{\alpha_{i1}\beta_j - \alpha_{j1}\beta_i}.$$

Also let

$$m_1 = \max\{m_{12}, m_{13}, m_{23}\},$$
$$m_2 = \min\{m_{23} + \sqrt{m_{23}^2 - M_{23}}; m_{13} + \sqrt{m_{13}^2 - M_{13}}\},$$
$$d = \alpha_{11}(\beta_2\gamma_3 - \beta_3\gamma_2) - \beta_1(\alpha_{21}\gamma_3 - \alpha_{31}\gamma_2) + \gamma_1(\alpha_{21}\beta_3 - \alpha_{31}\beta_2),$$
$$l_i = \alpha_{i1}\sigma(s)\sigma^2(s) - (\sigma(s) + \sigma^2(s))\beta_i + \gamma_i \quad \text{for } i = 1, 2, 3.$$

Let us assume that

- (A1) $f : [t_1, \sigma^3(t_3)] \times \mathbb{R}^{+^3} \to \mathbb{R}^+$ is continuous;
- (A2) $\alpha_{11} > 0$, $\alpha_{21} > 0$, $\alpha_{31} > 0$ and $\frac{\alpha_{12}}{\alpha_{11}} > \frac{\alpha_{22}}{\alpha_{21}} > \frac{\alpha_{32}}{\alpha_{31}}$;
- (A3) $m_1 \le t_1 < t_2 < t_3 \le m_2$; $2\alpha_{13}\alpha_{11} > \alpha_{12}^2$, $2\alpha_{23}\alpha_{21} > \alpha_{22}^2$, $2\alpha_{33}\alpha_{31} > \alpha_{32}^2$;
- (A4) $m_{23}^2 > M_{23}$, $m_{12}^2 < M_{12}$, $m_{13}^2 > M_{13}$, $d > 0$ and
- (A5) The point $t \in [t_1, \sigma^3(t_3)]$ is not left dense and right scattered at the same time.

Define the nonnegative extended real numbers $f_0, f^0, f_\infty, f^\infty$ by

$$f_0 = \lim_{y\to 0^+, y^\Delta\to 0^+, y^{\Delta^2}\to 0^+} \min_{t\in[t_1,\sigma^3(t_3)]} \frac{f(t, y, y^\Delta, y^{\Delta^2})}{y},$$
$$f^0 = \lim_{y\to 0^+, y^\Delta\to 0^+, y^{\Delta^2}\to 0^+} \max_{t\in[t_1,\sigma^3(t_3)]} \frac{f(t, y, y^\Delta, y^{\Delta^2})}{y},$$
$$f_\infty = \lim_{y\to \infty, y^\Delta\to \infty, y^{\Delta^2}\to \infty} \min_{t\in[t_1,\sigma^3(t_3)]} \frac{f(t, y, y^\Delta, y^{\Delta^2})}{y},$$
$$f^\infty = \lim_{y\to \infty, y^\Delta\to \infty, y^{\Delta^2}\to \infty} \max_{t\in[t_1,\sigma^3(t_3)]} \frac{f(t, y, y^\Delta, y^{\Delta^2})}{y}$$

and assume that they will exist. By an interval we mean the intersection of the real interval with a given time scale.

This paper is organized as follows. In Section 2, we construct Green's function for the corresponding homogeneous problem of (1.1)-(1.2) and estimate bounds of the Green's function. In Section 3, we present a lemma which is needed in further discussion and determine eigenvalue intervals for which (1.1)-(1.2) has at least one positive solution, by using Krasnosel'skii fixed point theorem. Finally as an application, we give an example to demonstrate our result.

2. Green's function and Bounds

In this section, we construct the Green's function for the corresponding homogeneous problem of (1.1)-(1.2) in six different intervals and we estimate the bounds for the Green's function.

