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A GLOBAL CURVE OF STABLE, POSITIVE SOLUTIONS FOR A p-LAPLACIAN PROBLEM

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Abstract. We consider the boundary-value problem

$$-\phi_p(u'(x))' = \lambda f(x, u(x)), \quad x \in (0,1),$$
$$u(0) = u(1) = 0,$$

where $p > 1$ ($p \neq 2$), $\phi_p(s) := |s|^{p-1} \operatorname{sign} s$, $s \in \mathbb{R}$, $\lambda \geq 0$, and the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies

$$f(x, \xi) > 0, \quad (x, \xi) \in [0,1] \times \mathbb{R},$$
$$(p-1) f(x, \xi) \geq f_\xi(x, \xi)\xi, \quad (x, \xi) \in [0,1] \times (0, \infty).$$

These assumptions on f imply that the trivial solution $(\lambda, u) = (0, 0)$ is the only solution with $\lambda = 0$ or $u = 0$, and if $\lambda > 0$ then any solution u is *positive*, that is, $u > 0$ on $(0,1)$.

We prove that the set of nontrivial solutions consists of a C^1 curve of positive solutions in $(0, \lambda_{\max}) \times C^0[0,1]$, with a parametrisation of the form $\lambda \to (\lambda, u(\lambda))$, where u is a C^1 function defined on $(0, \lambda_{\max})$, and $\lambda_{\max}$ is a suitable weighted eigenvalue of the p-Laplacian ($\lambda_{\max}$ may be finite or ∞), and u satisfies

$$\lim_{\lambda \to 0} u(\lambda) = 0, \qquad \lim_{\lambda \to \lambda_{\max}} |u(\lambda)|_0 = \infty.$$

We also show that for each $\lambda \in (0, \lambda_{\max})$ the solution $u(\lambda)$ is globally asymptotically stable, with respect to positive solutions (in a suitable sense).

1. Introduction

We consider the boundary-value problem

$$-\phi_p(u'(x))' = \lambda f(x, u(x)), \quad x \in (0,1), \tag{1.1}$$
$$u(0) = u(1) = 0, \tag{1.2}$$

where $p > 1$ ($p \neq 2$), $\phi_p(s) := |s|^{p-1} \operatorname{sign} s$, $s \in \mathbb{R}$, $\lambda \geq 0$, and the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies

$$f(x, \xi) > 0, \quad (x, \xi) \in [0,1] \times \mathbb{R}, \tag{1.3}$$
$$(p-1) f(x, \xi) \geq f_\xi(x, \xi)\xi, \quad (x, \xi) \in [0,1] \times (0, \infty). \tag{1.4}$$

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The condition (1.3) ensures that the trivial solution $(\lambda, u) = (0, 0)$ is the only solution with $\lambda = 0$ or $u = 0$, and if $\lambda > 0$ then any solution u is *positive*, that is, $u > 0$ on $(0, 1)$.

In the semilinear case ($p = 2$) the problem (1.1)–(1.2) has been considered in many papers, under various hypotheses on f, see for example [9, 12, 14, 17, 21, 22, 23, 24, 26, 27]. When f is independent of x, detailed results for this case are obtained in [9] and [24]. These papers use quadrature to derive explicit formulae for a C^1 curve of solutions in $[0, \infty) \times C^0[0, 1]$, passing through $(\lambda, u) = (0, 0)$, with a parametrisation of the form $s \to (\lambda(s), u(s))$, where the parameter $s = |u(s)|_0$. The results on the shape of the solution curve are then obtained by investigating the function $s \to \lambda(s)$. However, when f depends on x, such a formula for the solutions is not available. Despite this, curves of solutions, with similar properties to those in the x independent case, have been constructed in, for example, [12, Section 4], and [17, 21, 22, 23, 24, 26, 27] (again under a variety of hypotheses on f). In these papers the strategy is to use the implicit function theorem to construct a solution curve in $[0, \infty) \times C^0[0, 1]$ by continuation away from the solution $(\lambda, u) = (0, 0)$, and then investigate the structure of this curve directly.

The case of general $p > 1$ ($p \neq 2$) with f independent of x has been considered in many recent papers using the quadrature method, see for example, [2, 1, 4, 10, 20, 28]. In this paper we consider the general p case, with f dependent on x, and we use the continuation approach to prove the following results. There exists $\lambda_{\max} > 0$ ($\lambda_{\max}$ may be finite or ∞) such that the set of nontrivial solutions of (1.1)–(1.2) consists of a C^1 curve of globally stable, positive solutions in $(0, \lambda_{\max}) \times C^0[0, 1]$, with a parametrisation of the form $\lambda \to (\lambda, u(\lambda))$, where u is a C^1 function defined on the interval $(0, \lambda_{\max})$. Furthermore,

$$\lim_{\lambda \to 0} \lambda^{-p^*} u(\lambda) = -\Delta_p^{-1}(f(0)), \quad \lim_{\lambda \to \lambda_{\max}} |u(\lambda)|_0 = \infty,$$

(where Δ_p^{-1} is the inverse of the p-Laplacian operator, and $p^* := (p-1)^{-1}$), so the curve meets the point $(0, 0)$. We also characterise the value of $\lambda_{\max}$ as a weighted eigenvalue of the p-Laplacian

Under the hypotheses (1.3), (1.4), similar results have been obtained in the semilinear case $p = 2$, and in the general p case with f independent of x. Other hypotheses on f yield so called 'S-shaped' curves of solutions (the form of the parametrisation described above shows that S-shaped curves are precluded by (1.3), (1.4)), see for example [9, 22, 28], and the references therein.

The continuation approach relies on the use of the implicit function theorem. In order to apply the implicit function theorem to the above problem we require some recent results on differentiability of the inverse of the p-Laplacian. These results will be described in Section 2.1, and the solution curve will then be constructed in Section 3. In Section 4 it will be shown that the solutions on this curve are globally asymptotically stable with respect to positive solutions (in a sense to be made precise below). Finally, in Section 5, we briefly consider the situation when we change the condition (1.3) to allow $f(\cdot, 0) = 0$ (while retaining (1.4)), and we show that a similar C^1 curve of solutions exists.

