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EXISTENCE AND CONCENTRATION OF POSITIVE SOLUTIONS FOR A QUASILINEAR ELLIPTIC EQUATION IN $\mathbb{R}$

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Abstract. We study the existence and concentration of positive solutions for the quasilinear elliptic equation

$$-\varepsilon^2 u'' - \varepsilon^2 (u^2)'' u + V(x)u = h(u)$$

in $\mathbb{R}$ as $\varepsilon \to 0$, where the potential $V : \mathbb{R} \to \mathbb{R}$ has a positive infimum and $\inf_{\partial\Omega} V > \inf_{\Omega} V$ for some bounded domain Ω in $\mathbb{R}$, and h is a nonlinearity without having growth conditions such as Ambrosetti-Rabinowitz.

1. Introduction

In this article, we consider the quasilinear elliptic equation

$$-\varepsilon^2 u'' - \varepsilon^2 (u^2)'' u + V(x)u = h(u) \quad \text{in } \mathbb{R} \tag{1.1}$$

where $\varepsilon > 0$ is a small real parameter. Here our goal is to prove, by a variational approach, the existence and concentration of positive weak solutions. We say that $u \in H^1(\mathbb{R})$ is a (weak) solution of (1.1) if

$$\begin{aligned} &\varepsilon^2 \int_{\mathbb{R}^N} (1 + 2u^2) u' \varphi' \, \mathrm{d}x + 2\varepsilon^2 \int_{\mathbb{R}^N} |u'|^2 u \varphi \, \mathrm{d}x + \int_{\mathbb{R}^N} V(x) u \varphi \, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} h(u) \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}). \end{aligned}$$

Solutions of equations like (1.1) are related with existence of standing wave solutions for quasilinear equations of the form

$$i\frac{\partial \psi}{\partial t} = -\varepsilon^2 \psi'' + W(x)\psi - \eta(|\psi|^2)\psi - \varepsilon^2 \kappa [\rho(|\psi|^2)]'' \rho'(|\psi|^2)\psi \tag{1.2}$$

where $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, κ is a positive constant, $W : \mathbb{R} \to \mathbb{R}$ is a given potential and $\eta, \rho : \mathbb{R}^+ \to \mathbb{R}$ are suitable functions. Quasilinear equations of the form (1.2) arise in several areas of physics in correspondence to different type of functions ρ. For physical motivations and developing of the physical aspects we refer to [20] and references therein.

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Here we consider the case where $\rho(s) = s$. Looking for standing wave solutions of (1.2) we set $\psi(t,x) = e^{-i\xi t}u(x)$, where $\xi \in \mathbb{R}$ and $u > 0$ is a real function. So one obtains a corresponding equation of elliptic type which has the formal variational structure given by (1.1), where without loss of generality we set $\kappa = 1$.

Motivated by the physical aspects, equation (1.1) has recently attracted a lot of attention and existence results have been obtained in the case of a bounded potential $V(x)$ or in the coercive case. Direct variational methods by using constrained minimization arguments were used in [20] to provide existence of positive solutions up to an unknown Lagrange multiplier. The authors study the following problem

$$-u'' + V(x)u - (u^2)''u = \theta |u|^{p-1}u, \quad x \in \mathbb{R}. \tag{1.3}$$

Ambrosetti and Wang in [1], by using variational methods, proved the existence of positive solutions for the following class of quasilinear elliptic equations

$$-u'' + (1 + \varepsilon a(x))u - (1 + \varepsilon b(x))(u^2)''u = (1 + \varepsilon c(x))u^p, \quad u \in H^1(\mathbb{R})$$

for $p > 1$ and $\varepsilon > 0$ sufficiently small, where $a(x)$, $b(x)$ and $c(x)$ are real functions satisfying certain hypotheses. Subsequently a general existence result for (1.1) was derived in [19]. In this paper, which deals also with higher dimensions, to overcome the undefiniteness of natural functional associated to the equation the idea is to introduce a change of variable and to rewrite the functional with this new variable which turns the problem into finding solutions of an auxiliary semilinear equation. Then critical points are search in an associated Orlicz space and existence results are given in the case of bounded, coercive or radial potentials. Following the strategy developed in [10] on a related problem the authors in [11] also make use of a change of unknown and define an associated equation that they call dual. A simple and shorter proof of the results in [19] is presented for bounded potentials, which does not use Orlicz spaces and permit to cover a different class of nonlinearities. We observe that this change of variables is not necessary in dimension one because in this case the functional associated is well defined. We mention some works that study problem (1.1) without make this change of variables [2], [3] and [21]. In [2] and [21] the authors study (1.3) for p-laplacian or more general operator and $\theta = 1$. In [3] the authors study existence and concentration of positive solutions for equation (1.1) with $h(t) = t^p$, $p \geq 3$. There the potential $V : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following conditions:

(V1) V is bounded from below by a positive constant; that is,

$$\inf_{x \in \mathbb{R}} V(x) = V_0 > 0;$$

(V2) there exists a bounded domain Ω in $\mathbb{R}$ such that

$$m \equiv \inf_{x \in \Omega} V(x) < \inf_{x \in \partial\Omega} V(x).$$

We should also mention that equation (1.1) has been also considered in $\mathbb{R}^N$ for $N \geq 2$, we refer the reader to the works of [9, 10, 11, 16, 19] among others and references therein.

Here we also assume that $V \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies the assumptions (V1)-(V2). Hereafter we use the following notation:

$$\mathcal{M} \equiv \{x \in \Omega : V(x) = m\}$$

and without loss of generality we may assume that $0 \in \mathcal{M}$. We emphasize that besides the local condition (V2), introduced in [12] and so far well known for semilinear elliptic problems, we do not require any global condition other than (V1). We also suppose that $h : \mathbb{R}_+ \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying:

- (H1) $\lim_{t\to 0^+} h(t)/t = 0$;
- (H2) there exists $T > 0$ such that
$$h(T) > mT, \quad H(T) = \frac{m}{2}T^2, \quad H(t) < \frac{m}{2}t^2 \quad \text{for all } t \in (0, T)$$
where $H(t) = \int_0^t h(s)\,\mathrm{d}s$.

Similar hypothesis on the nonlinearity were used in [7] for the semilinear case. Following the strategy developed there, using variational methods, we shall prove existence and concentration of positive solutions for (1.1) without assuming Ambrosetti-Rabinowitz and monotonicity conditions on h. In particular we improve the results in [3] where h is a pure power.

Next we state our main result.

Theorem 1.1. *Suppose that* (V1)–(V), (H1)-(H2) *hold. Then there exists $\varepsilon_0 > 0$ such that* (1.1) *has a positive solution $u_\varepsilon \in C^{1,\alpha}_{\rm loc}(\mathbb{R})$ for all $0 < \varepsilon < \varepsilon_0$, satisfying the following:*

- (i) *u_ε admits a maximum point x_ε such that $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0$ and for any sequence $\varepsilon_n \to 0$ there exist $x_0 \in \mathcal{M}$ and a solution u_0 of*
$$-u'' - (u^2)''u + mu = h(u), \quad u > 0, \quad u \in H^1(\mathbb{R}) \tag{1.4}$$
such that, up to subsequences,
$$x_{\varepsilon_n} \to x_0 \quad \text{and} \quad u_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n}) \to u_0 \quad \text{in } H^1(\mathbb{R}) \text{ as } \quad n \to \infty.$$
- (ii) *There exist positive constants C and ζ such that*
$$u_\varepsilon(x) \le C \exp\big(-\frac{\zeta}{\varepsilon}(|x - x_\varepsilon|)\big) \quad \text{for all } x \in \mathbb{R}.$$

The proof of this theorem relies on the study of a semilinear equation obtained after making the chance of variables introduced in [19]. In order to prove existence of solutions for this equation we study some properties of the least energy solutions for a limit equation obtained from (1.4) by the same change of variables. Using these properties, after some technical lemmata, we can find a bounded Palais-Smale sequence in a suitable space for the associated functional. Thus we obtain a solution for the semilinear equation which gives us a solution for the original problem (1.1).

This paper is organized as follows: In Section 2 we a change of variables and study some properties of the functional, J_ε, associated to the new semilinear equation obtained from (1.1), and of the space where it is defined. Section 3 is devoted to prove that the mountain pass level of J_ε is well defined and converges to the least energy level of the functional associated to the limit problem. In Section 4 we prove the existence of a nontrivial critical point for J_ε and finally Section 5 brings the results that complete the proof of Theorem 1.1.

2. Preliminaries results

Since we are looking for positive solutions we define $h(t) = 0$ for $t < 0$. Observe that defining $v(x) = u(\varepsilon x)$ equation (1.1) becomes equivalent to
$$-v'' - (v^2)''v + V(\varepsilon x)v = h(v), \quad v > 0 \text{ in } \mathbb{R}. \tag{2.1}$$

The natural energy functional associated with (2.1), namely

$$I_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}} [(1+2v^2)|v'|^2 + V(\varepsilon x)v^2]\,\mathrm{d}x - \int_{\mathbb{R}} H(v)\,\mathrm{d}x,$$

is well defined on

$$H_\varepsilon := \big\{v \in H^1(\mathbb{R}) : \int_{\mathbb{R}} V(\varepsilon x)v^2\,\mathrm{d}x < \infty\big\}$$

due the imbedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (V1). Despite this, following the strategy developed in [9], [11], [14] and [19] on a related problem for higher dimensions, we introduce a change of variables $u = f^{-1}(v)$ where f is a C^∞ function defined by

$$f'(t) = \big(1 + 2f^2(t)\big)^{-1/2} \quad \text{if } t > 0, \quad f(0) = 0, \quad \text{and} \quad f(t) = -f(-t) \quad \text{if } t < 0.$$

This change of variables allows us to consider more general nonlinearities. To make easier the reference we list here some properties of $f(t)$ whose proofs can be found in [14, Lemma 2.1] (see also [11] and [19]). The proof of the last item is found in [16].

Lemma 2.1. *The function $f(t)$ satisfies:*

1. *f is C^∞, invertible and uniquely defined;*
2. *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$;*
3. *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*
4. *$f(t)/t \to 1$ as $t \to 0$;*
5. *$f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to +\infty$;*
6. *$f(t)/2 \le tf'(t) \le f(t)$ for all $t \ge 0$;*
7. *$|f(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;*
8. *The function $f^2(t)$ is strictly convex;*
9. *There exists a positive constant C such that*
$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1 \\ C|t|^{1/2}, & |t| \ge 1; \end{cases}$$
10. *$|f(t)f'(t)| \le 1/\sqrt{2}$ for all $t \in \mathbb{R}$;*
11. *For each $\lambda > 1$ we have $f^2(\lambda t) \le \lambda^2 f^2(t)$ for all $t \in \mathbb{R}$.*

After this change of variable from I_ε we obtain a new functional

$$P_\varepsilon(u) = I_\varepsilon(f(u)) = \frac{1}{2}\int_{\mathbb{R}} [|u'|^2 + V(\varepsilon x)f^2(u)]\,\mathrm{d}x - \int_{\mathbb{R}} H(f(u))\,\mathrm{d}x,$$

which is well defined on

$$E_\varepsilon := \big\{u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} V(\varepsilon x)f^2(u)\,\mathrm{d}x < \infty\big\}.$$

Using the properties of $f(t)$ we can see that E_ε is a normed space with norm

$$\|u\|_\varepsilon := \|u'\|_2 + \inf_{\lambda > 0} \lambda\big\{1 + \int_{\mathbb{R}} V(\varepsilon x)f^2(\lambda^{-1}u)\,\mathrm{d}x\big\} := \|u'\|_2 + |\|u\||_\varepsilon. \tag{2.2}$$

The following proposition is crucial to prove convergence results.