Let $G(t,s)$ be the Green's function for the problem $-y^{\Delta^3}(t)=0$ satisfying (1.2). After computation, the Green's function $G(t,s)$ can be obtained as

$$G(t,s)=\begin{cases} G_{11}(t,s), & t_1\le t<s<t_2<\sigma^3(t_3),\\ G_{12}(t,s), & t_1<\sigma(s)<t\le t_2<\sigma^3(t_3),\\ G_{13}(t,s), & t_1\le t<t_2<s<\sigma^3(t_3),\\ G_{21}(t,s), & t_1<t_2\le t<s<\sigma^3(t_3),\\ G_{22}(t,s), & t_1<t_2<\sigma(s)<t\le\sigma^3(t_3),\\ G_{23}(t,s), & t_1\le\sigma(s)<t_2<t<\sigma^3(t_3),\end{cases} \tag{2.1}$$

where

$$\begin{aligned}
G_{11}(t,s)=&\frac{1}{2d}[-(\beta_1\gamma_3-\beta_3\gamma_1)+t(\alpha_{11}\gamma_3-\alpha_{31}\gamma_1)-t^2(\alpha_{11}\beta_3-\alpha_{31}\beta_1)]l_2\\
&+\frac{1}{2d}[(\beta_1\gamma_2-\beta_2\gamma_1)-t(\alpha_{11}\gamma_2-\alpha_{21}\gamma_1)+t^2(\alpha_{11}\beta_2-\alpha_{21}\beta_1)]l_3\\
G_{12}(t,s)=&\frac{1}{2d}[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{21}\gamma_3-\alpha_{31}\gamma_2)-t^2(\alpha_{21}\beta_3-\alpha_{31}\beta_2)]l_1\\
G_{13}(t,s)=&\frac{1}{2d}[(\beta_1\gamma_2-\beta_2\gamma_1)-t(\alpha_{11}\gamma_2-\alpha_{21}\gamma_1)+t^2(\alpha_{11}\beta_2-\alpha_{21}\beta_1)]l_3\\
G_{21}(t,s)=&\frac{1}{2d}[(\beta_1\gamma_2-\beta_2\gamma_1)-t(\alpha_{11}\gamma_2-\alpha_{21}\gamma_1)+t^2(\alpha_{11}\beta_2-\alpha_{21}\beta_1)]l_3\\
G_{22}(t,s)=&\frac{1}{2d}[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{21}\gamma_3-\alpha_{31}\gamma_2)-t^2(\alpha_{21}\beta_3-\alpha_{31}\beta_2)]l_1\\
&+\frac{1}{2d}[(\beta_1\gamma_3-\beta_3\gamma_1)-t(\alpha_{11}\gamma_3-\alpha_{31}\gamma_1)+t^2(\alpha_{11}\beta_3-\alpha_{31}\beta_1)]l_2\\
G_{23}(t,s)=&\frac{1}{2d}[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{21}\gamma_3-\alpha_{31}\gamma_2)-t^2(\alpha_{21}\beta_3-\alpha_{31}\beta_2)]l_1
\end{aligned}$$

Figure 2 indicates that the Green's function for (1.1)-(1.2) should take the form of (2.1), where $s\in[t_1,t_3]$.

Theorem 2.1. *Assume that the conditions* (A2)-(A4) *are satisfied. Then*

$$\gamma G(\sigma(s),s)\le G(t,s)\le G(\sigma(s),s),\quad \text{for all } (t,s)\in[t_1,\sigma^3(t_3)]\times[t_1,t_3], \tag{2.2}$$

where

$$0<\gamma=\min\Big\{\frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)},\frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)},\frac{G_{11}(t_1,s)}{G_{11}(\sigma^3(t_3),s)},\frac{G_{11}(\sigma^3(t_3),s)}{G_{11}(t_1,s)}\Big\}<1.$$