2. Preliminaries

For any integer $r \geq 0$, $C^r[0, 1]$ will denote the standard Banach space of real valued, r-times continuously differentiable functions defined on $[0, 1]$, with the norm

$|u|_r = \sum_{i=0}^r |u^{(i)}|_0$, where $|\cdot|_0$ denotes the usual sup-norm on $C^0[0,1]$ (throughout, all function spaces will be real). For any $q \geq 1$, $L^q(0,1)$ will denote the standard Banach space of real valued functions on $[0,1]$ whose qth power is integrable, with norm $\|\cdot\|_q$. We let $W^{1,q}(0,1)$, with norm $\|\cdot\|_{1,q}$, denote the usual Sobolev space of absolutely continuous functions u on $[0,1]$, with derivative $u' \in L^q(0,1)$, while $W_0^{1,q}(0,1)$ denotes the set of functions in $W^{1,q}(0,1)$ satisfying (1.2).

If $F : X \to Z$ is a function between Banach spaces X and Z, then $Df(x) : X \to Z$ will denote the Fréchet derivative of F at x; partial Fréchet derivatives will be indicated by subscripts, for example, $D_xG(x,y)$, $D_yG(x,y)$ will denote the partial derivatives of a function G depending on x and y.

2.1. The p-Laplacian and its inverse. Letting

$$\mathcal{D}_p : \{u \in C^1[0,1] : u \text{ satisfies (1.2) and } \phi_p(u') \in W^{1,1}(0,1)\}\ ,$$

we define the p-Laplacian operator $\Delta_p : \mathcal{D}_p \to L^1(0,1)$ by

$$\Delta_p(u) = \phi_p(u')', \quad u \in \mathcal{D}_p.$$

This operator is $(p-1)$-*homogeneous*, that is, $\Delta_p(tu) = t^{p-1}\Delta_p(u)$, for any $t \in \mathbb{R}$ and $u \in \mathcal{D}_p$. The following invertibility result is well known — see, for example, [6, Theorem 3.1], [18, Theorem 20] (these references prove the result for periodic boundary conditions, but the proof can readily be modified to deal with Dirichlet boundary conditions).

Theorem 2.1. *For any $h \in L^1(0,1)$, the problem*

$$\Delta_p(u) = h, \quad h \in L^1(0,1), \tag{2.1}$$

has a unique solution $u = \Delta_p^{-1}(h) \in \mathcal{D}_p$. The operator $\Delta_p^{-1} : L^1(0,1) \to C^1[0,1]$ is continuous and p^*-homogeneous (recall that $p^* := (p-1)^{-1}$). The operator $\Delta_p^{-1} : L^1(0,1) \to C^0[0,1]$ is compact.

Next we discuss the differentiability of the operator Δ_p^{-1}. The following result is proved in [6, Theorem 3.4] for the periodic case; the proof in the Dirichlet case is similar (but simpler). A similar result is described in Theorem 5 and Corollary 6 of [15], however, the arguments in the proofs in [15] seem to be incomplete.

Theorem 2.2. *For $h \in L^1(0,1)$, let $u = u(h) := \Delta_p^{-1}(h)$.*

(A) *Suppose that $p > 2$ and $h \in C^0[0,1]$ is such that $u'(x) = 0 \implies h(x) \neq 0$, for $x \in [0,1]$. Then there exists a neighbourhood V of h in $C^0[0,1]$ such that:*
  (a) *for $h \in V$, $|u(h)'|^{2-p} \in L^1(0,1)$;*
  (b) *the mapping $h \to |u(h)'|^{2-p} : V \to L^1(0,\pi_p)$ is continuous;*
  (c) *the mapping $\Delta_p^{-1} : V \to W_0^{1,1}(0,1)$ is C^1 and*

$$w = D\Delta_p^{-1}(h)\bar{h} \implies (|u'|^{p-2}w')' = p^*\bar{h}, \quad \bar{h} \in C^0[0,1]. \tag{2.2}$$

(B) *Suppose that $1 < p < 2$. Then the mapping $\Delta_p^{-1} : L^1(0,1) \to C^1[0,1]$ is C^1, and (2.2) holds for $\bar{h} \in L^1(0,1)$.*

Remark 2.3. In either case (A) or case (B) of Theorem 2.2, the implication in (2.2) is that the function $|u'|^{p-2}w' \in W^{1,1}(0,1)$, so that the derivative $(|u'|^{p-2}w')'$ is defined in the L^1 sense (at least). This remark also applies to the function w in Lemma 3.3 below.

2.2. **Principal eigenvalues of the p-Laplacian.** We briefly consider the weighted, nonlinear eigenvalue problem

$$-\Delta_p(w) = \mu\rho\phi_p(w), \quad w \in \mathcal{D}_p, \tag{2.3}$$

where $\mu \in \mathbb{R}$ and the weight function $\rho \in L^1(0,1)$ satisfies $\rho \ge 0$ on $[0,1]$, and $\rho > 0$ on a set of positive measure. A *principal eigenvalue* of (2.3) is an eigenvalue μ for which there is a corresponding eigenfunction $w_\mu \ge 0$. The following result is well known — see, for example, [13, Sections 3-4].

Lemma 2.4. *Under the above hypotheses on the weight function ρ the eigenvalue problem (2.3) has a unique principal eigenvalue $\mu_0(\rho)$. In addition, $\mu_0(\rho) > 0$, $w_{\mu_0(\rho)}$ is positive and*

$$\int_0^1 |w'|^p \ge \mu_0(\rho)\int_0^1 \rho|w|^p, \quad w \in W_0^{1,p}(0,1).$$

If ρ_1, ρ_2 are two such weight functions, with

$$\rho_1 \le \rho_2 \text{ on } [0,1] \text{ and } \rho_1 < \rho_2 \text{ on a set of positive measure,}$$

then $\mu_0(\rho_1) > \mu_0(\rho_2)$.