Proposition 2.2. *There exists $C > 0$ independent of $\varepsilon > 0$ such that*

$$\int_{\mathbb{R}} V(\varepsilon x)f^2(u)\,\mathrm{d}x \le C|\|u\||_\varepsilon\Big[1 + \Big(\int_{\mathbb{R}} V(\varepsilon x)f^2(u)\,\mathrm{d}x\Big)^{1/2}\Big] \tag{2.3}$$

for all $u \in E_\varepsilon$.

The proof of the above proposition is the same as in [14, Proposition 2.1], since the constant C that appearing there depends only on f.

From this result we obtain that E_ε is a Banach space and the embedding $E_\varepsilon \hookrightarrow H^1(\mathbb{R})$ is continuous. Also can be proved that the space $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in E_ε (see [9], [13], [14] and [19] for details). Moreover due to the imbedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we can see that the functional P_ε is of class $\mathcal{C}^1$ on E_ε. This does not occurs in general for higher dimensions. For $N \geq 2$ some regularity results can be found in [9, 13, 14] where the authors prove that P_ε is continuous in E_ε and Gâteaux differentiable with derivative given by

$$\langle P_\varepsilon'(u), \varphi\rangle = \int_{\mathbb{R}^N} \nabla u \nabla \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} f'(u)\,[V(\varepsilon x) f(u) - h(f(u))]\,\varphi \,\mathrm{d}x.$$

They also prove that P_ε' is continuous from the norm topology of E_ε to the weak-* topology of E_ε'; i.e., if $u_n \to u$ strongly in E_ε then

$$\langle P_\varepsilon'(u_n), \varphi\rangle \to \langle P_\varepsilon'(u), \varphi\rangle \quad \text{for each } \varphi \in E_\varepsilon.$$

In our case, for $N = 1$, we have P_ε of class $\mathcal{C}^1$ and for each $\varphi \in E_\varepsilon$ it holds

$$\langle P_\varepsilon'(u), \varphi\rangle = \int_{\mathbb{R}} u'\varphi' \,\mathrm{d}x + \int_{\mathbb{R}} f'(u)\,[V(\varepsilon x) f(u) - h(f(u))]\,\varphi \,\mathrm{d}x.$$

We observe that nontrivial critical points for P_ε are weak solutions for

$$-u'' = f'(u)\,[h(f(u)) - V(\varepsilon x) f(u)] \quad \text{in } \mathbb{R}. \tag{2.4}$$

In Proposition 2.3 below we relate the solutions of (2.4) to the solutions of (2.1). From now on, for any set $A \subset \mathbb{R}$ and $\varepsilon > 0$, we define $A_\varepsilon \equiv \{x \in \mathbb{R} : \varepsilon x \in A\}$. We define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \Omega_\varepsilon \\ \varepsilon^{-1} & \text{if } x \notin \Omega_\varepsilon, \end{cases}$$

and

$$Q_\varepsilon(u) = \Big(\int_{\mathbb{R}} \chi_\varepsilon(x) u^2 \,\mathrm{d}x - 1\Big)_+^2.$$

The functional $Q_\varepsilon : H^1(\mathbb{R}) \to \mathbb{R}$ is of class $\mathcal{C}^1$ with Frechet derivative given by

$$\langle Q_\varepsilon'(u), \varphi\rangle = 4\Big(\int_{\mathbb{R}} \chi_\varepsilon(x) u^2 \,\mathrm{d}x - 1\Big)_+ \int_{\mathbb{R}} \chi_\varepsilon(x) u\varphi \,\mathrm{d}x.$$

It will act as a penalization to force the concentration phenomena to occur inside Ω. This type of penalization was first introduced in [8] for the semilinear case in $\mathbb{R}^N$ with $N \geq 2$. Finally let $J_\varepsilon : E_\varepsilon \to \mathbb{R}$ be given by

$$J_\varepsilon(u) = P_\varepsilon(u) + Q_\varepsilon(u).$$

The next proposition relates solutions of (2.1) and (2.4).

Proposition 2.3. (i) *If $u \in E_\varepsilon$ is a critical point of P_ε then $v = f(u) \in E_\varepsilon$ is a weak solution of* (2.1)*;*

(ii) *If u is a classical solution of* (2.4) *then $v = f(u)$ is a classical solution of* (2.1).

Proof. The second claim was proved in [11] and to prove (i) we follow the same idea. If $v = f(u)$ by Lemma 2.1 we have $|v| \le |u|$ and $|v'| = f'(u)|u'| \le |u'|$ which imply $v \in E_\varepsilon$. Since u is a critical point for P_ε, u is a weak solution for (2.4). So

$$\int_{\mathbb{R}} u'\varphi' \,\mathrm{d}x = \int_{\mathbb{R}} f'(u)\left[h(f(u)) - V(\varepsilon x)f(u)\right]\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in E_\varepsilon. \tag{2.5}$$

Since $(f^{-1})'(t) = [f'(f^{-1}(t))]^{-1}$, it follows that

$$(f^{-1})'(t) = \left[1 + 2f^2(f^{-1}(t))\right]^{1/2} = (1+2t^2)^{1/2}, \quad (f^{-1})''(t) = \frac{2t}{(1+2t^2)^{1/2}}$$

which yields

$$u' = (f^{-1})'(v)v' = (1+2v^2)^{1/2}v'.$$

For each $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ we have $\varphi := (f'(u))^{-1}\psi = (f^{-1})'(v)\psi \in E_\varepsilon$ with

$$\varphi' = \frac{2v\psi}{(1+2v^2)^{1/2}}v' + (1+2v^2)^{1/2}\psi'.$$

Hence by (2.5) we obtain

$$\int_{\mathbb{R}} \Big[2|v'|^2 v\psi + (1+2v^2)^{1/2}v'\psi\Big]\,\mathrm{d}x = \int_{\mathbb{R}} \left[h(v) - V(\varepsilon x)v\right]\psi \,\mathrm{d}x$$

and concludes the proof of (i). ☐

Following this result, to prove existence of solutions for (1.1), we shall look for critical points to J_ε for which ones Q_ε is zero. Initially we will study the limiting problem (1.4).

2.1. The limiting problem.

In this subsection we shall study some properties of the solutions of (1.4), namely

$$-v'' - (v^2)''v + mv = h(v), \quad v > 0 \quad \text{in } \mathbb{R}.$$

Using the same change of variables f, we will do it dealing with classical solutions for the problem

$$-u'' = g(u), \quad \lim_{|x|\to\infty} u(x) = 0, \quad u(x_0) > 0 \quad \text{for some } x_0 \in \mathbb{R}, \tag{2.6}$$

where $g(t) = f'(t)[h(f(t)) - mf(t)]$ for $t \ge 0$ and $g(t) = -g(-t)$ for $t < 0$. Like in Proposition 2.3 we see that if $u \in H^1(\mathbb{R})$ is a classical solution of (2.6) then $v = f(u)$ is a classical solution for (1.4). From assumptions on h and Lemma 2.1 we can see that the function $g(t)$ is locally Lipschitz continuous and satisfies:

- (G1) $\lim_{t\to 0} g(t)/t = -m < 0$;
- (G2) for $\tilde{T} = f^{-1}(T)$ and $G(t) = \int_0^t g(s)\,\mathrm{d}s$ it holds $\tilde{T} > 0$ and
$$G(\tilde{T}) = 0, \quad g(\tilde{T}) > 0, \quad G(t) < 0 \quad \text{for all } t \in (0, \tilde{T}). \tag{2.7}$$

In [4, Theorem 5], the authors prove that (2.7) is a necessary and sufficient condition for the existence of a solution of (2.6). They also show some properties of this solutions when they there exist. Thus from [4, Theorem 5 and Remark 6.3] we have the following result.

Theorem 2.4. *Assume* (H1), (H2). *Then* (2.6) *has a solution* $U \in \mathcal{C}^2(\mathbb{R})$, *which is unique up to translation, positive and satisfies:*

- (i) $U(0) = \tilde{T}$, *U is radially symmetric and decreases with respect to* $|x|$*;*

(ii) *U together with its derivatives up to order 2 have exponential decay at infinity*
$$0 \le U(x) + |U'(x)| + |U''(x)| \le C \exp(-c|x|) \quad \textit{for all } x \in \mathbb{R};$$
(iii) $-[U'(x)]^2 = 2G(U(x))$ *for all* $x \in \mathbb{R}$.

Now we consider $L_m : H^1(\mathbb{R}) \to \mathbb{R}$, the functional associated to equation (2.6),
$$L_m(u) = \frac{1}{2}\int_{\mathbb{R}} \big(|\nabla u|^2 + mf^2(u)\big)\, \mathrm{d}x - \int_{\mathbb{R}} H(f(u))\, \mathrm{d}x$$
which is well defined and of class $\mathcal{C}^1$. Let
$$E_m := L_m(U).$$
Since U is unique up to translation we have $L_m(w) = E_m$ for each solution w of (2.6). By a result of Jeanjean and Tanaka [17] we know that these solutions have a mountain pass characterization, that is
$$L_m(w) = c_m := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} L_m(\gamma(t)) \tag{2.8}$$
where $\Gamma = \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R})) : \gamma(0) = 0 \text{ and } L_m(\gamma(1)) < 0\}$. Using the same arguments as in [7, Proposition 2] we prove the next result.

Proposition 2.5. *There exist $t_0 > 1$ and a continuous path $\theta : [0, t_0] \to H^1(\mathbb{R})$ satisfying:*

(i) $\theta(0) = 0$, $L_m(\theta(t_0)) < -1$ *and* $\max_{t\in[0,t_0]} L_m(\theta(t)) = E_m$;
(ii) $\theta(1) = U$ *and* $L_m(\theta(t)) < E_m$ *for all* $t \ne 1$;
(iii) *there exist $C, c > 0$ such that for any $t \in [0, t_0]$ it holds*
$$|\theta(t)(x)| + |[\theta(t)]'(x)| \le C \exp(-c|x|) \quad x \in \mathbb{R}.$$

3. The mountain pass level

For the rest of this article, we fix $\beta = \operatorname{dist}(\mathcal{M}, \mathbb{R}^N\backslash\Omega)/10$ and choose a cut-off function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $|x| \le \beta$ and $\varphi(x) = 0$ for $|x| \ge 2\beta$. We define $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$ and for $z \in \mathcal{M}^\beta$
$$U_\varepsilon^z(x) := \varphi_\varepsilon(x - z/\varepsilon)U(x - z/\varepsilon), \quad x \in \mathbb{R}.$$
For sufficiently small ε we will find a solution near the set
$$X_\varepsilon := \{U_\varepsilon^z : \ z \in \mathcal{M}^\beta\}.$$