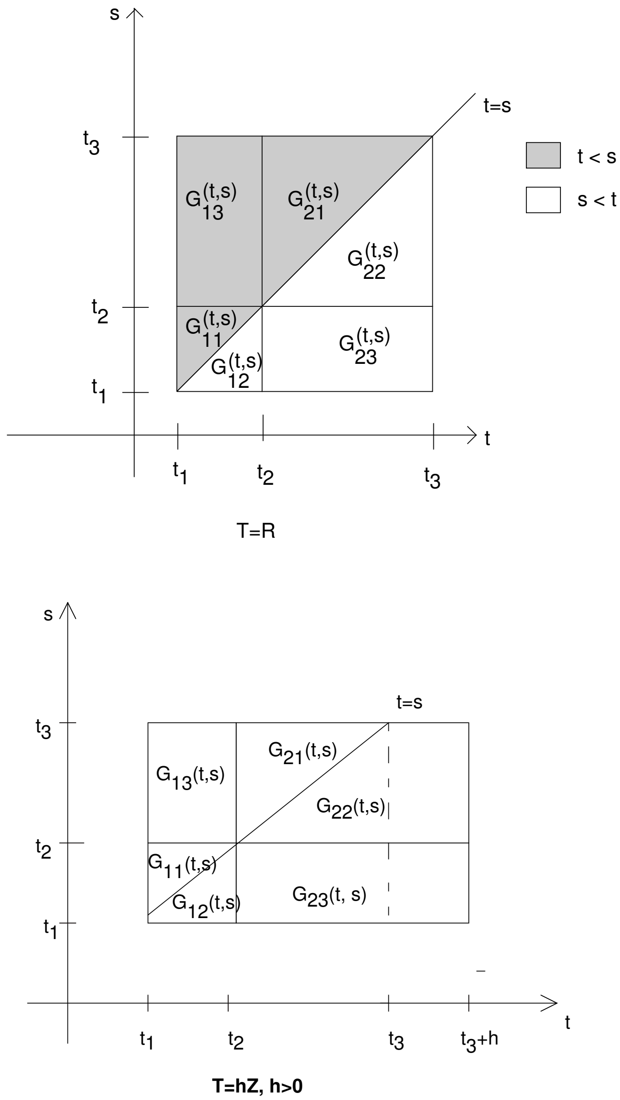

FIGURE 1. Representation of Green's function in six intervals

Proof. The Green's function $G(t,s)$ is given in (2.1) in six different cases. In each case we prove the inequality as in (2.2). Clearly

$$G(t,s) > 0 \quad \text{on } [t_1, \sigma^3(t_3)] \times [t_1, t_3]. \tag{2.3}$$

Case (i). For $t_1 < \sigma(s) < t \le t_2 < \sigma^3(t_3)$,

$$\begin{aligned}
\frac{G(t,s)}{G(\sigma(s),s)} &= \frac{G_{12}(t,s)}{G_{12}(\sigma(s),s)} \\
&= \frac{[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{21}\gamma_3 - \alpha_{31}\gamma_2) - t^2(\alpha_{21}\beta_3 - \alpha_{31}\beta_2)]}{[-(\beta_2\gamma_3 - \beta_3\gamma_2) + \sigma(s)(\alpha_{21}\gamma_3 - \alpha_{31}\gamma_2) - (\sigma(s))^2(\alpha_{21}\beta_3 - \alpha_{31}\beta_2)]},
\end{aligned}$$

from (A3) and (A4), we have $G_{12}(t,s) \le G_{12}(\sigma(s),s)$. Therefore,

$$G(t,s) \le G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3].$$

And also, from (A2), we have

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{12}(t,s)}{G_{12}(\sigma(s),s)} \ge \frac{G_{12}(t,s)}{G_{12}(t_1,s)} \ge \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)}.$$

Therefore,

$$G(t,s) \ge \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)} G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3].$$