We can now define the number $\lambda_{\max}$ which will be shown to characterise the right-hand end point of the curve of solutions. Define $g : [0,1] \times (0,\infty) \to [0,\infty)$ by

$$g(x,\xi) := f(x,\xi)/\xi^{p-1}, \quad (x,\xi) \in [0,1] \times (0,\infty). \tag{2.4}$$

It follows from (1.4) that

$$g_\xi(x,\xi) = (f_\xi(x,\xi)\xi - (p-1)f(x,\xi))/\xi^p \le 0, \quad (x,\xi) \in [0,1] \times (0,\infty), \tag{2.5}$$

which implies that the following limits exist

$$\gamma_\infty(x) := \lim_{\xi\to\infty} g(x,\xi) \ge 0, \quad x \in [0,1], \tag{2.6}$$

and $\gamma_\infty \in L^\infty(0,1)$. Now let

$$\lambda_{\max} := \begin{cases} \mu_0(\gamma_\infty) < \infty, & \text{if } \gamma_\infty \ne 0, \\ \infty, & \text{if } \gamma_\infty = 0. \end{cases}$$

3. Main results

For any $u \in C^0[0,1]$, we define $f(u) \in C^0[0,1]$ by $f(u)(x) = f(x,u(x))$, $x \in [0,1]$ (that is, f will denote both a function and its corresponding Nemitskii operator). Then (1.1)–(1.2) can be rewritten as

$$-\Delta_p(u) = \lambda f(u), \quad (\lambda,u) \in [0,\infty) \times \mathcal{D}_p. \tag{3.1}$$

Clearly, $(\lambda,u) = (0,0)$ is a solution of (3.1) and, by Theorem 2.1 and the fact that $f(0) > 0$, the only solution of (3.1) with $\lambda = 0$ or $u = 0$ is $(0,0)$. Let

$$\mathcal{S} := \{(\lambda,u) \in (0,\infty) \times \mathcal{D}_p \text{ satisfying (3.1)}\}.$$

We first prove some basic positivity properties of solutions of (3.1).

Lemma 3.1. *Every solution $(\lambda,u) \in \mathcal{S}$ satisfies:*

(a) $u > 0$ *on* $(0,1)$;
(b) $u'(0) > 0$, $u'(1) < 0$.

Proof. For any $(\lambda, u) \in \mathcal{S}$, it follows from the differential equation (1.1) and the positivity condition (1.3) that u cannot have a local minimum in the interval $(0,1)$, so $u \ge 0$ on $[0,1]$, and $u'(0) \ge 0$, $u'(1) \le 0$. If $u'(0) = 0$ then by (1.1) there exists $\delta > 0$ such that $u' < 0$ on $(0,\delta)$, which contradicts the fact that $u \ge 0$. Hence, $u'(0) > 0$ and similarly $u'(1) < 0$, and it then follows, by the preceding argument, that $u > 0$ on $(0,1)$. □

We now prove our main result on the structure of $\mathcal{S}$.

Theorem 3.2. *There exists a C^1 function $u : (0, \lambda_{\max}) \to C^0[0,1]$ such that:*

(a) $\lim_{\lambda\to 0} \lambda^{-p^*} u(\lambda) = -\Delta_p^{-1}(f(0))$ *and* $\lim_{\lambda\to\lambda_{\max}} |u(\lambda)|_0 = \infty$ (*if* $p^* > 1$ *then* u *extends to a* C^1 *function through* 0);

(b) *if* $\lambda \in (0, \lambda_{\max})$ *then* $u(\lambda)$ *is the unique solution of* (3.1)*;*

(c) *if* $\lambda \ge \lambda_{\max}$ *then* (3.1) *has no solution.*

Hence, $\mathcal{S}$ consists precisely of the curve of solutions $\{(\lambda, u(\lambda)) : \lambda \in (0, \lambda_{\max})\}$.

Theorem 3.2 will be proved by a series of intermediate steps. We first note that, by Theorem 2.1, equation (3.1) is equivalent to the equation

$$F(\lambda, u) := u + \lambda^{p^*} \Delta_p^{-1}(f(u)) = 0, \quad (\lambda, u) \in [0,\infty) \times C^0[0,1], \tag{3.2}$$

and the function $F : \mathbb{R} \times C^0[0,1] \to C^0[0,1]$ is continuous. Since $f(u) > 0$, for any $u \in C^0[0,1]$, Theorem 2.2 yields additional properties of F which, for reference, we state in the following lemma. This lemma also unifies the cases $1 < p < 2$ and $p > 2$ in Theorem 2.2, and is tailored to our use of the implicit function theorem below (the lemma would be trivial in the linear case $p = 2$).

Lemma 3.3. *If $(\lambda, u) \in \mathcal{S}$ then F is C^1 on a neighbourhood of (λ, u) in $(0,\infty) \times C^0[0,1]$. The derivative $D_uF(\lambda, u) = I + \lambda^{p^*} K_u$, where I denotes the identity on $C^0[0,1]$ and $K_u : C^0[0,1] \to C^0[0,1]$ is defined by*

$$K_u w := D\Delta_p^{-1}(f(u))(f_\xi(u)w), \quad w \in C^0[0,1].$$

The operator K_u is compact, so $D_uF(\lambda, u)$ is singular if and only the null space $N(D_uF(\lambda, u)) \ne \{0\}$. Also, $w \in N(D_uF(\lambda, u))$ if and only if $w \in W_0^{1,1}(0,1)$ and satisfies the linear, weighted Sturm-Liouville problem

$$\begin{gathered} -(|u'|^{p-2}w')' = p^*\lambda f_\xi(u)w, \\ w(0) = w(1) = 0 \end{gathered} \tag{3.3}$$

Remark 3.4. By Theorem 2.2, the coefficient function $|u'|^{p-2}$ in (3.3) satisfies the standard linear Sturm-Liouville hypothesis: $1/|u'|^{p-2} \in L^1(0,1)$. Hence, in particular, if w is a non-trivial solution of (3.3) then $|u'|^{p-2}w' \in W^{1,1}(0,1)$ and $|u'|^{p-2}w'|_i \ne 0$, $i = 0,1$ (where $|u'|^{p-2}w'|_i$ denotes the value of the continuous function $|u'|^{p-2}w'$ at $x = i$).