Remark 3.1. For $\varepsilon \in (0, 10)$ we have X_ε uniformly bounded and moreover for each ε it is compact in E_ε. Indeed, let $U_\varepsilon^z \in X_\varepsilon$ for some $z \in \mathcal{M}^\beta$. So
$$\begin{aligned}
\|U_\varepsilon^z\|_\varepsilon &\le \Big[\int_{\mathbb{R}} |(\varphi_\varepsilon U)'|^2\, \mathrm{d}x\Big]^{1/2} + \Big[1 + \int_{\mathbb{R}} V(\varepsilon x + z) f^2(\varphi_\varepsilon U)\, \mathrm{d}x\Big] \\
&\le \Big[2\int_{\mathbb{R}} \Big(\varepsilon^2|\varphi'(\varepsilon x)|^2 U^2 + \varphi_\varepsilon^2 |U'|^2\Big]\, \mathrm{d}x\Big]^{1/2} + 1 + \sup_{x\in\Omega} V(x) \int_{\mathbb{R}} (\varphi_\varepsilon U)^2\, \mathrm{d}x \\
&\le c\|U\| + \tilde{c}\|U\|^2 + 1 \le C
\end{aligned}$$
independently of $z \in \mathcal{M}^\beta$ and $\varepsilon \in (0, 10)$. This proves the uniform boundedness of X_ε. Now let $\{U_\varepsilon^{z_n}\}$ be a sequence in X_ε. The compactness of $\mathcal{M}^\beta$ implies the

existence of $z_0 \in \mathcal{M}^\beta$ such that $z_n \to z_0$ in $\mathbb{R}$, up to subsequences. Hence $U_\varepsilon^{z_0} \in X_\varepsilon$ and due to the exponential decay of $U + |U'|$ and the boundedness of $\{z_n\}$ we get

$$\int_{\mathbb{R}} V(\varepsilon x) f^2\left(U_\varepsilon^{z_n} - U_\varepsilon^{z_0}\right) \mathrm{d}x \le \sup_{\Omega} V(x) \int_{\mathbb{R}} \left|U_\varepsilon^{z_n} - U_\varepsilon^{z_0}\right|^2 \mathrm{d}x \to 0,$$
$$\int_{\mathbb{R}} |\left(U_\varepsilon^{z_n} - U_\varepsilon^{z_0}\right)'|^2 \mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Now for $\lambda \in (0,1)$ it follows from (ii) in Lemma 2.1 that

$$\lambda\Big\{1 + \int_{\mathbb{R}} V(\varepsilon x) f^2(\lambda^{-1}(U_\varepsilon^{z_n} - U_\varepsilon^{z_0}))\,\mathrm{d}x\Big\} \le \lambda + \lambda^{-1} \int_{\mathbb{R}} V(\varepsilon x) f^2\left(U_\varepsilon^{z_n} - U_\varepsilon^{z_0}\right) \mathrm{d}x.$$

Thus $|||U_\varepsilon^{z_n} - U_\varepsilon^{z_0}|||_\varepsilon \le 2\lambda$ for large n which proves that $U_\varepsilon^{z_n} \to U_\varepsilon^{z_0}$ in E_ε as $n \to \infty$.

Lemma 3.2. *We have*

$$\sup_{t \in [0,t_0]} |J_\varepsilon(\varphi_\varepsilon \theta(t)) - L_m(\theta(t))| \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Since $\operatorname{supp}(\varphi_\varepsilon\theta(t)) \subset \Omega_\varepsilon$ and $\operatorname{supp}(\chi_\varepsilon) \subset \mathbb{R}\backslash\Omega_\varepsilon$ we have $Q_\varepsilon(\varphi_\varepsilon\theta(t)) = 0$ and $J_\varepsilon(\varphi_\varepsilon\theta(t)) = P_\varepsilon(\varphi_\varepsilon\theta(t))$. Then for $t \in (0, t_0]$ we get

$$\begin{aligned}
&|P_\varepsilon(\varphi_\varepsilon\theta(t)) - L_m(\theta(t))| \\
&\le \frac{1}{2}\Big|\int_{\mathbb{R}} \left[|\left(\varphi_\varepsilon\theta(t)\right)'|^2 - |\theta(t)'|^2 + V(\varepsilon x) f^2(\varphi_\varepsilon\theta(t)) - m f^2(\theta_t)\right] \mathrm{d}x\Big| \\
&\quad + \int_{\mathbb{R}} |H(f(\varphi_\varepsilon\theta(t))) - H(f(\theta(t)))|\,\mathrm{d}x.
\end{aligned}$$

At first, using a change of variables and the exponential decay of $\theta(t)$, $\theta(t)'$, we get

$$\int_{\mathbb{R}} |\left(\varphi_\varepsilon\theta(t)\right)' - \theta(t)'|^2\,\mathrm{d}x \le C \int_{\mathbb{R}} \left[\varepsilon^2 + (1-\varphi_\varepsilon)^2\right] \exp(-c|x|)\,\mathrm{d}x$$

for all $t \in (0, t_0]$. Now since $f(t)f'(t) < 2^{-1/2}$ for all $t \in [0, t_0]$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \left|V(\varepsilon x) f^2(\varphi_\varepsilon\theta(t)) - m f^2(\theta(t))\right| \mathrm{d}x \\
&\le \int_{\mathbb{R}} |V(\varepsilon x) - m|\, f^2(\varphi_\varepsilon\theta(t))\,\mathrm{d}x + m \int_{\mathbb{R}} \left|f^2(\varphi_\varepsilon\theta(t)) - f^2(\theta(t))\right| \mathrm{d}x \\
&\le 2^{1/2} C \int_{\mathbb{R}} \left[|V(\varepsilon x) - m| \chi_{\{|x| \le 2\beta/\varepsilon\}} + m(1-\varphi_\varepsilon)\right] \exp(-c|x|)\,\mathrm{d}x.
\end{aligned}$$

Recalling that

$$H(f(a+b)) - H(f(a)) = b \int_0^1 f'(a+sb) h(f(a+sb))\,\mathrm{d}s \tag{3.1}$$

due to the imbedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the boundedness of $\{\theta(t)\}$ in $L^\infty(\mathbb{R})$ it follows from (H1) that

$$\begin{aligned}
\int_{\mathbb{R}} |H(f(\varphi_\varepsilon\theta(t))) - H(f(\theta(t)))|\,\mathrm{d}x &\le C \int_{\mathbb{R}} |\varphi_\varepsilon\theta(t) - \theta(t)| \left[\theta(t) + \varphi_\varepsilon\theta(t)\right] \mathrm{d}x \\
&\le C \int_{\mathbb{R}} (1-\varphi_\varepsilon) \exp\left(-c|x|\right) \mathrm{d}x
\end{aligned}$$

for $t \in (0, t_0]$. Therefore, $J_\varepsilon(\varphi_\varepsilon\theta(t)) \to L_m(\theta(t))$ as $\varepsilon \to 0$, uniformly in $t \in [0, t_0]$. This is the end of the proof. □

For Lemma 3.2 there exists ε_0 sufficiently small such that

$$|J_\varepsilon(\varphi_\varepsilon\theta(t_0)) - L_m(\theta(t_0))| \le -L_m(\theta(t_0)) - 1$$

and so $J_\varepsilon(\varphi_\varepsilon\theta(t_0)) < -1$ for all $\varepsilon \in (0, \varepsilon_0)$. From now on we consider $\varepsilon \in (0, \varepsilon_0)$. We define the minimax level

$$C_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{s\in[0,1]} J_\varepsilon(\gamma(s)),$$

where

$$\Gamma_\varepsilon = \{\gamma \in \mathcal{C}([0, 1], E_\varepsilon) : \gamma(0) = 0,\ \gamma(1) = \varphi_\varepsilon\theta(t_0)\}.$$

Proposition 3.3. *C_ε converges to E_m as ε goes to zero.*

Proof. At first we will prove that

$$\limsup_{\varepsilon\to0} C_\varepsilon \le E_m.$$

Since $\theta : [0, t_0] \to H^1(\mathbb{R})$ is a continuous function using arguments as in Remark 3.1 we prove that $\gamma_\varepsilon : [0, 1] \to E_\varepsilon$ given by

$$\gamma_\varepsilon(s) := \varphi_\varepsilon\theta(st_0) \quad \text{for } s \in [0, 1] \tag{3.2}$$

is continuous. So $\gamma_\varepsilon \in \Gamma_\varepsilon$ and by Lemma 3.2 and Proposition 2.5 we obtain

$$\begin{aligned}\limsup_{\varepsilon\to0} C_\varepsilon &\le \limsup_{\varepsilon\to0} \max_{s\in[0,1]} J_\varepsilon(\gamma_\varepsilon(s))\\ &= \limsup_{\varepsilon\to0} \max_{t\in[0,t_0]} J_\varepsilon(\varphi_\varepsilon\theta(t))\\ &\le \max_{t\in[0,t_0]} L_m(\theta(t)) = E_m\end{aligned}$$

which concludes the first part of the proof. Next we are going to prove that

$$\liminf_{\varepsilon\to0} C_\varepsilon \ge E_m. \tag{3.3}$$

Let us assume $\liminf_{\varepsilon\to0} C_\varepsilon < E_m$ instead. Then there exist $\alpha > 0$, $\varepsilon_n \to 0$ and $\gamma_n \in \Gamma_{\varepsilon_n}$ satisfying $\max_{s\in[0,1]} J_{\varepsilon_n}(\gamma_n(s)) < E_m - \alpha$. Take ε_n such that

$$\frac{m}{2}\varepsilon_n \Big[1 + (1 + E_m)^{1/2}\Big] < \min\{\alpha, 1\}.$$

Denoting ε_n by ε and γ_n by γ, since $P_\varepsilon(\gamma(0)) = 0$ and $P_\varepsilon(\gamma(1)) = J_\varepsilon(\varphi_\varepsilon\theta(t_0)) < -1$ we can find $s_0 \in (0, 1)$ such that

$$P_\varepsilon(\gamma(s_0)) = -1 \quad \text{and} \quad P_\varepsilon(\gamma(s)) \ge -1 \quad \text{for} \quad s \in [0, s_0].$$

Then

$$Q_\varepsilon(\gamma(s)) \le J_\varepsilon(\gamma(s)) + 1 < E_m - \alpha + 1 < E_m + 1$$

which implies

$$\int_{\mathbb{R}\setminus\Omega_\varepsilon} f^2(\gamma(s))\,\mathrm{d}x \le \int_{\mathbb{R}\setminus\Omega_\varepsilon} |\gamma(s)|^2\,\mathrm{d}x \le \varepsilon\Big[1 + (1 + E_m)^{1/2}\Big],$$

for all $s \in [0, s_0]$. So it follows that

$$\begin{aligned}P_\varepsilon(\gamma(s)) &\ge L_m(\gamma(s)) - \frac{m}{2}\int_{\mathbb{R}\setminus\Omega_\varepsilon} f^2(\gamma(s))\,\mathrm{d}x\\ &\ge L_m(\gamma(s)) - \frac{m}{2}\varepsilon\Big[1 + (1 + E_m)^{1/2}\Big] \quad \text{for all } s \in [0, s_0].\end{aligned}$$

In particular for s_0, we have

$$L_m(\gamma(s_0)) \leq \frac{m}{2}\varepsilon[1 + (1 + E_m)^{1/2}] - 1 < 0.$$

Recalling that the mountain pass level for equation (2.6) corresponds to the least energy level (see [17]) we have $\max_{s\in[0,s_0]} L_m(\gamma(s)) \geq E_m$. Since

$$E_m - \alpha > \max_{s\in[0,1]} J_\varepsilon(\gamma(s)) \geq \max_{s\in[0,s_0]} P_\varepsilon(\gamma(s)),$$

by the estimates above we obtain

$$E_m - \alpha > E_m - \frac{m}{2}\varepsilon[1 + (1 + E_m)^{1/2}] > E_m - \alpha.$$

This contradiction completes the proof. □

At this point, denoting

$$D_\varepsilon \equiv \max_{s\in[0,1]} J_\varepsilon(\gamma_\varepsilon(s))$$

where γ_ε was defined in (3.2), we see that $C_\varepsilon \leq D_\varepsilon$ and also $\lim_{\varepsilon\to 0} D_\varepsilon = E_m$.

4. Existence of a critical point for J_ε

We define

$$J_\varepsilon^\alpha \equiv \{u \in E_\varepsilon : J_\varepsilon(u) \leq \alpha\}, \quad A^\alpha \equiv \{u \in E_\varepsilon : \inf_{v\in A} \|u - v\|_\varepsilon \leq \alpha\}$$

for any $A \subset E_\varepsilon$ and $\alpha > 0$. Moreover in the next propositions, for any $\varepsilon > 0$ and $R > 0$, we consider the functional J_ε restricted to the space $H_0^1((-R/\varepsilon, R/\varepsilon))$ endowed with the norm

$$\|v\|_\varepsilon = \|v'\|_{L^2((-R/\varepsilon,R/\varepsilon))} + \inf_{\lambda>0} \lambda\Big\{1 + \int_{-R/\varepsilon}^{R/\varepsilon} V(\varepsilon x) f^2(\lambda^{-1}v)\,dx\Big\}.$$

We will denote this space by E_ε^R. We can see that E_ε^R is a Banach space and J_ε is of class $\mathcal{C}^1$ on E_ε^R.