Case (ii). For $t_1 \le t < t_2 < s < \sigma^3(t_3)$

$$\begin{aligned}
\frac{G(t,s)}{G(\sigma(s),s)} &= \frac{G_{13}(t,s)}{G_{13}(\sigma(s),s)} \\
&= \frac{[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{11}\gamma_2 - \alpha_{21}\gamma_1) + t^2(\alpha_{11}\beta_2 - \alpha_{21}\beta_1)]}{[(\beta_1\gamma_2 - \beta_2\gamma_1) - \sigma(s)(\alpha_{11}\gamma_2 - \alpha_{21}\gamma_1) + (\sigma(s))^2(\alpha_{11}\beta_2 - \alpha_{21}\beta_1)]},
\end{aligned}$$

from, (A3) and (A4), we have $G_{13}(t,s) \le G_{13}(\sigma(s),s)$. Therefore,

$$G(t,s) \le G(\sigma(s),s) \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3].$$

Also, from (A2), we have

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{13}(t,s)}{G_{13}(\sigma(s),s)} \ge \frac{G_{13}(t,s)}{G_{13}(\sigma^3(t_3),s)} \ge \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}.$$

Therefore,

$$G(t,s) \ge \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)} G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3].$$

Case (iii). For $t_1 \le t < s < t_2 < \sigma^3(t_3)$. From (A3) and Case (ii), we have $G_{11}(t,s) \le G_{11}(\sigma(s),s)$. Therefore,

$$G(t,s) \le G(\sigma(s),s) \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3].$$

Also, from (A2), we have

$$\frac{G(t,s)}{G(\sigma(s),s)} \ge \min\Big\{\frac{G_{11}(\sigma^3(t_3),s)}{G_{11}(t_1,s)}, \frac{G_{11}(t_1,s)}{G_{11}(\sigma^3(t_3),s)}, \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}\Big\}.$$

Therefore,

$$G(t,s) \ge \min\Big\{\frac{G_{11}(\sigma^3(t_3),s)}{G_{11}(t_1,s)}, \frac{G_{11}(t_1,s)}{G_{11}(\sigma^3(t_3),s)}, \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}\Big\} G(\sigma(s),s),$$

for all $(t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3]$.

Case (iv). For $t_1 < t_2 < \sigma(s) < t \le \sigma^3(t_3)$. From Case (i) and Case (ii), we have

$$G(t,s) \le G(\sigma(s),s) \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3],$$

and

$$G(t,s) \ge \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)} G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3].$$

Case (v). For $t_1 < t_2 \le t < s < \sigma^3(t_3)$. From Case (ii), we have

$$G(t,s) \le G(\sigma(s),s) \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3],$$

and

$$G(t,s) \geq \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)} G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3].$$

Case (vi). For $t_1 \leq \sigma(s) < t_2 < t < \sigma^3(t_3)$. From Case (i), we have

$$G(t,s) \leq G(\sigma(s),s) \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3],$$

and

$$G(t,s) \geq \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)} G(\sigma(s),s), \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3].$$

By consolidating all the above cases, we have

$$\gamma G(\sigma(s),s) \leq G(t,s) \leq G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1,\sigma^3(t_3)] \times [t_1,t_3],$$

where

$$0 < \gamma = \min\Big\{\frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)}, \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}, \frac{G_{11}(t_1,s)}{G_{11}(\sigma^3(t_3),s)}, \frac{G_{11}(\sigma^3(t_3),s)}{G_{11}(t_1,s)}\Big\} < 1.$$

□

3. Existence of Positive Solutions

In this section, first we prove a lemma which is needed in our main result and establish a criteria to determine eigenvalue intervals for which there exists at least one positive solution of (1.1)-(1.2).

Definition 3.1. Let X be a Banach space. A nonempty closed convex set κ is called a *cone* of X, if it satisfies the following conditions:

(1) $\alpha_1 u + \alpha_2 v \in \kappa$, for all $u, v \in \kappa$ and $\alpha_1, \alpha_2 \geq 0$,
(2) $u \in \kappa$ and $-u \in \kappa$, implies $u = 0$.