Proposition 3.5. *If $(\lambda, u) \in \mathcal{S}$ then $D_uF(\lambda, u)$ is non-singular.*

Proof. Suppose that $D_uF(\lambda, u)$ is singular. By Lemma 3.3, there exists a non-trivial $w \in W^{1,1}(0,1)$ satisfying (3.3), with $|u'|^{p-2}w'|_0 > 0$. Let

$$x_0 := \inf\{x \in (0,1] : w(x) = 0\}$$

($w(1) = 0$, so the set on the right is non-empty). Clearly, $x_0 > 0$, $w(x_0) = 0$, $w(x) > 0$ for $x \in (0, x_0)$ and $|u'|^{p-2}w'|_{x_0} < 0$. Now let

$$W := -\phi_p(u')w + u|u'|^{p-2}w'.$$

By the properties of u and w, we have $W \in W^{1,1}(0,1)$, $W(0) = W(1) = 0$, and, by (1.2), (1.4), and (3.3)

$$W' = \lambda(f(u) - p^* u f_\xi(u))w \geq 0, \quad \text{on } (0, x_0), \tag{3.4}$$

with strict inequality when $x > 0$ is sufficiently close to 0 (by (1.3)). Combining these results yields

$$0 < W(x_0) = u|u'|^{p-2}w'|_{x_0} \leq 0,$$

and this contradiction proves Proposition 3.5. □

Proposition 3.6. *Suppose that $(\lambda_n, u_n) \in \mathcal{S}$, $n = 1, 2, \dots$, satisfies $|u_n|_0 \to \infty$. Then $\lambda_n \to \lambda_{\max}$.*

Proof. Without loss of generality we may suppose that $\lambda_n \to \lambda_\infty$, for some $\lambda_\infty \geq 0$ (we allow $\lambda_\infty = \infty$), and $|u_n|_0 > 1$ for each $n \geq 1$.

We first suppose that $\lambda_\infty < \infty$. Let

$$\widetilde{u}_n := u_n/|u_n|_0, \quad \widetilde{f}_n := f(u_n)/|u_n|_0^{p-1}, \quad n \geq 1.$$

Dividing (3.2) by $|u_n|_0$ and using the p^*-homogeneity of Δ_p^{-1} shows that

$$-\widetilde{u}_n = \Delta_p^{-1}(\lambda_n \widetilde{f}_n) = \Delta_p^{-1}(\lambda_n g(u_n)\phi_p(\widetilde{u}_n)), \quad n \geq 1. \tag{3.5}$$

By (2.5),

$$0 \leq \widetilde{f}_n \leq |g(\cdot, 1)|_0 + \max\{f(x,\xi) : (x,\xi) \in [0,1]^2\}, \quad n \geq 1, \tag{3.6}$$

so by (3.5), Theorem 2.1, and the compactness of the embedding $C^1[0,1] \to C^0[0,1]$, we may suppose that $\widetilde{u}_n \to \widetilde{u}_\infty$ in $C^0[0,1]$, for some $\widetilde{u}_\infty \in C^0[0,1]$ with $|\widetilde{u}_\infty|_0 = 1$.

We now claim that

$$\widetilde{f}_n \to \gamma_\infty \phi_p(\widetilde{u}_\infty), \quad \text{in } L^1(0,1). \tag{3.7}$$

Results of this form are well known, so we simply sketch a proof. By (2.6), for any $x \in [0,1]$ and $\epsilon > 0$, there exists $C(x,\epsilon) > 0$ such that, for $n \geq 1$,

$$\widetilde{f}_n(x) \leq \frac{C(x,\epsilon) + (\gamma_\infty(x) + \epsilon)u_n(x)^{p-1}}{|u_n|_0^{p-1}} \to (\gamma_\infty(x) + \epsilon)\widetilde{u}_\infty(x)^{p-1}.$$

Together with a similar lower bound, and dominated convergence (recall (3.6)), this proves (3.7).

Now, letting $n \to \infty$ in (3.5) (using Theorem 2.1 and (3.7)), yields

$$-\widetilde{u}_\infty = \Delta_p^{-1}(\lambda_\infty \gamma_\infty \phi_p(\widetilde{u}_\infty)).$$

Since $\widetilde{u}_\infty \neq 0$, this implies that $\gamma_\infty \neq 0$, and so we have shown that $\lambda_\infty < \infty \implies \gamma_\infty \neq 0$ (and $\lambda_\infty \neq 0$). Furthermore, since $\widetilde{u}_\infty \geq 0$, it follows from Lemma 2.4 that $\lambda_\infty = \mu_0(\gamma_\infty) = \lambda_{\max}$.

Now suppose that $\gamma_\infty \neq 0$. Choose a closed subinterval $I \subset (0,1)$ such that the restriction $\gamma_{\infty,I} := \gamma_\infty|_I \neq 0$ in $L^\infty(I)$. Applying Lemma 2.4 on the interval I, there exists a principal eigenvalue $\mu_{\infty,I} > 0$ and a positive eigenfunction $w_{\infty,I} \in C^1(I)$ satisfying the eigenvalue problem

$$\begin{gathered} -\phi_p(w_{\infty,I}')' = \mu_{\infty,I}\gamma_{\infty,I}\phi_p(w_{\infty,I}), \quad \text{on } I, \\ w_{\infty,I} = 0, \quad \text{on } \partial I. \end{gathered} \tag{3.8}$$

Now, for any $n \geq 1$, $g(u_n) \geq \gamma_{\infty,I}$ on I, so comparing (3.5) and (3.8) shows that if $\lambda_n > \mu_{\infty,I}$ then w_n must have a zero in I (by the p-Laplacian form of the Sturm comparison theorem, see [18, Theorem 6]). However, this contradicts the positivity of u_n, so in fact $\lambda_n \le \mu_{\infty,I}$ for all $n \ge 1$. Thus, we have shown that $\gamma_\infty \ne 0 \implies \lambda_\infty < \infty$, which completes the proof of Proposition 3.6. □

Proposition 3.7. *There exists a C^1 function $u : (0, \lambda_{\max}) \to C^0[0,1]$ having the properties (a)-(b) described in Theorem 3.2.*

Proof. We first search for solutions of (3.2) near to $(\lambda, u) = (0, 0)$. Writing $\lambda^{p^*} = \widetilde{\lambda}$, equation (3.2) becomes

$$\widetilde{F}(\widetilde{\lambda}, u) := u + \widetilde{\lambda}\Delta_p^{-1}(f(u)) = 0, \quad (\widetilde{\lambda}, u) \in \mathbb{R} \times C^0[0,1].$$

By Theorem 2.2, $\widetilde{F}$ is C^1 near to $(0,0)$ (since $f(0) > 0$), and clearly $\widetilde{F}(0,0) = 0$, $D_u\widetilde{F}(0,0) = I$. Hence, by the implicit function theorem, there exists $\epsilon > 0$ and a C^1 function $\widetilde{u} : (-\epsilon, \epsilon) \to C^0[0,1]$, such that $\widetilde{F}(\widetilde{\lambda}, \widetilde{u}(\widetilde{\lambda})) = 0$, $\widetilde{\lambda} \in (-\epsilon, \epsilon)$. Also, it is clear that $\widetilde{u}'(0) = -\Delta_p^{-1}(f(0))$. Defining $u : (0, \epsilon) \to C^0[0,1]$ by $u(\lambda) = \widetilde{u}(\lambda^{p^*})$, this function has the required properties on the interval $(0, \epsilon)$. We now need to extend this function to $(0, \lambda_{\max})$.