Proposition 4.1. *There exist $d > 0$ sufficiently small such that if $\varepsilon_n \to 0$, $R_n \to \infty$ and $u_n \in X_{\varepsilon_n}^d \cap E_{\varepsilon_n}^{R_n}$ satisfy*

$$\lim_{n\to\infty} J_{\varepsilon_n}(u_n) \leq E_m, \quad \lim_{n\to\infty} \|J'_{\varepsilon_n}(u_n)\|_{(E_{\varepsilon_n}^{R_n})'} = 0$$

then, up to subsequences, there exist $\{y_n\} \subset \mathbb{R}$ and $z_0 \in \mathcal{M}$ satisfying

$$\lim_{n\to\infty} |\varepsilon_n y_n - z_0| = 0, \quad \lim_{n\to\infty} \|u_n - \varphi_{\varepsilon_n}(\cdot - y_n)U(\cdot - y_n)\|_{\varepsilon_n} = 0.$$

Proof. From now on we suppose $d \in (0, 10)$. Since $u_n \in X_{\varepsilon_n}^d$ by definition of $X_{\varepsilon_n}^d$ there exists $v_n \in X_{\varepsilon_n}$ such that

$$\|u_n - v_n\|_{\varepsilon_n} \leq d. \tag{4.1}$$

We have $v_n(x) = \varphi_{\varepsilon_n}(x - z_n/\varepsilon_n)U(x - z_n/\varepsilon_n)$, $x \in \mathbb{R}$, for $\{z_n\} \subset \mathcal{M}^\beta$. From Remark 3.1 we have

$$\|u_n\|_{\varepsilon_n} \leq C \quad \text{for all } n \in \mathbb{N},\ d \in (0, 10).$$

By compactness of $\mathcal{M}^\beta$, up to subsequences, we may assume that $z_n \to z_0$ in $\mathbb{R}$ for some $z_0 \in \mathcal{M}^\beta$. We divide the proof of this proposition in five steps.

Step 1: For small $d > 0$, defining $A(y; r_1, r_2) = \{x \in \mathbb{R} : r_1 \le |y - x| \le r_2\}$ for $0 < r_1 < r_2$ and $y \in \mathbb{R}$, we obtain

$$\lim_{n\to\infty} \sup_{z \in A\left(\frac{z_n}{\varepsilon_n}; \frac{\beta}{2\varepsilon_n}, \frac{3\beta}{\varepsilon_n}\right)} \int_{z-R}^{z+R} |u_n|^2 \,\mathrm{d}x = 0 \quad \text{for any } R > 0.$$

Indeed, suppose that there exist $R > 0$ and a sequence $\{\tilde{z}_n\}$ satisfying

$$\tilde{z}_n \in A\Big(\frac{z_n}{\varepsilon_n}; \frac{\beta}{2\varepsilon_n}, \frac{3\beta}{\varepsilon_n}\Big), \quad \lim_{n\to\infty} \int_{\tilde{z}_n - R}^{\tilde{z}_n + R} |u_n|^2 \,\mathrm{d}x > 0.$$

Since Remark 3.1 implies that X_ε^d is uniformly bounded on $\varepsilon \in (0, \varepsilon_0)$ and $d \in (0, 10)$, due to Proposition 2.2 and the imbedding $H^1(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ we get $\{u_n'\}_n$ bounded in $L^2(\mathbb{R})$ and

$$\begin{aligned}
\int_{\mathbb{R}} |u_n|^2 \,\mathrm{d}x &\le C \int_{\mathbb{R}} \left[f^2(u_n) + f^4(u_n)\right] \,\mathrm{d}x \\
&\le C \int_{\mathbb{R}} V(\varepsilon x) f^2(u_n) \,\mathrm{d}x + C\|f(u_n)\|_{H^1}^4 \\
&\le C\Big\{\|u_n\|_{\varepsilon_n} + \Big[\int_{\mathbb{R}} \left(|u_n'|^2 + V(\varepsilon x) f^2(u_n)\right) \,\mathrm{d}x\Big]^2\Big\} \\
&\le C\left(\|u_n\|_{\varepsilon_n} + \|u_n\|_{\varepsilon_n}^2 + \|u_n\|_{\varepsilon_n}^4\right) \le \tilde{C}.
\end{aligned}$$

Consequently $\{u_n\}$ is bounded in $H^1(\mathbb{R})$. Hence we may assume that $\varepsilon_n \tilde{z}_n \to \tilde{z}_0$ and that $\tilde{w}_n := u_n(\cdot + \tilde{z}_n) \rightharpoonup \tilde{w}$ in $H^1(\mathbb{R})$ for some $\tilde{z}_0 \in A\left(z_0; \beta/2, 3\beta\right)$ and $\tilde{w} \in H^1(\mathbb{R})$. By the compactness of the imbedding $H^1((-R, R)) \hookrightarrow \mathcal{C}([-R, R])$ we get

$$\int_{-R}^{R} |\tilde{w}|^2 \,\mathrm{d}x = \lim_{n\to\infty} \int_{-R}^{R} |\tilde{w}_n|^2 \,\mathrm{d}x = \lim_{n\to\infty} \int_{\tilde{z}_n - R}^{\tilde{z}_n + R} |u_n|^2 \,\mathrm{d}x > 0$$

and so $\tilde{w} \ne 0$. Now given $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ let $\phi_n(x) = \phi(x - \tilde{z}_n)$, $n \in \mathbb{N}$. We have $\varepsilon_n \tilde{z}_n \in \mathcal{M}^{4\beta}$ and so we obtain $\phi_n \in E_{\varepsilon_n}^{R_n}$ for large n. Since $\|J_{\varepsilon_n}'(u_n)\|_{(E_{\varepsilon_n}^{R_n})'} \to 0$ and $\|\phi_n\|_{\varepsilon_n} \le C$ we have

$$\lim_{n\to\infty} \langle J_{\varepsilon_n}'(u_n), \phi_n\rangle = 0.$$

Consequently the boundedness of $\mathrm{supp}(\phi)$ implies that

$$\int_{\mathbb{R}} [\tilde{w}'\phi' + V(\tilde{z}_0) f'(\tilde{w}) f(\tilde{w})\phi]) \,\mathrm{d}x = \int_{\mathbb{R}} f'(\tilde{w}) h(f(\tilde{w}))\phi \,\mathrm{d}x.$$

Since ϕ is arbitrary it follows that $\tilde{w}$ satisfies

$$-\tilde{w}'' = f'(\tilde{w})[h(f(\tilde{w})) - V(\tilde{z}_0) f(\tilde{w})] = g_0(\tilde{w}), \quad \tilde{w} \ge 0 \quad \text{in } \mathbb{R}. \tag{4.2}$$

By assumptions on h we get g_0 locally Lipschitz continuous, $g_0(0) = 0$ and so due to ([4], Theorem 5) we know that the function g_0 must satisfy (2.7) for some $T > 0$. Thus Theorem 2.4 hods for problem (4.2) and $\tilde{w}(x) = w_0(x + c)$ where w_0 is radial. Then for $L_{V(\tilde{z}_0)}$ defined as L_m with $V(\tilde{x}_0)$ instead of m we denote $E_{V(\tilde{z}_0)} = L_{V(\tilde{z}_0)}(\tilde{w})$. By ([5], Theorem 2.1) we obtain $\tilde{w}_n'(x) \to \tilde{w}'(x)$ a.e. in A for any set $A \subset \mathbb{R}$. So using the Fatou's Lemma for $R > 0$ sufficiently large we get

$$\frac{1}{2} \int_{\mathbb{R}} |\tilde{w}'|^2 \,\mathrm{d}x \le \int_{-R}^{R} |\tilde{w}'|^2 \,\mathrm{d}x \le \liminf_{n\to\infty} \int_{-R}^{R} |\tilde{w}_n'|^2 \,\mathrm{d}x = \liminf_{n\to\infty} \int_{\tilde{z}_n - R}^{\tilde{z}_n + R} |u_n'|^2 \,\mathrm{d}x.$$

Since $V(\tilde{z}_0) \geq m$ and the least energy levels for equations (2.6) and (4.2) are equal to the mountain pass levels (see [17]) we have $E_{V(\tilde{z}_0)} \geq E_m$. Using item (iii) in Theorem 2.4 we see that

$$\int_{\mathbb{R}} |\tilde{w}'|^2 \, \mathrm{d}x = L_{V(\tilde{z}_0)}(\tilde{w}).$$

Thus we obtain

$$\liminf_{n\to\infty} \int_{\tilde{z}_n - R}^{\tilde{z}_n + R} |u_n'|^2 \, \mathrm{d}x \geq \frac{1}{2} L_{V(\tilde{z}_0)}(\tilde{w}) \geq \frac{1}{2} E_m > 0.$$

On the other hand, from (4.1) we have

$$\int_{\tilde{z}_n - R}^{\tilde{z}_n + R} |u_n'|^2 \, \mathrm{d}x \leq 4d^2$$

for large n ($n \geq n_0(d)$). Then

$$\frac{1}{2} E_m \leq \liminf_{n\to\infty} \int_{\tilde{z}_n - R}^{\tilde{z}_n + R} |u_n'|^2 \, \mathrm{d}x \leq 4d^2$$

which is impossible for $d \in (0, \sqrt{E_m/8})$. This proves Step 1.

Step 2: Defining $u_{n,1} = \varphi_{\varepsilon_n}(\cdot - z_n/\varepsilon_n) u_n$ and $u_{n,2} = u_n - u_{n,1}$ we have

$$J_{\varepsilon_n}(u_n) \geq J_{\varepsilon_n}(u_{n,1}) + J_{\varepsilon_n}(u_{n,2}) + o(1) \tag{4.3}$$

where $o(1)$ indicates the quantity that vanishes as $n \to \infty$.

Indeed, we can see that $Q_{\varepsilon_n}(u_{n,1}) = 0$ and $Q_{\varepsilon_n}(u_n) = Q_{\varepsilon_n}(u_{n,2})$. Then the boundedness of $\{u_n\}$ and the convexity of f^2 imply that

$$\begin{aligned}
&J_{\varepsilon_n}(u_{n,1}) + J_{\varepsilon_n}(u_{n,2}) \\
&= J_{\varepsilon_n}(u_n) + \frac{1}{2} \int_{\mathbb{R}} \Big\{ \varphi_{\varepsilon_n}^2(x - z_n/\varepsilon_n) + [1 - \varphi_{\varepsilon_n}(x - z_n/\varepsilon_n)]^2 - 1 \Big\} |u_n'|^2 \, \mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} V(\varepsilon_n x) \left[f^2(u_{n,1}) + f^2(u_{n,2}) - f^2(u_n) \right] \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} [H(f(u_n)) - H(f(u_{n,1})) - H(f(u_{n,2}))] \, \mathrm{d}x + o(1) \\
&\leq J_{\varepsilon_n}(u_n) + \int_{\mathbb{R}} [H(f(u_n)) - H(f(u_{n,1})) - H(f(u_{n,2}))] \, \mathrm{d}x + o(1).
\end{aligned}$$

To conclude Step 2 we need to estimate this last integral. We have

$$\begin{aligned}
&\int_{\mathbb{R}} [H(f(u_n)) - H(f(u_{n,1})) - H(f(u_{n,2}))] \, \mathrm{d}x \\
&= \int_{A\left(\frac{z_n}{\varepsilon_n}; \frac{\beta}{\varepsilon_n}, \frac{2\beta}{\varepsilon_n}\right)} [H(f(u_n)) - H(f(u_{n,1})) - H(f(u_{n,2}))] \, \mathrm{d}x.
\end{aligned}$$