Let $y(t)$ be the solution of (1.1)-(1.2), given by

$$y(t) = \lambda \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s)) \Delta s, \quad \text{for all } t \in [t_1, \sigma^3(t_3)]. \tag{3.1}$$

Define

$$X = \big\{u \in C^3[t_1 : \sigma^3(t_3)]\big\},$$

with norm $\|u\| = \max_{t\in[t_1,\sigma^3(t_3)]} |u(t)|$. Then $(X, \|\cdot\|)$ is a Banach space. Define a set

$$\kappa = \big\{u \in X : u(t) \geq 0 \text{ on } [t_1,\sigma^3(t_3)] \text{ and } \min_{t\in[t_1,\sigma^3(t_3)]} u(t) \geq \gamma\|u\|\big\}. \tag{3.2}$$

Then it is easy to see that κ is a positive cone in X.

Definition 3.2. Let X and Y be Banach spaces and $T : X \to Y$. T is said to be completely continuous, if T is continuous, and for each bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a convergent subsequence.

Now we define the operator $T : \kappa \to X$ by

$$(Ty)(t) = \lambda \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s)) \Delta s, \quad \text{for all } t \in [t_1, \sigma^3(t_3)]. \tag{3.3}$$

If $y \in \kappa$ is a fixed point of T, then y satisfies (3.1) and hence y is a positive solution of (1.1)-(1.2). We seek a fixed point of the operator T in the cone κ.

Lemma 3.3. *The operator T defined in* (3.3) *is a self map on κ.*

Proof. Let $y \in \kappa$. From (2.3), we have $(Ty)(t) \geq 0$, for all $t \in [t_1, \sigma^3(t_3)]$, and

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\leq \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s
\end{aligned}$$

so that

$$\|Ty\| \leq \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s$$

Next, if $y \in \kappa$, then by the above inequality we have

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\geq \gamma\lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\geq \gamma \|Ty\|.
\end{aligned}$$

Hence $T : \kappa \to \kappa$. Standard arguments involving the Arzela-Ascoli theorem shows that T is completely continuous. □

To establish eigenvalue intervals we will employ the following fixed point theorem due to Krasnosel'skii [20].

Theorem 3.4. *Let X be a Banach space, $K \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.5. *Assume that conditions (A1)-(A5) are satisfied. Then, for each λ satisfying*

$$\frac{1}{[\gamma^2 \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s] f_\infty} < \lambda < \frac{1}{[\int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s] f^0}, \tag{3.4}$$

there exists at least one positive solution of (1.1)-(1.2) *that lies in κ.*

Proof. Let λ be given as in (3.4). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{[\gamma^2 \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s](f_\infty - \epsilon)} \leq \lambda \leq \frac{1}{[\int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s](f^0 + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined in (3.3). By the definition of f^0, there exists $H_{1i} > 0$, $i = 0, 1, 2$ such that

$$\max_{t \in [t_1, \sigma^3(t_3)]} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y} \leq (f^0 + \epsilon)$$

for $0 < y \le H_{10}, 0 < y^{\Delta} \le H_{11}, 0 < y^{\Delta^2} \le H_{12}$. Let $H_1 = \min\{H_{1i} : i = 0, 1, 2\}$. It follows that, $f(t, y, y^{\Delta}, y^{\Delta^2}) \le (f^0 + \epsilon)y$, for $0 < y, y^{\Delta}, y^{\Delta^2} \le H_1$. So choosing $y \in \kappa$ with $\|y\| = H_1$, then from (2.2) we have

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\le \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\le \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)(f^0+\epsilon) y(s)\Delta s \\
&\le \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)(f^0+\epsilon) \|y\|\Delta s \\
&\le \|y\|, \quad t \in [t_1, \sigma^3(t_3)].
\end{aligned}$$