Suppose that $(\lambda_0, u_0) \in \mathcal{S}$. By Proposition 3.5 the derivative $D_uF(\lambda_0, u_0)$ is non-singular so, by the implicit function theorem, there exists a C^1 function u, defined on a neighbourhood of λ_0, such that $u(\lambda_0) = u_0$ and $F(\lambda, u(\lambda)) \equiv 0$. Hence, the function u constructed above near to $\lambda = 0$ extends in a C^1 manner to a maximal interval $(0, \bar{\lambda})$, for some $\bar{\lambda} \in (0, \infty]$.

Lemma 3.8. *If (λ_n) is a sequence in $(0, \bar{\lambda})$ with $\lambda_n \to \bar{\lambda}$, then $|u(\lambda_n)|_0 \to \infty$.*

Proof. Suppose that the sequence $(|u(\lambda_n)|_0)$ is bounded (the same argument deals with the case where a subsequence is bounded). First suppose that $\bar{\lambda} < \infty$. Then, by (3.2) and the compactness of the operator $\Delta_p^{-1} : L^1(0,1) \to C^0[0,1]$ (Theorem 2.1), we may suppose that $u(\lambda_n) \to u_\infty$ in $C^0[0,1]$, and hence, by the continuity of F, $(\bar{\lambda}, u_\infty) \in \mathcal{S}_0$. But now, by the implicit function theorem, the parametrisation can be extended to the right of $\bar{\lambda}$, which contradicts the maximality of the interval $(0, \bar{\lambda})$ and shows that the case $\bar{\lambda} < \infty$ cannot occur.

Now suppose that $\bar{\lambda} = \infty$. Then by (1.3) and the boundedness of the sequence $(|u(\lambda_n)|_0)$, there exists $\delta > 0$ such that $f(u_n) \ge 4\delta$ on $[0,1]$. For each n, the function u_n attains its maximum at a point x_n. Suppose that $x_n \ge 1/2$ (the case $x_n < 1/2$ is similar). Then, since $u_n'(x_n) = 0$ and (λ_n, u_n) satisfies (1.1), we have

$$\phi_p(u_n') \ge \lambda_n\delta, \quad \text{on } [0, \tfrac{1}{4}],$$

and hence, by (1.2), $u_n(\frac{1}{4}) \ge \frac{1}{4}\phi_{p^*}(\lambda_n\delta)$. Since $\lambda_n \to \infty$, this contradicts the boundedness of $(|u(\lambda_n)|_0)$, and so proves the lemma. □

Combining Proposition 3.6 with Lemma 3.8 shows that $\bar{\lambda} = \lambda_{\max}$, which completes the proof of Proposition 3.7. □

Let $\mathcal{S}_0$ denote the curve of solution $\{(\lambda, u(\lambda)) : \lambda \in (0, \lambda_{\max})\}$. We now show that this is the only curve of solutions.

Proposition 3.9. $\mathcal{S} = \mathcal{S}_0$.

Proof. Suppose, on the contrary, that there exists a solution $(\lambda_1, u_1) \in \mathcal{S} \setminus \mathcal{S}_0$, and let $\mathcal{S}_1$ denote the connected component of $\mathcal{S}$ containing (λ_1, u_1). Then the proof of Proposition 3.7 shows that $\mathcal{S}_1$ is a C^1 curve in $(0, \infty) \times C^0[0,1]$, having a parametrisation of the form $\lambda \to (\lambda, \widetilde{u}(\lambda))$ defined on a maximal open interval

$I_1 \subset (0,\infty)$, and the only possible non-zero end point of I_1 is $\lambda_{\max}$. Hence, the left end point of I must be at $\lambda = 0$. However, Proposition 3.6 shows that the function $\widetilde{u}$ is bounded near to $\lambda = 0$, and so, by (3.2), $\lim_{\lambda\to 0}\widetilde{u}(\lambda) = 0$. However, by the implicit function theorem, this would imply that $\mathcal{S}_1 = \mathcal{S}_0$ (recall the first part of the proof of Proposition 3.7), contradicting the assumption that $(\lambda_1, u_1) \in \mathcal{S}\setminus\mathcal{S}_0$. $\square$

Combining Propositions 3.7 and 3.9 completes the proof of Theorem 3.2.

4. Stability

We now consider the following time-dependent, initial value problem

$$v_t(t) = \Delta_p(v(t)) + \lambda f(v(t)), \quad t \ge 0, \tag{4.1}$$

$$v(0) = v_0 \ge 0, \tag{4.2}$$

for $\lambda \ge 0$. Clearly, any positive solution of (3.1) is an equilibrium solution of (4.1)-(4.2), so by Theorem 3.2, for each $\lambda \in [0,\lambda_{\max})$, (4.1)-(4.2) has a unique equilibrium solution $u(\lambda)$ (here, we write $u(0) = 0$). In this section we will briefly consider the stability of this solution.

To state precisely what we mean by a solution of (4.1)-(4.2), we define the space

$$S_p := C([0,\infty), W^{1,p}_{0,\mathrm{w}}(0,1)) \cap W^{1,2}((0,\infty), L^2(0,1)),$$

where $C([0,\infty), W^{1,p}_{0,\mathrm{w}}(0,1))$ denotes the space of $W^{1,p}_0(0,1)$-valued, weakly continuous functions on $[0,\infty)$. The compactness of the embedding $W^{1,p}_0(0,1) \to C^0[0,1]$ implies that $C([0,\infty), W^{1,p}_{0,\mathrm{w}}(0,1)) \subset C([0,\infty), C^0[0,1])$.