Choose $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $A(0; \beta, 2\beta)$ and $\psi \equiv 0$ on $\mathbb{R} \backslash A(0; \beta/2, 3\beta)$. Setting $\psi_n(x) = \psi(\varepsilon_n x - z_n) u_n(x)$, for large n we get

$$\begin{aligned}
\sup_{y \in A\left(\frac{z_n}{\varepsilon_n}; \frac{\beta}{2\varepsilon_n}, \frac{3\beta}{\varepsilon_n}\right)} \int_{y-R}^{y+R} |u_n|^2 \, \mathrm{d}x &\geq \sup_{y \in A\left(\frac{z_n}{\varepsilon_n}; \frac{\beta}{2\varepsilon_n}, \frac{3\beta}{\varepsilon_n}\right)} \int_{y-R}^{y+R} |\psi_n|^2 \, \mathrm{d}x \\
&= \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} |\psi_n|^2 \, \mathrm{d}x.
\end{aligned}$$

Using Step 1 and a result of Lions [18, Lemma 1.1], we see that $\psi_n \to 0$ in $L^p(\mathbb{R})$ as $n \to \infty$ for all $p \in (2, \infty)$. Since $\psi_n = u_n$ in $A(z_n/\varepsilon_n; \beta/\varepsilon_n, 2\beta/\varepsilon_n)$ we obtain

$$\lim_{n\to\infty} \int_{A\left(\frac{z_n}{\varepsilon_n};\frac{\beta}{\varepsilon_n},\frac{2\beta}{\varepsilon_n}\right)} |u_n|^p \,\mathrm{d}x = 0.$$

Thus for $p > 2$ fixed using the fact that $|u_{n,1}|,\ |u_{n,2}| \le |u_n|$ and (H1) we see that given $\sigma > 0$ there exists $c = c(\sigma, p) > 0$ such that

$$\begin{aligned}
&\int_{A\left(\frac{z_n}{\varepsilon_n};\frac{\beta}{\varepsilon_n},\frac{2\beta}{\varepsilon_n}\right)} |H(f(u_n)) - H(f(u_{n,1})) - H(f(u_{n,2}))| \,\mathrm{d}x \\
&\le \sigma \|u_n\|_{L^2} + c\int_{A\left(\frac{z_n}{\varepsilon_n};\frac{\beta}{\varepsilon_n},\frac{2\beta}{\varepsilon_n}\right)} |u_n|^p \,\mathrm{d}x \le C\sigma
\end{aligned}$$

for large n. So (4.3) is proved.

Step 3: Given $d > 0$ sufficiently small there exists $n_0 = n_0(d)$ such that

$$J_{\varepsilon_n}(u_{n,2}) \ge \frac{1}{8}\Big[\int_{\mathbb{R}} \big(|u_{n,2}'|^2 + V(\varepsilon_n x) f^2(u_{n,2})\big)\,\mathrm{d}x\Big] \quad \text{for all } n \ge n_0.$$

In fact, using (4.1) we can see that there exists $n_0 = n_0(d)$ such that

$$\begin{aligned}
\|u_{n,2}'\|_{L^2} &\le \|[1 - \varphi_{\varepsilon_n}(\cdot - z_n/\varepsilon_n)]' u_n\|_{L^2} + \|u_n' - v_n'\|_{L^2} + \|(1 - \varphi_{\varepsilon_n})(\varphi_{\varepsilon_n} U)'\|_{L^2} \\
&\le o(1) + d \le 2d \quad \text{for all } n \ge n_0
\end{aligned}$$

where $v_n = \varphi_{\varepsilon_n}(\cdot - z_n/\varepsilon_n)U(\cdot - z_n/\varepsilon_n)$. Moreover by Proposition 2.2 we get

$$\int_{\mathbb{R}} V(\varepsilon_n x) f^2(u_{n,2})\,\mathrm{d}x \le c_0 d \quad \text{for all } n \ge n_0$$

for large n_0. Since $\{u_{n,2}\}$ is bounded in $H^1(\mathbb{R})$ it is also bounded in $L^\infty(\mathbb{R})$. So by (H1) we get

$$H(f(u_{n,2})) \le (V_0/4) f^2(u_{n,2}) + C f^4(u_{n,2}).$$

Due to the imbedding $H^1(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ and (V1) we see that

$$\int_{\mathbb{R}} H(f(u_{n,2})) \le \frac{1}{4}\int_{\mathbb{R}} V(\varepsilon x) f^2(u_{n,2})\,\mathrm{d}x + C\Big[\int_{\mathbb{R}} \big(|u_{n,2}'|^2 + V(\varepsilon x) f^2(u_{n,2})\big)\,\mathrm{d}x\Big]^2.$$

Hence we obtain

$$\begin{aligned}
J_{\varepsilon_n}(u_{n,2}) &\ge \frac{1}{2}\|u_{n,2}'\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}} V(\varepsilon_n x) f^2(u_{n,2})\,\mathrm{d}x - C\|f(u_{n,2})\|_{H^1}^4 \\
&\ge \big(\frac{1}{2} - C(2d)^2\big)\|u_{n,2}'\|_{L^2}^2 + \big(\frac{1}{4} - C(c_0 d)\big)\int_{\mathbb{R}} V(\varepsilon_n x) f^2(u_{n,2})\,\mathrm{d}x
\end{aligned}$$

for $n \ge n_0$. This proves Step 3 for small $d > 0$.

Step 4: We have $\lim_{n\to\infty} J_{\varepsilon_n}(u_{n,1}) = E_m$ and $z_0 \in \mathcal{M}$.

Indeed, let $w_n := u_{n,1}(\cdot + z_n/\varepsilon_n)$. After extracting a subsequence, we may assume $w_n \rightharpoonup w$ in $H^1(\mathbb{R})$, $w_n(x) \to w(x)$ for almost every $x \in \mathbb{R}$ and $w_n \to w$ in

$L^2((0,1))$. As we see in Step 3 using (8) and (11) of Lemma 2.1 and (2.3) it follows from (4.1)

$$\begin{aligned}
&\frac{V_0}{2}\int_0^1 f^2(\varphi_{\varepsilon_n}U)\,\mathrm{d}x - V_0\int_0^1 f^2(w_n)\,\mathrm{d}x\\
&\le V_0\int_0^1 f^2(w_n-\varphi_{\varepsilon_n}U)\,\mathrm{d}x\\
&\le \int_{\mathbb{R}} V(\varepsilon_n x) f^2(u_{n,1}-v_n)\,\mathrm{d}x\\
&\le 2\int_{\mathbb{R}} V(\varepsilon_n x)\left[f^2(u_n-v_n)+f^2(u_{n,2})\right]\,\mathrm{d}x \le c_0 d
\end{aligned}$$

for large n. Since $\varphi_{\varepsilon_n}U = U$ in $[0,1]$ for large n, we obtain

$$\int_0^1 f^2(w)\,\mathrm{d}x = \lim_{n\to\infty}\int_0^1 f^2(w_n)\,\mathrm{d}x \ge c\int_0^1 f^2(U)\,\mathrm{d}x - cd > 0$$

for small d. Consequently $w \neq 0$. Moreover for any $r>0$ it follows that

$$u_{n,1}(x+z_n/\varepsilon_n) = u_n(x+z_n/\varepsilon_n) \quad \text{in } (-r,r)$$

for large n. Then as in Step 1, we can see that w satisfies

$$-w'' = f'(w)\left[h(f(w)) - V(z_0)f(w)\right], \quad w>0 \quad \text{in } \mathbb{R}.$$

Now we shall consider two cases:

Case 1: $\lim_{n\to\infty}\sup_{z\in\mathbb{R}}\int_{z-1}^{z+1}|w_n-w|^2\,\mathrm{d}x = 0$.

Case 2: $\lim_{n\to\infty}\sup_{z\in\mathbb{R}}\int_{z-1}^{z+1}|w_n-w|^2\,\mathrm{d}x > 0$.

If Case 1 occurs we have that $w_n \to w$ in $L^p(\mathbb{R})$ for all $p\in(2,\infty)$. By (H1), (3.1) and the boundedness of $\|w_n\|_\infty$, given $\sigma>0$ there exists $C=C(\sigma)$ such that

$$\begin{aligned}
&\int_{\mathbb{R}} |H(f(w_n)) - H(f(w))|\mathrm{d}x\\
&\le \int_{\mathbb{R}} |w_n-w|\left[\sigma\left(|w|+|w_n|\right) + C\left(|w|^3+|w_n-w|^3\right)\right]\mathrm{d}x\\
&\le c\sigma + C\left(\|w_n-w\|_{L^4} + \|w_n-w\|_{L^4}^4\right) \le (c+1)\sigma
\end{aligned}$$

for large n. Thus

$$\int_{\mathbb{R}} H(f(w_n))\,\mathrm{d}x \to \int_{\mathbb{R}} H(f(w))\,\mathrm{d}x \quad \text{as } n\to\infty. \tag{4.4}$$

Now if Case 2 occurs there exists $\{\hat z_n\}\subset\mathbb{R}$ such that

$$\lim_{n\to\infty}\int_{\hat z_n-1}^{\hat z_n+1}|w_n-w|^2\,\mathrm{d}x > 0.$$

Since $w_n \rightharpoonup w$ in $H^1(\mathbb{R})$ we have

$$|\hat z_n| \to \infty. \tag{4.5}$$

Therefore,

$$\lim_{n\to\infty}\int_{\hat z_n-1}^{\hat z_n+1}|w|^2\,\mathrm{d}x = 0 \quad \text{and so} \quad \lim_{n\to\infty}\int_{\hat z_n-1}^{\hat z_n+1}|w_n|^2\,\mathrm{d}x > 0.$$

Since $w_n(x) = \varphi_{\varepsilon_n}(x)u_n(x + z_n/\varepsilon_n)$, it is easily seen that $|\hat{z}_n| \le 3\beta/\varepsilon_n$ for large n. If $|\hat{z}_n| \ge \beta/2\varepsilon_n$ for a subsequence from Step 1, we would have

$$0 < \lim_{n\to\infty} \int_{\hat{z}_n-1}^{\hat{z}_n+1} |w_n|^2\,\mathrm{d}x \le \lim_{n\to\infty} \sup_{z\in A\left(\frac{z_n}{\varepsilon_n};\frac{\beta}{2\varepsilon_n},\frac{3\beta}{\varepsilon_n}\right)} \int_{z-1}^{z+1} |u_n|^2\,\mathrm{d}x = 0$$

which is impossible. So $|\hat{z}_n| \le \beta/2\varepsilon_n$ for large n. We may assume that

$$\varepsilon_n\hat{z}_n \to \hat{z}_0 \quad \text{and} \quad u_{n,1}(\cdot + \hat{z}_n + z_n/\varepsilon_n) \rightharpoonup \hat{w},$$

and we see that $|\hat{z}_0| \le \beta/2$ and $\hat{w} \in H^1(\mathbb{R})\backslash\{0\}$. Then, given any $r > 0$ we have

$$u_{n,1}(\cdot + \hat{z}_n + z_n/\varepsilon_n) = u_n(\cdot + \hat{z}_n + z_n/\varepsilon_n) \quad \text{in } [-r, r]$$

for large n. Consequently as in Step 1 it follows that $\hat{w}$ satisfies

$$-\hat{w}'' = f'(\hat{w})\left[h(f(\hat{w})) - V(\hat{z}_0 + z_0)f(\hat{w})\right], \quad \hat{w} > 0 \quad \text{in } \mathbb{R}.$$

Analogous to Step 1, (4.5) leads us to a contradiction with (4.1) if $d > 0$ is sufficiently small. At this point we have proved that Case 2 does not hold and so Case 1 takes place. Now from ([5], Theorem 2.1) we see that $w_n'(x) \to w'(x)$ a.e. in $\mathbb{R}$. Then by (4.4) and Fatou's Lemma we have

$$\begin{aligned}
&\liminf_{n\to\infty} J_{\varepsilon_n}(u_{n,1})\\
&= \liminf_{n\to\infty} \Big\{\frac{1}{2}\int_{\mathbb{R}} \left[|w_n'|^2 + V(\varepsilon_n x + z_n)f^2(w_n)\right]\,\mathrm{d}x - \int_{\mathbb{R}} H(f(w_n))\,\mathrm{d}x\Big\}\\
&\ge \frac{1}{2}\int_{\mathbb{R}} \left[|w'|^2 + V(z_0)f^2(w)\right]\,\mathrm{d}x - \int_{\mathbb{R}} H(f(w))\,\mathrm{d}x\\
&\ge L_{V(z_0)}(w) \ge E_{V(z_0)} \ge E_m.
\end{aligned}$$

On the other hand, since $\lim_{n\to\infty} J_{\varepsilon_n}(u_n) \le E_m$ and $J_{\varepsilon_n}(u_{n,2}) \ge 0$ because of (4.3) we get

$$\limsup_{n\to\infty} J_{\varepsilon_n}(u_{n,1}) \le E_m.$$

Hence $E_{V(z_0)} = E_m$ and $\lim_{n\to\infty} J_{\varepsilon_n}(u_{n,1}) = E_m$. Moreover from the mountain pass characterization to the least energy solution and Proposition 2.5 we can see that $a > b$ implies $E_a > E_b$. So $V(z_0) = m$ and this concludes the proof of Step 4.