Consequently, $\|Ty\| \le \|y\|$. So, if we define $\Omega_1 = \{y \in X : \|y\| < H_1\}$, then

$$\|Ty\| \le \|y\|, \quad \text{for } y \in \kappa \cap \partial\Omega_1. \tag{3.5}$$

By the definition of f_∞, there exists $\overline{H}_{2i} > 0$, $i = 0, 1, 2$ such that

$$\min_{t \in [t_1, \sigma^3(t_3)]} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y} \ge (f_\infty - \epsilon),$$

for $y \ge \overline{H}_{20}$, $y^{\Delta} \ge \overline{H}_{21}$, $y^{\Delta^2} \ge \overline{H}_{22}$. Let $\overline{H}_2 = \min\{\overline{H}_{2i} : i = 0, 1, 2\}$. It follows that,

$$f(t, y, y^{\Delta}, y^{\Delta^2}) \ge (f_\infty - \epsilon)y, \quad \text{for } y, y^{\Delta}, y^{\Delta^2} \ge \overline{H}_2.$$

Let

$$H_2 = \max\big\{2H_1, \frac{1}{\gamma}\overline{H}_2\big\}, \quad \Omega_2 = \{y \in X : \|y\| < H_2\}.$$

Now choose $y \in \kappa \cap \partial\Omega_2$ with $\|y\| = H_2$, so that

$$\min_{t \in [t_1, \sigma^3(t_3)]} y(t) \ge \gamma\|y\| \ge \overline{H}_2.$$

Consider

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\ge \lambda \int_{t_1}^{\sigma(t_3)} \gamma G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\ge \gamma\lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)(f_\infty-\epsilon) y(s)\Delta s \\
&\ge \gamma^2\lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)(f_\infty-\epsilon) \|y\|\Delta s \\
&\ge \|y\|.
\end{aligned}$$

Thus,

$$\|Ty\| \ge \|y\|, \quad \text{for } y \in \kappa \cap \partial\Omega_2. \tag{3.6}$$

An application of Theorem 3.4 to (3.5) and (3.6) yields that T has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is the positive solution of (1.1)-(1.2) for the given λ. ☐

Theorem 3.6. *Assume that conditions* (A1)-(A5) *are satisfied. Then, for each* λ *satisfying*

$$\frac{1}{[\gamma^2 \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s] f_0} < \lambda < \frac{1}{[\int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s] f^\infty}, \tag{3.7}$$

there exists at least one positive solution of (1.1)-(1.2) *that lies in* κ.

Proof. Let λ be given in (3.7), and choose $\epsilon > 0$ such that

$$\frac{1}{[\gamma^2 \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s](f_0 - \epsilon)} \le \lambda \le \frac{1}{[\int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s](f^\infty + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.3). By the definition of f_0, there exists $J_{1i} > 0$, $i = 0,1,2$ such that

$$\min_{t\in[t_1,\sigma^3(t_3)]} \frac{f(t,y,y^\Delta,y^{\Delta^2})}{y} \ge (f_0 - \epsilon),$$

for $0 < y \le J_{10}$, $0 < y^\Delta \le J_{11}$, $0 < y^{\Delta^2} \le J_{12}$. Let $J_1 = \min\{J_{1i} : i = 0,1,2\}$. It follows that,

$$f(t,y,y^\Delta,y^{\Delta\Delta}) \ge (f_0 - \epsilon)y, \quad \text{for } 0 < y, y^\Delta, y^{\Delta^2} \le J_1.$$

So, choose $y \in \kappa$ with $\|y\| = J_1$, then

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{\sigma(t_3)} G(t,s) f(s,y(s),y^\Delta(s),y^{\Delta^2}(s))\Delta s \\
&\ge \lambda \int_{t_1}^{\sigma(t_3)} \gamma G(\sigma(s),s) f(s,y(s),y^\Delta(s),y^{\Delta^2}(s))\Delta s \\
&\ge \gamma\lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)(f_0 - \epsilon) y(s)\Delta s \\
&\ge \gamma^2\lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)(f_0 - \epsilon)\|y\|\Delta s \\
&\ge \|y\|.
\end{aligned}$$