Definition 4.1. For any $\lambda \ge 0$ and $v_0 \in \mathcal{D}_p$, a *solution* of (4.1)-(4.2) is a function $v(\lambda, v_0) \in S_p$ satisfying (4.2), such that, for almost all $t \ge 0$, $v(\lambda, v_0)(t) \in \mathcal{D}_p$ and (4.1) holds (in the $L^2(0,1)$ sense).

We now prove that $u(\lambda)$ is globally asymptotically stable, with respect to positive solutions, in the sense of the following theorem.

Theorem 4.2. *For each $\lambda \in (0,\lambda_{\max})$ and $0 \le v_0 \in \mathcal{D}_p$, the problem* (4.1)-(4.2) *has a unique solution $v(\lambda, v_0) \in S_p$. In addition, $v(\lambda, v_0)$ satisfies:*
(*a*) $0 \le v(\lambda, v_0)(t) \in \mathcal{D}_p$, *for each $t \ge 0$;*
(*b*) $\lim_{t\to\infty} v(\lambda, v_0)(t) = u(\lambda)$, *in $C^0[0,1]$.*

Proof. We consider fixed $\lambda \in (0,\lambda_{\max})$ and $0 \le v_0 \in \mathcal{D}_p$, and suppose, for now, that a solution $v(\lambda, v_0) \in S_p$ exists. Then the non-negativity property in (a) is well-known, but for convenience we give a short proof. For any function w, let $w^-(x) := \min\{w(x), 0\}$. By the definition of S_p, the function $\|v(\lambda, v_0)^-(\cdot)\|_2$ is absolutely continuous on $[0,\infty)$, with $\|v(\lambda, v_0)^-(0)\|_2 = 0$ and, writing v for $v(\lambda, v_0)(t)$,

$$\frac{1}{2}\frac{d}{dt}\|v^-\|_2^2 = \int_0^1 v_t v^- = \int_0^1 \Delta_p(v)v^- + \lambda\int_0^1 f(v)v^- \le 0, \quad \text{a.e. } t \ge 0$$

(the differentiability properties follow from [29, Prob 30.9] and a slight modification of [16, Lemma 7.6]; the final inequality follows from an integration by parts and (1.3)). Hence, $\|v(\lambda, v_0)^-(\cdot)\|_2 \equiv 0$, which proves the non-negativity.

By the preceding argument, we need only search for non-negative solutions. We now define the set $W_{0,+}^{1,p}(0,1) := \{w \in W_0^{1,p}(0,1) : w \ge 0\}$, and the functions

$$\begin{aligned} F(x,\xi) &:= \int_0^\xi f(x,s)\,ds, \quad (x,\xi) \in [0,1] \times [0,\infty), \\ E(w) &:= \frac{1}{p}\int_0^1 |w'|^p - \lambda \int_0^1 F(w), \quad w \in W_{0,+}^{1,p}(0,1). \end{aligned}$$

The function $E : W_{0,+}^{1,p}(0,1) \to \mathbb{R}$ is continuous.

Lemma 4.3. *There exists an increasing function $M : \mathbb{R} \to (0,\infty)$ such that,*

$$|w|_0 + \|w\|_{1,p} < M(E(w)), \quad w \in W_{0,+}^{1,p}(0,1).$$

Proof. Suppose that there exists $R \in \mathbb{R}$ and $0 \ne w_n \in W_{0,+}^{1,p}(0,1)$, $n = 1,2,\dots$, such that $E(w_n) \le R$ and $\|w_n\|_{1,p} \to \infty$; let $\widetilde{w}_n := w_n/\|w_n\|_{1,p}$. By the compactness of the embedding $W_0^{1,p}(0,1) \to C^0[0,1]$, we may assume that $\widetilde{w}_n \to \widetilde{w}_\infty$ in $C^0[0,1]$, for some $0 \le \widetilde{w}_\infty \in C^0[0,1]$, and it suffices to show that this leads to a contradiction.

A similar argument to the proof of (3.7) shows that

$$\int_0^1 \frac{F(w_n)}{\|w_n\|_{1,p}^p} \to \frac{1}{p}\int_0^1 \gamma_\infty \widetilde{w}_\infty^p. \tag{4.3}$$

Now suppose that $\int_0^1 \gamma_\infty \widetilde{w}_\infty^p > 0$. Then, by Lemma 2.4, for each $n \ge 1$,

$$\int_0^1 |\widetilde{w}_n'|^p \ge \mu_0(\gamma_\infty)\int_0^1 \gamma_\infty \widetilde{w}_n^p \to \mu_0(\gamma_\infty)\int_0^1 \gamma_\infty \widetilde{w}_\infty^p > 0, \tag{4.4}$$

and combining (4.3) with (4.4) shows that $E(w_n) \to \infty$ (since $\lambda < \lambda_{\max} = \mu_0(\gamma_\infty)$). However, this contradicts the initial assumption that $E(w_n) \le R$, $n \ge 1$. Next, suppose that $\int_0^1 \gamma_\infty \widetilde{w}_\infty^p = 0$, but $\widetilde{w}_\infty^p \ne 0$. Then, by Lemma 2.4,

$$\int_0^1 |\widetilde{w}_n'|^p \ge \mu_0(\mathbf{1})\|\widetilde{w}_n\|_p^p \to \mu_0(1)\|\widetilde{w}_\infty\|_p^p > 0 \tag{4.5}$$

(where $\mathbf{1}$ denotes the weight function that is identically 1 on $[0,1]$), and combining (4.3) with (4.5) again yields the contradiction $E(w_n) \to \infty$. Finally, suppose that $\widetilde{w}_\infty^p = 0$. Since $\|\widetilde{w}_n\|_{1,p} = 1$, $n \ge 1$, this implies that $\int_0^1 |\widetilde{w}_n'|^p \to 1$, which again yields a contradiction, and so completes the proof of the lemma. ∎