Step 5: Conclusion. From Step 4, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \left[|w_n'|^2 + V(\varepsilon_n x + z_n)f^2(w_n)\right]\mathrm{d}x = \int_{\mathbb{R}} \left(|w'|^2 + mf^2(w)\right)\,\mathrm{d}x.$$

Since w is a solution for (2.6) there exists $\zeta \in \mathbb{R}$ such that $w = U(\cdot - \zeta)$. We have $w_n(x) \to w(x)$ and $w_n'(x) \to w'(x)$ a.e. in $\mathbb{R}$ which imply the following convergence results

$$\int_A |w_n'|^2\,\mathrm{d}x \to \int_A |w'|^2\,\mathrm{d}x, \quad \int_A V(\varepsilon_n x + z_n)f^2(w_n)\,\mathrm{d}x \to \int_A mf^2(w)\,\mathrm{d}x,$$
$$\int_A V(\varepsilon_n x + z_n)f^2(\varphi_{\varepsilon_n}(x - \zeta)w)\,\mathrm{d}x \to \int_A mf^2(w)\,\mathrm{d}x$$

for any $A \subset \mathbb{R}$. Then given $\sigma > 0$ there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{\{|x|\ge R\}} V(\varepsilon_n x + z_n)\left[f^2(w_n) + f^2(\varphi_{\varepsilon_n}(x - \zeta)w)\right]\,\mathrm{d}x \le \frac{\sigma}{4}$$

for all $n \geq n_0$. On the other hand, due the convergence $w_n \to w$ in $L^2((-R,R))$ we obtain

$$\int_{-R}^{R} V(\varepsilon_n x + z_n) f^2(w_n - \varphi_{\varepsilon_n}(x-\zeta)w)\,\mathrm{d}x \leq \frac{\sigma}{2} \quad \text{for all } n \geq n_0$$

for large n_0 . This implies

$$\int_{\mathbb{R}} V(\varepsilon_n x + z_n) f^2(w_n - \varphi_{\varepsilon_n}(x-\zeta)w)\,\mathrm{d}x \leq \sigma \quad \text{for all } n \geq n_0.$$

By the definition of $|\|\cdot\||_{\varepsilon_n}$ (see also Remark 3.1), we obtain

$$|\|u_{n,1} - \varphi_{\varepsilon_n}(\cdot - \zeta - z_n/\varepsilon_n)w(\cdot - z_n/\varepsilon_n)\||_{\varepsilon_n} \to 0.$$

Now let $y_n := z_n/\varepsilon_n + \zeta$. Since $w_n'(x) \to w'(x)$ a.e. in $\mathbb{R}$ and $\|w_n'\|_{L^2} \to \|w'\|_{L^2}$ from Brezis-Lieb Lemma (see [6]) it follows that $w_n' \to w'$ in $L^2(\mathbb{R})$. Consequently $[u_{n,1} - \varphi_{\varepsilon_n}(\cdot - y_n)U(\cdot - y_n)]' \to 0$ in $L^2(\mathbb{R})$. Hence

$$\|u_{n,1} - \varphi_{\varepsilon_n}(\cdot - y_n)U_0(\cdot - y_n)\|_{\varepsilon_n} \to 0 \quad \text{as } n \to \infty.$$

On the other hand, using Steps 2, 3, and 4, we obtain

$$E_m \geq \lim_{n\to\infty} J_{\varepsilon_n}(u_n) \geq E_m + \frac{1}{8}\limsup_{n\to\infty}\int_{\mathbb{R}} [|u_{n,2}'|^2 + V(\varepsilon_n x) f^2(u_{n,2})]\,\mathrm{d}x,$$

which implies that $\|u_{n,2}\|_{\varepsilon_n} \to 0$. This completes the proof. □

We observe that the result of Proposition 4.1 holds for $d \in (0, d_0)$, with $d_0 > 0$ sufficiently small, independently of the sequences satisfying the assumptions.

Corollary 4.2. *For any $d \in (0, d_0)$ there exist constants ω_d, R_d, $\varepsilon_d > 0$ such that*

$$\|J_\varepsilon'(u)\|_{(E_\varepsilon^R)'} \geq \omega_d$$

for any $u \in E_\varepsilon^R \cap J_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^{d_0} \backslash X_\varepsilon^d)$, $R \geq R_d$ and $\varepsilon \in (0, \varepsilon_d)$.

Proof. By contradiction we suppose that for some $d \in (0, d_0)$ there exist sequences $\{\varepsilon_n\}$, $\{R_n\}$ and $\{u_n\}$ such that

$$R_n \geq n, \quad \varepsilon_n \leq 1/n, \quad u_n \in E_{\varepsilon_n}^{R_n} \cap J_{\varepsilon_n}^{D_{\varepsilon_n}} \cap (X_{\varepsilon_n}^{d_0} \backslash X_{\varepsilon_n}^d), \quad \|J_{\varepsilon_n}'(u_n)\|_{(E_{\varepsilon_n}^{R_n})'} < \frac{1}{n}.$$

By Proposition 4.1 there exist $\{y_n\} \subset \mathbb{R}$ and $z_0 \in \mathcal{M}$ such that

$$\lim_{n\to\infty} |\varepsilon_n y_n - z_0| = 0, \quad \lim_{n\to\infty} \|u_n - \varphi_{\varepsilon_n}(\cdot - y_n)U(\cdot - y_n)\|_{\varepsilon_n} = 0.$$

So for sufficiently large n, we have $\varepsilon_n y_n \in \mathcal{M}^\beta$ and then, by the definition of X_{ε_n} and $X_{\varepsilon_n}^d$, we obtain $\varphi_{\varepsilon_n}(\cdot - y_n)U(\cdot - y_n) \in X_{\varepsilon_n}$ and $u_n \in X_{\varepsilon_n}^d$. This contradicts $u_n \in X_{\varepsilon_n}^{d_0} \backslash X_{\varepsilon_n}^d$ and completes the proof. □

The next lemmas are necessary to obtain a suitable bounded Palais-Smale sequence in E_ε^R.

Lemma 4.3. *Given $\lambda > 0$ there exist ε_0 and $d_0 > 0$ small enough such that*

$$J_\varepsilon(u) > E_m - \lambda \quad \textit{for all } u \in X_\varepsilon^{d_0} \;\; \varepsilon \in (0, \varepsilon_0).$$

Proof. For $u \in X_\varepsilon$ we have $u(x) = \varphi_\varepsilon(x - z/\varepsilon)U(x - z/\varepsilon)$, $x \in \mathbb{R}$, for some $z \in \mathcal{M}^\beta$. Since $L_m(U) = E_m$ by (V2) we obtain

$$\begin{aligned} J_\varepsilon(u) - E_m \geq \frac{1}{2}\int_{\mathbb{R}} \big[\big(|(\varphi_\varepsilon U)'|^2 - |U'|^2\big) + m\left(f^2(\varphi_\varepsilon U) - f^2(U)\right)\big]\,\mathrm{d}x \\ - \int_{\mathbb{R}} |H(f(\varphi_\varepsilon U)) - H(f(U))|\,\mathrm{d}x \end{aligned}$$

independently of $z \in \mathcal{M}^\beta$. It is easily seen that $\varphi_\varepsilon U \to U$ in $H^1(\mathbb{R})$ as $\varepsilon \to 0$. Hence using (3.1) we can see that there exists $\varepsilon_0 > 0$ such that

$$J_\varepsilon(u) - E_m > -\frac{\lambda}{2} \quad \text{for all } u \in X_\varepsilon,\ \varepsilon \in (0, \varepsilon_0).$$

Now, if $v \in X_\varepsilon^d$ there exists $u \in X_\varepsilon$ such that $\|u - v\|_\varepsilon \leq d$. We have $v = u + w$ with $\|w\|_\varepsilon \leq d$. Since $Q_\varepsilon(u) = 0$ we see that

$$\begin{aligned} J_\varepsilon(v) - J_\varepsilon(u) \geq \frac{1}{2}\int_{\mathbb{R}} \big[|(u+w)'|^2 - |u'|^2 + V(\varepsilon x)\left(f^2(u+w) - f^2(u)\right)\big]\,\mathrm{d}x \\ - \int_{\mathbb{R}} [H(f(u+w)) - H(f(u))]\,\mathrm{d}x. \end{aligned}$$

From (2.3) and Lemma 2.1 we obtain

$$\begin{aligned} &\int_{\mathbb{R}} V(\varepsilon x)\big|f^2(u+w) - f^2(u)\big|\,\mathrm{d}x \\ &\leq \int_{\{|w|\leq 1\}} V(\varepsilon x)\,|f(u+w) - f(u)|\,|f(u+w) + f(u)|\,\mathrm{d}x \\ &\quad + \int_{\{|w|>1\}} V(\varepsilon x)\,\big|f^2(u+w) - f^2(u)\big|\,\mathrm{d}x \\ &\leq C(\||w\||_\varepsilon^{1/2} + \||w\||_\varepsilon) \\ &\leq Cd \leq \frac{\lambda}{6} \end{aligned}$$

provided d is small enough. With the same arguments as used before we see that there exists small $d_0 > 0$ such that

$$J_\varepsilon(v) > J_\varepsilon(u) - \frac{\lambda}{2} > E_m - \lambda \quad \text{for all } v \in X_\varepsilon^{d_0},\ \varepsilon \in (0, \varepsilon_0).$$

This completes the proof. □

Following Corollary 4.2 and Lemma 4.3, we fix $d_0 > 0$, $d_1 \in (0, d_0/3)$ and corresponding $\omega > 0$, $R_0 > 0$ and $\varepsilon_0 > 0$ satisfying

$$\begin{gathered} \|J_\varepsilon'(u)\|_{(E_\varepsilon^R)'} \geq \omega \quad \text{for all } u \in E_\varepsilon^R \cap J_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^{d_0} \backslash X_\varepsilon^{d_1}), \\ J_\varepsilon(u) > E_m/2 \quad \text{for all } u \in X_\varepsilon^{d_0} \end{gathered} \tag{4.6}$$

for any $R \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$. Thus we obtain the following result.

Lemma 4.4. *There exists $\alpha > 0$ such that $|s - 1/t_0| \leq \alpha$ implies $\gamma_\varepsilon(s) \in X_\varepsilon^{d_1}$ for all $\varepsilon \in (0, \varepsilon_0)$, where γ_ε is given by* (3.2).