Consequently, $\|Ty\| \ge \|y\|$. So, if we define $\Omega_1 = \{y \in X : \|y\| < J_1\}$, then

$$\|Ty\| \ge \|y\|, \quad \text{for } y \in \kappa \cap \partial\Omega_1. \tag{3.8}$$

It remains for us to consider f^∞. By the definition of f^∞, there exists $\overline{J}_{2i} > 0$, $i = 0,1,2$ such that

$$\max_{t\in[t_1,\sigma^3(t_3)]} \frac{f(t,y,y^\Delta,y^{\Delta^2})}{y} \le (f^\infty + \epsilon),$$

for $y \ge \overline{J}_{20}$, $y^\Delta \ge \overline{J}_{21}$, $y^{\Delta^2} \ge \overline{J}_{22}$, it follows that

$$f(t,y,y^\Delta,y^{\Delta^2}) \le (f^\infty + \epsilon)y, \quad \text{for } y, y^\Delta, y^{\Delta^2} \ge \overline{J}_2.$$

There are two possible cases.

Case(i). f is bounded. Suppose $L > 0$ and $\max_{t\in[t_1,\sigma^3(t_3)]} f(t, y, y^\Delta, y^{\Delta^2}) \le L$, for all $0 < y, y^\Delta, y^{\Delta^2} < \infty$. Let

$$J_2 = \max\big\{2J_1, L\lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s), s)\Delta s\big\}.$$

Then, for $y \in \kappa$ with $\|y\| = J_2$, we have

$$\begin{aligned}(Ty)(t) &= \lambda \int_{t_1}^{\sigma(t_3)} G(t, s)f(s, y(s), y^\Delta(s), y^{\Delta^2}(s))\Delta s\\ &\le \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s), s)f(s, y(s), y^\Delta(s), y^{\Delta^2}(s))\Delta s\\ &\le \lambda L \int_{t_1}^{\sigma(t_3)} G(\sigma(s), s)\Delta s\\ &\le \|y\|,\ t \in [t_1, \sigma^3(t_3)],\end{aligned}$$

so that $\|Ty\| \le \|y\|$. So, if we define $\Omega_2 = \{y \in X : \|y\| < J_2\}$, then

$$\|Ty\| \le \|y\|, \quad \text{for } y \in \kappa \cap \partial\Omega_2. \tag{3.9}$$

Case(ii). f is unbounded. Let $J_{2i} > \max\{2J_{1i}, \overline{J}_{2i}\}$, $i = 0, 1, 2$ be such that $f(t, y, y^\Delta, y^{\Delta^2}) \le f(t, J_{20}, J_{21}, J_{22})$, for $0 < y \le J_{20}$, $0 < y^\Delta \le J_{21}$, $0 < y^{\Delta^2} \le J_{22}$. Let $J_2 = \max\{J_{2i} : i = 0, 1, 2\}$. Let $y \in \kappa$ with $\|y\| = J_2$. Then

$$\begin{aligned}(Ty)(t) &= \lambda \int_{t_1}^{\sigma(t_3)} G(t, s)f(s, y(s), y^\Delta(s), y^{\Delta^2}(s))\Delta s\\ &\le \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s), s)f(s, y(s), y^\Delta(s), y^{\Delta^2}(s))\Delta s\\ &\le \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s), s)f(s, J_{20}, J_{21}, J_{22})\Delta s\\ &\le \lambda \int_{t_1}^{\sigma(t_3)} G(\sigma(s), s)(f^\infty + \epsilon)J_2\Delta s\\ &\le J_2\\ &= \|y\|, \quad t \in [t_1, \sigma^3(t_3)].\end{aligned}$$

Thus, $\|Ty\| \le \|y\|$. For this case, if we define $\Omega_2 = \{y \in X : \|y\| < J_2\}$, then

$$\|Ty\| \le \|y\|, \quad \text{for } y \in \kappa \cap \partial\Omega_2. \tag{3.10}$$