By the definition of S_p, [8, Lemma 3.3], and the argument in the proof of [11, Lemma 3.1], the function $E(v(\lambda,v_0)(\cdot))$ is absolutely continuous and decreasing, so by Lemma 4.3, any solution $v(\lambda,v_0)$ of (4.1)-(4.2) satisfies

$$|v(\lambda,v_0)(t)|_0 < M(E(v_0)), \quad t \ge 0. \tag{4.6}$$

Now, choosing a decreasing function $\theta_0 \in C^\infty(\mathbb{R},\mathbb{R})$, with

$$\theta_0(s) = \begin{cases} 1, & s \le M(E(v_0)), \\ 0, & s \ge 2M(E(v_0)), \end{cases}$$

and defining $\hat{f}_0 : [0,1] \times \mathbb{R} \to (0,\infty)$ by

$$\hat{f}_0(x,\xi) = f(x,\theta_0(\xi)\xi), \quad (x,\xi) \in [0,1] \times \mathbb{R},$$

we consider the initial value problem

$$\hat{v}_t(t) = \Delta_p(\hat{v}(t)) + \lambda \hat{f}_0(\hat{v}(t)), \quad t \ge 0, \tag{4.7}$$

$$\hat{v}(0) = v_0. \tag{4.8}$$

It follows from the form of θ_0 and $\hat{f}_0$, and (4.6), that any solution of (4.1)-(4.2) is a solution of (4.7)-(4.8), and vice versa. Now, since $\hat{f}_0$ is bounded and Lipschitz, (4.7)-(4.8) has a unique solution $\hat{v}(\lambda, v_0) \in S_p$, see [3, Theorem 3.4] (the existence and properties of solutions under the hypothesis that $\hat{f}_0$ is bounded and Lipschitz is also discussed in [11, Remark 2.2]). Hence, $\hat{v}(\lambda, v_0)$ is in fact a solution of (4.1)-(4.2), which we will denote by $v(\lambda, v_0)$. In addition, [3, Theorem 3.4] also shows that $v(\lambda, v_0)(t) \in \mathcal{D}_p$, for all $t \ge 0$.

Finally, suppose that there exists $\epsilon > 0$ and $t_n > 0$, $n = 1, 2, \dots$, such that $t_n \to \infty$ and $|v(\lambda, v_0)(t_n) - u(\lambda)|_0 \ge \epsilon$. By Lemma 4.3 and compactness we may suppose that $v(\lambda, v_0)(t_n) \to v_\infty$ in $C^0[0,1]$, for some $v_\infty \ge 0$. By the argument in the proof of [11, Lemma 3.1], this now implies that v_∞ must be an equilibrium solution of (4.1), so that $v_\infty = u(\lambda)$. However, this contradicts the choice of the sequence (t_n) and so completes the proof of Theorem 4.2. □

5. The case when $f(\cdot, 0) = 0$

In this final section we briefly consider the situation when we replace the condition (1.3) with the condition

$$f(x, 0) = 0, \quad f(x, \xi) > 0, \quad (x, \xi) \in [0,1] \times (0, \infty). \tag{5.1}$$

We continue to assume that (1.4) holds so that, for each $x \in [0,1]$, the function $g(x, \cdot)$ is still decreasing on $(0, \infty)$ and the following limit exists

$$\gamma_0(x) := \lim_{\xi \to 0^+} g(x, \xi) \ge \gamma_\infty(x), \quad x \in [0,1] \tag{5.2}$$

(recall that g was defined in (2.4)). We also impose the additional assumption:

$$\gamma_0 \in L^1(0,1) \quad \text{and} \quad \lim_{\xi \to 0^+} g_\xi(x, \xi) < 0 \text{ for } x \text{ in a set of positive measure.} \tag{5.3}$$

This implies that $\gamma_0 > \gamma_\infty$ in a set of positive measure, and so by Lemma 2.4,

$$\lambda_{\min} := \mu_0(\gamma_0) < \lambda_{\max} = \mu_0(\gamma_\infty).$$

Under the assumption (5.1), $(\lambda, 0)$ is a solution of (3.1) for all $\lambda \in \mathbb{R}$, and solutions need not be positive. Hence, we define the set of positive solutions

$$\mathcal{S}^+ := \{(\lambda, u) \in \mathcal{S} \colon u \ne 0 \text{ and } u \ge 0\}.$$

We will show that with these assumptions a C^1 curve of positive solutions of (3.1) exists, similar to that constructed in Theorem 3.2, but now defined over the interval $(\lambda_{\min}, \lambda_{\max})$.

It will be useful to observe that, by the above hypotheses on f (and hence g) any solution $(\lambda, u) \in \mathcal{S}^+$ of (3.1) satisfies the equation

$$-\Delta_p(u) = \lambda g(u) \phi_p(u), \tag{5.4}$$

with $g(u) \in L^1(0,1)$ and

$$\gamma_0 \ge g(u) \ge \gamma_\infty, \tag{5.5}$$

and these inequalities are strict on (possibly different) sets of positive measure.

Lemma 5.1. *The results of Lemma 3.1 hold for every $(\lambda, u) \in \mathcal{S}^+$.*

Proof. This follows from the arguments in the proof of Lemma 3.1 and the uniqueness of solutions of the initial value problem for the differential equation (5.4), see for example [5, Lemma 3.1]. □

We now have the following analogue of Theorem 3.2

Theorem 5.2. *Suppose that f satisfies* (1.4), (5.1) *and* (5.3). *There exists a C^1 function $u : (\lambda_{\min}, \lambda_{\max}) \to C^0[0,1]$ such that:*

(a) $\lim_{\lambda\to\lambda_{\min}} u(\lambda) = 0$ *and* $\lim_{\lambda\to\lambda_{\max}} |u(\lambda)|_0 = \infty$;
(b) *if* $\lambda \in (\lambda_{\min}, \lambda_{\max})$ *then* $u(\lambda)$ *is the unique positive solution of* (3.1);
(c) *if* $\lambda \in (0,\infty) \setminus (\lambda_{\min}, \lambda_{\max})$ *then* (3.1) *has no positive solution.*

Hence, $\mathcal{S}^+$ consists precisely of the curve of solutions $\{(\lambda, u(\lambda)) : \lambda \in (\lambda_{\min}, \lambda_{\max})\}$.

Proof. The proof is similar to the proof of Theorem 3.2, so we will merely sketch the necessary modifications to the preceding arguments. By (5.1) and Lemma 5.1, if $(\lambda, u) \in \mathcal{S}^+$ then $f(u) > 0$ in $(0,1)$ and $f(u)(0) = f(u)(1) = 0$, so it follows from Theorem 2.2 and Lemma 3.1 that Lemma 3.3 holds here. Also, Propositions 3.5, 3.6 and their proofs hold unchanged, while a similar argument to the proof of Proposition 3.6 also proves the following result.