Proof. At first we observe that

$$\begin{aligned}\|\varphi_\varepsilon v\|_\varepsilon &\le \|(\varphi_\varepsilon v)'\|_{L^2} + \|v\|_{L^2}\Big\{1 + \int_{\mathbb{R}} V(\varepsilon x) f^2\left(\|v\|_{L^2}^{-1}\varphi_\varepsilon v\right)\,\mathrm{d}x\Big\}\\ &\le \|\varepsilon\varphi'(\varepsilon\cdot)v + \varphi_\varepsilon v'\|_{L^2} + \|v\|_{L^2}\big(1 + \sup_\Omega V(x)\big)\\ &\le C_0\|v\|_{H^1} \quad \text{for all } \varepsilon \in (0,\varepsilon_0),\ v \in H^1(\mathbb{R}).\end{aligned}$$

Since the function $\theta : [0, t_0] \to H^1(\mathbb{R})$ given by Proposition 2.5 is continuous and $\theta(1) = U$ there exists $\sigma > 0$ such that

$$|t-1| \le \sigma \quad \Rightarrow \quad \|\theta(t) - U\|_{H^1} < \frac{d_1}{C_0}.$$

So if $|st_0 - 1| \le \sigma$, which means $|s - 1/t_0| \le \sigma/t_0 =: \alpha$, this inequality yields

$$\|\gamma_\varepsilon(s) - \varphi_\varepsilon U\|_\varepsilon = \|\varphi_\varepsilon[\theta(st_0) - U]\|_\varepsilon \le C_0\|\theta(st_0) - U\| < d_1 \quad \text{for } \varepsilon \in (0,\varepsilon_0).$$

Since $\varphi_\varepsilon U \in X_\varepsilon$ we have $\gamma_\varepsilon(s) \in X_\varepsilon^{d_1}$. □

Lemma 4.5. *For α given in Lemma 4.4 there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that*

$$J_\varepsilon(\gamma_\varepsilon(s)) < E_m - \rho \quad \textit{for any } \varepsilon \in (0,\varepsilon_0),\ |s - 1/t_0| \ge \alpha.$$

Proof. By Proposition 2.5 we have $L_m(\theta(t)) < E_m$ for all $t \ne 1$. So there exists $\rho > 0$ satisfying

$$L_m(\theta(t)) < E_m - 2\rho \quad \text{for all } t \in [0,t_0] \text{ such that } |t-1| \ge t_0\alpha.$$

From Lemma 3.2 we know that there exists $\varepsilon_0 > 0$ such that

$$\sup_{t\in[0,t_0]} |J_\varepsilon(\varphi_\varepsilon\theta(t)) - L_m(\theta(t))| < \rho \quad \text{for } \varepsilon \in (0,\varepsilon_0).$$

So for $|t-1| \ge t_0\alpha$ and $\varepsilon \in (0,\varepsilon_0)$ we obtain

$$J_\varepsilon(\varphi_\varepsilon\theta(t)) \le L_m(\theta(t)) + |J_\varepsilon(\varphi_\varepsilon\theta(t)) - L_m(\theta(t))| < E_m - 2\rho + \rho = E_m - \rho\,.$$

The proof is complete. □

Proposition 4.6. *For sufficiently small $\varepsilon > 0$ and large $R > 0$ there exists a sequence $\{u_n^R\} \subset E_\varepsilon^R \cap X_\varepsilon^{d_0} \cap J_\varepsilon^{D_\varepsilon}$ such that $J'_\varepsilon(u_n^R) \to 0$ in $\big(E_\varepsilon^R\big)'$ as $n \to \infty$.*

Proof. We take $R_0 > 0$ such that $\Omega \subset B(0,R_0)$. Then $\gamma_\varepsilon([0,1]) \subset E_\varepsilon^R$ for all $R \ge R_0$. Suppose that the statement of Proposition 4.6 does not hold. Then for small $\varepsilon > 0$ and large $R > R_0$ there exists $a(\varepsilon,R) > 0$ such that

$$\|J'_\varepsilon(u)\|_{(E_\varepsilon^R)'} \ge a(\varepsilon,R) \quad \text{on } E_\varepsilon^R \cap X_\varepsilon^{d_0} \cap J_\varepsilon^{D_\varepsilon}.$$

From (4.6) that there exists ω independent of $\varepsilon \in (0,\varepsilon_0)$ and $R > R_0$ satisfying

$$\|J'_\varepsilon(u)\|_{(E_\varepsilon^R)'} \ge \omega \quad \text{on} \quad E_\varepsilon^R \cap (X_\varepsilon^{d_0}\backslash X_\varepsilon^{d_1}) \cap J_\varepsilon^{D_\varepsilon}.$$

So there exists a pseudo-gradient vector field, T_ε^R, for J_ε on a neighborhood $Z_\varepsilon^R \subset E_\varepsilon^R$ of $E_\varepsilon^R \cap X_\varepsilon^{d_0} \cap J_\varepsilon^{D\varepsilon}$. We refer to [22] for details. Let $\tilde Z_\varepsilon^R \subset Z_\varepsilon^R$ for which one $\|J'_\varepsilon(u)\|_{(E_\varepsilon^R)'} > a(\varepsilon,R)/2$ and take a Lipschitz continuous function on E_ε^R, η_ε^R, such that

$$0 \le \eta_\varepsilon^R \le 1, \quad \eta_\varepsilon^R \equiv 1 \text{ on } E_\varepsilon^R \cap X_\varepsilon^{d_0} \cap J_\varepsilon^{D_\varepsilon}, \quad \text{and} \quad \eta_\varepsilon^R \equiv 0 \text{ on } E_\varepsilon^R\backslash\tilde Z_\varepsilon^R.$$

Letting $\xi : \mathbb{R} \to \mathbb{R}^+$ be a Lipschitz continuous function such that

$$\xi \le 1, \quad \xi(a) = 1 \quad \text{if } |a - E_m| \le E_m/2, \quad \text{and} \quad \xi(a) = 0 \quad \text{if } |a - E_m| \ge E_m$$

and defining

$$e_\varepsilon^R(u) = \begin{cases} -\eta_\varepsilon^R(u)\xi(J_\varepsilon(u))T_\varepsilon^R(u) & \text{if } u \in Z_\varepsilon^R \\ 0 & \text{if } u \in E_\varepsilon^R \backslash Z_\varepsilon^R, \end{cases}$$

there exists a global solution $\Psi_\varepsilon^R : E_\varepsilon^R \times \mathbb{R} \to E_\varepsilon^R$, which is unique, of the initial value problem

$$\begin{gathered} \frac{d}{dt}\Psi_\varepsilon^R(u,t) = e_\varepsilon^R(\Psi_\varepsilon^R(u,t)) \\ \Psi_\varepsilon^R(u,0) = u. \end{gathered} \tag{4.7}$$

Since $\lim_{\varepsilon\to 0} D_\varepsilon = E_m$, we have $D_\varepsilon \le E_m + (1/2)\min\{E_m, \omega^2 d_1\}$ for small $\varepsilon > 0$. Hence, by the choice of d_0 and d_1, Ψ_ε^R has the following properties:

(i) $\Psi_\varepsilon^R(u,t) = u$ if $t = 0$ or $u \in E_\varepsilon^R \backslash Z_\varepsilon^R$ or $J_\varepsilon(u) \notin (0, 2E_m)$.
(ii) $\|\frac{d}{dt}\Psi_\varepsilon^R(u,t)\| \le 2$ for all (u,t).
(iii) $\frac{d}{dt}\left(J_\varepsilon\left(\Psi_\varepsilon^R(u,t)\right)\right) \le 0$ for all (u,t).
(iv) $\frac{d}{dt}\left(J_\varepsilon(\Psi_\varepsilon^R(u,t))\right) \le -\omega^2$ if $\Psi_\varepsilon^R(u,t) \in E_\varepsilon^R \cap (X_\varepsilon^{d_0} \backslash X_\varepsilon^{d_1}) \cap J_\varepsilon^{D_\varepsilon}$.
(v) $\frac{d}{dt}(J_\varepsilon(\Psi_\varepsilon^R(u,t))) \le -(a(\varepsilon,R))^2$ if $\Psi_\varepsilon^R(u,t) \in E_\varepsilon^R \cap X_\varepsilon^{d_1} \cap J_\varepsilon^{D_\varepsilon}$.

Due to Lemmas 4.4 and 4.5, there exist α and $\rho > 0$ such that

$$|s - 1/t_0| \le \alpha \Longrightarrow \gamma_\varepsilon(s) \in X_\varepsilon^{d_1} \quad \text{and} \quad |s - 1/t_0| > \alpha \Longrightarrow J_\varepsilon(\gamma_\varepsilon(s)) < E_m - \rho$$

for all $\varepsilon \in (0, \varepsilon_0)$. Defining $\gamma_\varepsilon^R(s) = \Psi_\varepsilon^R(\gamma_\varepsilon(s), t_\varepsilon^R)$ we shall prove that

$$J_\varepsilon(\gamma_\varepsilon^R(s)) \le E_m - \min\left\{\rho, \frac{\omega^2 d_1}{2}\right\} \quad \text{for all } s \in [0,1], \tag{4.8}$$

for t_ε^R sufficiently large. Note that by (iii) above if $|s - 1/t_0| > \alpha$ it follows that

$$J_\varepsilon(\Psi_\varepsilon^R(\gamma_\varepsilon(s), t)) \le J_\varepsilon(\gamma_\varepsilon(s)) < E_m - \rho \quad \text{for any } t > 0.$$

So (4.8) holds for any t_ε^R. Now, if $s \in I := [1/t_0 - \alpha, 1/t_0 + \alpha]$, we get $\gamma_\varepsilon(s) \in X_\varepsilon^{d_1}$ and two distinct cases are considered:

(a) $\Psi_\varepsilon^R(\gamma_\varepsilon(s), t) \in X_\varepsilon^{d_0}$ for all $t \in [0, \infty)$.
(b) $\Psi_\varepsilon^R(\gamma_\varepsilon(s), t_s) \notin X_\varepsilon^{d_0}$ for some $t_s > 0$.

If $s \in I$ satisfies (a), then (i), (iv) and (v) yield

$$\begin{aligned} J_\varepsilon(\Psi_\varepsilon^R(\gamma_\varepsilon(s), t)) &= J_\varepsilon(\gamma_\varepsilon(s)) + \int_0^t \frac{d}{d\tau}\left(J_\varepsilon(\Psi_\varepsilon^R(\gamma_\varepsilon(s), \tau))\right) \mathrm{d}\tau \\ &\le D_\varepsilon - \min\left\{\omega^2, (a(\varepsilon,R))^2\right\} t \end{aligned}$$

and so $J_\varepsilon(\Psi_\varepsilon^R(\gamma_\varepsilon(s), t)) \to -\infty$ as $t \to \infty$ which is in contradiction with (4.6). Thus any $s \in I$ satisfies (b). We fix s_0 and a neighborhood $I^{s_0} = I^{s_0}(\varepsilon, R) \subset I$ such that $\Psi_\varepsilon^R(\gamma_\varepsilon(s), t_{s_0}) \notin X_\varepsilon^{d_0}$ for all $s \in I^{s_0}$. Since $\gamma_\varepsilon(s) \in X_\varepsilon^{d_1}$ for any $s \in I^{s_0}$, we can observe from $(i) - (v)$ that there exists an interval $[t_s^1, t_s^2] \subset [0, t_{s_0}]$ for which one

$$\Psi_\varepsilon^R(\gamma_\varepsilon(s), t) \in X_\varepsilon^{d_0} \backslash X_\varepsilon^{d_1} \quad \text{for} \quad t \in [t_s^1, t_s^2] \quad \text{and} \quad |t_s^1 - t_s^2| \ge d_1.$$