Thus, an application of Theorem 3.4 to (3.8), (3.9) and (3.10) yields that T has fixed point $y(t) \in \kappa\cap(\overline{\Omega}_2\backslash\Omega_1)$. This fixed point is the positive solution of (1.1)-(1.2) for the given λ. □

4. Example

Now, we give an example to illustrate the above result. Consider the eigenvalue problem

$$y^{\Delta^3} + \lambda y(20-19.5e^{-7y})(30-29.5e^{-5y^\Delta})(61-60e^{-3y^{\Delta^2}}) = 0,\ t \in [0, \sigma^3(1)]\cap\mathbb{T} \tag{4.1}$$

where $\mathbb{T} = \{0\} \cup \{\frac{1}{2^{n+1}} : n \in \mathbb{N}\} \cup [\frac{1}{2}, \frac{3}{2}]$, subject to the boundary conditions

$$\begin{gathered}
y(0) + \frac{4}{3}y^{\Delta}(0) + \frac{5}{4}y^{\Delta^2}(0) = 0\\
y(\frac{1}{2}) + \frac{1}{2}y^{\Delta}(\frac{1}{2}) + y^{\Delta^2}(\frac{1}{2}) = 0\\
y(\sigma^3(1)) + \frac{1}{4}y^{\Delta}(\sigma^2(1)) + \frac{1}{2}y^{\Delta^2}(\sigma(1)) = 0
\end{gathered} \tag{4.2}$$

The Green's function is

$$G(t,s) = \begin{cases} G_{11}(t,s), & 0 \le t < s < \frac{1}{2} < \sigma^3(1)\\ G_{12}(t,s), & 0 < \sigma(s) < t \le \frac{1}{2} < \sigma^3(1)\\ G_{13}(t,s), & 0 \le t < \frac{1}{2} < s < \sigma^3(1)\\ G_{21}(t,s), & 0 < \frac{1}{2} \le t < s < \sigma^3(1)\\ G_{22}(t,s), & 0 < \frac{1}{2} < \sigma(s) < t \le \sigma^3(1)\\ G_{23}(t,s), & 0 \le \sigma(s) < \frac{1}{2} < t < \sigma^3(1), \end{cases}$$

where

$$\begin{aligned}
G_{11}(t,s) &= [\frac{-5}{6} + \frac{t^2}{3}][6\sigma(s)\sigma^2(s) - 6(\sigma(s) + \sigma^2(s)) + \frac{33}{2}]\\
&\quad + [14 - 3t - 4t^2][2\sigma(s)\sigma^2(s) - \frac{5}{2}(\sigma(s) + \sigma^2(s)) + 5]\\
G_{12}(t,s) = G_{23}(t,s) &= [\frac{15}{4} - t - t^2][6\sigma(s)\sigma^2(s) - 8(\sigma(s) + \sigma^2(s)) + 15]\\
G_{13}(t,s) = G_{21}(t,s) &= [14 - 3t - 4t^2][2\sigma(s)\sigma^2(s) - \frac{5}{2}(\sigma(s) + \sigma^2(s)) + 5]\\
G_{22}(t,s) &= [\frac{15}{4} - t - t^2][6\sigma(s)\sigma^2(s) - 8(\sigma(s) + \sigma^2(s)) + 15]\\
&\quad + [\frac{5}{6} - \frac{t^2}{3}][6\sigma(s)\sigma^2(s) - 6(\sigma(s) + \sigma^2(s)) + \frac{33}{2}].
\end{aligned}$$

We found that $\gamma = 0.4666$, $f_\infty = 36600$, and $f^0 = 0.25$. Employing Theorem 3.5, we obtain the optimal eigenvalue interval $0.0000089125 < \lambda < 0.566972$, for which (4.1)-(4.2) has a positive solution.

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