Proposition 5.3. *Suppose that* $(\lambda_n, u_n) \in \mathcal{S}^+$, $n = 1, 2, \dots,$ *satisfies* $|u_n|_0 \to 0$. *Then* $\lambda_n \to \lambda_{\min}$.

Proposition 5.4. *There exists a C^1 function $u : (\lambda_{\min}, \lambda_{\max}) \to C^0[0,1]$ having the properties* (a)-(c) *described in Theorem 5.2.*

Proof. For any $\lambda > 0$, it follows from [19, Lemma 3.3] that a positive solution u of (5.4) (if it exists) must be unique (using the fact that for each $x \in [0,1]$, the function $g(x,\cdot)$ is decreasing). Now, if $\lambda \notin (\lambda_{\min}, \lambda_{\max})$ then by (5.5) and Lemma 2.4, (5.4) has no positive solution. On the other hand, if $\lambda \in (\lambda_{\min}, \lambda_{\max})$ then [25, Theorem 6.4] shows that (3.1) has a positive solution. The preceding argument then constructs a suitable C^1 function u with the required properties. □

This completes the proof of Theorem 5.2. □

References

[1] I. Addou, S.-H. Wang, Exact multiplicity results for a p-Laplacian problem with concave-convex-concave nonlinearities, *Nonlinear Analysis* **53** (2003), 111–137.
[2] I. Addou, S.-H. Wang, Exact multiplicity results for a p-Laplacian positone problem with concave-convex-concave nonlinearities, *Electron. J. Diff. Equns.* **2004**, No. 72.
[3] G. Akagai, M. Otani, Evolution inclusions governed by subdifferentials in reflexive Banach spaces, *J. Evolution Equations* **4** (2004), 519–541.
[4] S. Bouguima, A. Lakmeche, Multiple solutions of a nonlinear problem involving the p-Laplacian, *Comm. Appl. Nonlinear Anal.* **7** (2000), 83–96.
[5] P. Binding, P. Drábek, Sturm-Liouville theory for the p-Laplacian, *Studia Sci. Math. Hungar.* **40** (2003), 375–396.
[6] P. A. Binding, B. P. Rynne, The spectrum of the periodic p-Laplacian, *J. Differential Equations* **235** (2007), 199–218.
[7] H. Brezis, S. Kamin, Sublinear elliptic equations in $\mathbb{R}^n$, *Manuscripta Math.* **74** (1992), 87–106.
[8] H. Brezis, *Opérateurs Maximaux Monotones et Semi-groupes de Contraction dans les Espace de Hilbert,* North-Holland, Vol. 5, North-Holland, Amsterdam (1973).
[9] K. J. Brown, M. M. A. Ibrahim, R. Shivaji, S-shaped bifurcation curves, *Nonlinear Anal.* **5** (1981), 475–486.

[10] J. Cheng, Y. Shao, The positive solutions of boundary value problems for a class of one-dimensional p-Laplacians, *Nonlinear Analysis* **68** (2008), 883–891.
[11] R. Chill, A. Fiorenza, Convergence and decay rate to equilibrium of bounded solutions of quasilinear parabolic equations, *J. Differential Equations* **228** (2006), 611–632.
[12] M. G. Crandall, P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues, and linearized stability, *Arch. Rat. Mech. Anal.* **52** (1973), 161–180.
[13] M. Cuesta, Eigenvalue problems for the p-Laplacian with indefinite weights, *Electron. J. Differential Equations* **2001** No. 33.
[14] E. N. Dancer, On the structure of solutions of an equation in catalysis theory when a parameter is large, *J. Differential Equations* **37** (1980), 404–437.
[15] J. Garcia-Melian, J. Sabina de Lis, A local bifurcation theorem for degenerate elliptic equations with radial symmetry, *J. Differential Equns.* **179** (2002), 27–43.
[16] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1983.
[17] J. Hernandez, Qualitative methods for nonlinear diffusion equations, *Nonlinear diffusion problems (Montecatini Terme, 1985),* 47–118, Lecture Notes in Math., 1224, Springer, Berlin, 1986.
[18] Y.-X. Huang, G. Metzen, The existence of solutions to a class of semilinear differential equations, *Differential Integral Equations* **8** (1995), 429–452.
[19] R. Kajikiya, Y-H. Lee, I. Sim, One-dimensional p-Laplacian with a strong singular indenite weight, I. Eigenvalue, *J. Differential Equations* **244** (2008), 1985–2019.
[20] J. Karatson, P. L. Simon, Exact multiplicity for degenerate two-point boundary value problems with p-convex nonlinearity, *Nonlinear Analysis* **52** (2003), 1569–1590.
[21] P. Korman, A global solution curve for a class of semilinear equations, *Electron. J. Differ. Equ. Conf.* **1** (1997), 119–127.
[22] P. Korman, Y. Li, On the exactness of an S-shaped bifurcation curve, *Proc. Amer. Math. Soc.* **127** (1999), 1011–1020.
[23] P. Korman, J. Shi, Instability and exact multiplicity of solutions of semilinear equations, *Electron. J. Differ. Equ. Conf.* **5** (2000), 311–322.
[24] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, *Indiana Univ. Mat. J.* **20** (1970), 1–13.
[25] B. P. Rynne, p-Laplacian problems with jumping nonlinearities, *J. Differential Equations* **226** (2006), 501–524.
[26] J. Shi, Blow up points of solution curves for a semilinear problem, *Topol. Methods Nonlinear Anal.* **15** (2000), 251–266.
[27] J. Shi, R. Shivaji, Global bifurcations of concave semipositone problems, *Evolution equations*, 385–398, Lecture Notes in Pure and Appl. Math., 234, Dekker, New York, 2003.
[28] S.-H. Wang, T.-Z. Yeh, Exact multiplicity of solutions and S-shaped bifurcation curves for the p-Laplacian perturbed Gelfand problem in one space variable, *J. Math. Anal. Appl.* **342** (2008), 1175–1191.
[29] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B*, Springer, New York (1990).

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