So (i), (iii) and (iv) lead to

$$\begin{aligned} J_\varepsilon\left(\Psi_\varepsilon^R(\gamma_\varepsilon(s),t_{s_0})\right) &\le J_\varepsilon\left(\gamma_\varepsilon(s)\right) + \int_{t_s^1}^{t_s^2} \frac{d}{d\tau}\left(J_\varepsilon(\Psi_\varepsilon^R(\gamma_\varepsilon(s),\tau))\right) \mathrm{d}\tau \\ &\le D_\varepsilon - \omega^2\left(t_s^2 - t_s^1\right) \\ &\le E_m - \frac{1}{2}\omega^2 d_1 \quad \text{for all } s \in I^{s_0}. \end{aligned}$$

By compactness there exist $s_1, \cdots, s_l$, $l = l(\varepsilon, R)$, such that $I = \bigcup_{i=1}^l I^{s_i}$. Let $t_\varepsilon^R = \max_{1\le i\le l} t_{s_i}$. Then for any $s \in I$ we have $s \in I^{s_i}$ for some i and so

$$J_\varepsilon(\Psi_\varepsilon^R(\gamma_\varepsilon(s), t_\varepsilon^R)) \le J_\varepsilon(\Psi_\varepsilon^R(\gamma_\varepsilon(s), t_{s_i})) \le E_m - \frac{1}{2}\omega^2 d_1.$$

Therefore, (4.8) holds. Since $\gamma_\varepsilon^R \in \Gamma_\varepsilon$ we obtain

$$C_\varepsilon \le \max_{s\in[0,1]} J_\varepsilon(\gamma_\varepsilon^R(s)) \le E_m - \min\left\{\rho, \frac{\omega^2 d_1}{2}\right\},$$

which is in contradiction with Proposition 3.3. This completes the proof. □

Proposition 4.7. *There exists a critical point $u_\varepsilon \in X_\varepsilon^{d_0} \cap J_\varepsilon^{D_\varepsilon}$ of J_ε if $\varepsilon > 0$ is sufficiently small.*

Proof. From Proposition 4.6 there exist $\varepsilon_0 > 0$ and $R_0 > 0$ for which ones we can find $\{u_n\}_n \subset E_\varepsilon^R \cap X_\varepsilon^{d_0} \cap J_\varepsilon^{D_\varepsilon}$ such that $J_\varepsilon'(u_n) \to 0$ in $\left(E_\varepsilon^R\right)'$ as $n \to \infty$, for each $R \ge R_0$ and $\varepsilon \in (0, \varepsilon_0)$. Since $\{u_n\}_n$ is bounded in E_ε^R it is also bounded in $H_0^1((-R/\varepsilon, R/\varepsilon))$ with the usual norm. So we may assume that $u_n \rightharpoonup u$ in $H_0^1((-R/\varepsilon, R/\varepsilon))$, $u_n \to u$ in $L^r((-R/\varepsilon, R/\varepsilon))$ for $r = 2$ and 4 and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}$ where $u = u_{\varepsilon,R}$. Because $\|J_\varepsilon'(u_n)\|_{(E_\varepsilon^R)'} \to 0$ we see that u is a nonnegative solution for

$$-u'' = f'(u)\left[h(f(u)) - V(\varepsilon x) f(u)\right] - g_{\varepsilon,R}(u)\chi_\varepsilon u \quad \text{in } (-R/\varepsilon, R/\varepsilon) \tag{4.9}$$

where

$$g_{\varepsilon,R}(u) = 4\Big(\int_{-R/\varepsilon}^{R/\varepsilon} \chi_\varepsilon |u|^2\,\mathrm{d}x - 1\Big)_+.$$

Then we can see that $u_n \to u$ in $H_0^1((-R/\varepsilon, R/\varepsilon))$ which implies

$$\int_{B(0,R/\varepsilon)} \left[|u_n' - u'|^2 + V(\varepsilon x) f^2(u_n - u)\right]\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty$$

and so $u_n \to u$ in E_ε. Thus $u \in X_\varepsilon^{d_0} \cap J_\varepsilon^{D_\varepsilon}$. Due to boundedness of $\{u_{\varepsilon,R}\}$ in $H^1(\mathbb{R})$ we get $\|u_{\varepsilon,R}\|_\infty \le C_0$ for all $R \ge R_0$ and $\varepsilon \in (0, \varepsilon_0)$. So from (H1) and Lemma 2.1 there exists $C > 0$ depending on C_0 such that

$$-u'' \le C f'(u) f(u)^2 \le Cu \quad \text{in } (-R/\varepsilon, R/\varepsilon).$$

Hence by [15, Theorem 9.26], there exists $C_0 = C_0(N, C)$ such that

$$\sup_{B(y,1)} u \le C_0\,\|u\|_{L^2(B(y,2))} \quad \text{for all } y \in \mathbb{R}. \tag{4.10}$$

Due to the boundedness of $\{\|u_{\varepsilon,R}\|_\varepsilon\}$ and $\{J_\varepsilon(u_{\varepsilon,R})\}$ we get $\{Q_\varepsilon(u_{\varepsilon,R})\}$ uniformly bounded on $R \ge R_0$ and $\varepsilon \in (0, \varepsilon_0)$. So there is $C_1 > 0$ such that

$$\int_{\{|x|\ge R_0/\varepsilon\}} |u_{\varepsilon,R}|^2\,\mathrm{d}x \le \varepsilon \int_{\mathbb{R}} \chi_\varepsilon |u_{\varepsilon,R}|^2\,\mathrm{d}x \le \varepsilon C_1 \tag{4.11}$$

for any $R \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$. Hence for sufficiently small ε_0 and $\varepsilon \in (0, \varepsilon_0)$ fixed, it follows from (4.10), (4.11) and by (H1)

$$h(f(u_{\varepsilon,R}(x))) \leq \frac{V_0}{2} f(u_{\varepsilon,R}(x)) \quad \text{for any } |x| \geq \frac{R_0}{\varepsilon} + 2,\ R \geq R_0.$$

Then after some calculations we obtain

$$\lim_{A \to \infty} \int_{\mathbb{R}^N \setminus B(0,A)} \left[|u'_{\varepsilon,R}|^2 + V(\varepsilon x) f^2(u_{\varepsilon,R})\right] \mathrm{d}x = 0 \tag{4.12}$$

uniformly on $R \geq R_0$. We take $R_k \to \infty$ and denote $u_k = u_{\varepsilon,R_k}$. We may assume $u_k \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R})$ as $k \to \infty$. Since u_k is a solution for (4.9), using (4.12) and ([5], Theorem 2.1) we see that

$$\int_{\mathbb{R}} |u'_k|^2 \, \mathrm{d}x \to \int_{\mathbb{R}} |u'_\varepsilon|^2 \, \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}} V(\varepsilon x) f^2(u_k - u_\varepsilon) \, \mathrm{d}x \to 0$$

as $k \to \infty$, up to subsequences. From this result we get $u_k \to u_\varepsilon$ in E_ε which implies that $u_\varepsilon \in X_\varepsilon^{d_0} \cap J_\varepsilon^{D_\varepsilon}$ and $J'_\varepsilon(u_\varepsilon) = 0$ in E'_ε. This completes the proof. □

5. Proof of Theorem 1.1

Until now we have proved the existence of a critical point for J_ε, $u_\varepsilon \in X_\varepsilon^{d_0} \cap J_\varepsilon^{D_\varepsilon}$, for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ and $d_0 > 0$ sufficiently small. We also have $u_\varepsilon \geq 0$ and $J_\varepsilon(u_\varepsilon) \geq (E_m/2)$ which imply $u_\varepsilon \neq 0$. The function u_ε satisfies

$$-u''_\varepsilon = f'(u_\varepsilon)\left[h(f(u_\varepsilon)) - V(\varepsilon x) f(u_\varepsilon)\right] - 4\Big(\int_{\mathbb{R}} \chi_\varepsilon |u|^2 \, \mathrm{d}x - 1\Big)_+ \chi_\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}. \tag{5.1}$$

Since $u_\varepsilon \in \mathcal{C}^{1,\alpha}_{\rm loc}(\mathbb{R})$ by the Maximum Principle we get $u_\varepsilon > 0$. Moreover from (5.1) we can see that there exists $\rho > 0$ such that $\|u_\varepsilon\|_{L^\infty} \geq \rho$ for small $\varepsilon > 0$. We observe that by Proposition 4.1 there exists $\{y_\varepsilon\} \subset \mathbb{R}$ such that $\varepsilon y_\varepsilon \in \mathcal{M}^{2\beta}$ and for any sequence $\varepsilon_n \to 0$ there exists $z_0 \in \mathcal{M}$ satisfying

$$\varepsilon_n y_{\varepsilon_n} \to z_0 \quad \text{and} \quad \|u_{\varepsilon_n} - \varphi_{\varepsilon_n}(\cdot - y_{\varepsilon_n}) U(\cdot - y_{\varepsilon_n})\|_{\varepsilon_n} \to 0,$$

and so

$$\|u_{\varepsilon_n}(\cdot + y_{\varepsilon_n}) - U\|_{H^1} \to 0.$$

Consequently given $\sigma > 0$ there exist $A > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \int_{\{|x| \geq A\}} u_\varepsilon^2(x + y_\varepsilon) \, \mathrm{d}x \leq \sigma. \tag{5.2}$$

Denoting $w_\varepsilon = u_\varepsilon(\cdot + y_\varepsilon)$, the equation (5.1) and the uniform boundedness of $\{u_\varepsilon\}$ in $L^\infty(\mathbb{R}^N)$ give us

$$-w''_\varepsilon \leq C w_\varepsilon \quad \text{in } \mathbb{R}.$$

Hence from [15, Theorem 8.17], there exists $C_0 = C_0(C)$ such that

$$\sup_{(y-1,y+1)} w_\varepsilon(x) \leq C_0 \, \|w_\varepsilon\|_{L^2((y-2,y+2))} \quad \text{for all } y \in \mathbb{R}.$$

From this inequality and by (5.2) we have $\lim_{|x| \to \infty} w_\varepsilon(x) = 0$ uniformly on ε. So we can prove the exponential decay of w_ε

$$w_\varepsilon(x) \leq C \exp(-c|x|) \quad \text{for all } x \in \mathbb{R},\ \varepsilon \in (0, \varepsilon_0)$$

for some $C, c > 0$. Now we consider $\zeta_\varepsilon \in \mathbb{R}$ a maximum point of w_ε. Since

$$w_\varepsilon(x) \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{and} \quad \|w_\varepsilon\|_\infty \geq \rho \quad \text{for all } \varepsilon \in (0, \varepsilon_0)$$

we conclude that $\{\zeta_\varepsilon\}$ is bounded. Hence $x_\varepsilon := \zeta_\varepsilon + y_\varepsilon$ is a maximum point for u_ε and the following exponential decay holds

$$u_\varepsilon(x) = w_\varepsilon(x - y_\varepsilon) \le C \exp\left(-c|x - x_\varepsilon|\right) \quad \text{for all } x \in \mathbb{R}. \tag{5.3}$$

So $Q_\varepsilon(u_\varepsilon) = 0$ for small ε and u_ε is a critical point for P_ε. From Proposition 2.3 we have $v_\varepsilon = f(u_\varepsilon)$ a positive solution for (2.1). Since f is increasing, x_ε is also a maximum point for v_ε. Moreover by the choice of $\{y_\varepsilon\}$ for any sequence $\varepsilon_n \to 0$ there are $z_0 \in \mathcal{M}$ and $\zeta_0 \in \mathbb{R}$ such that

$$\zeta_{\varepsilon_n} \to \zeta_0, \quad \varepsilon_n x_{\varepsilon_n} \to z_0 \quad \text{and} \quad \|u_{\varepsilon_n}(\cdot + x_{\varepsilon_n}) - U(\cdot + \zeta_0)\|_{H^1} \to 0, \tag{5.4}$$

up to subsequences. We observe that $U(\cdot + \zeta_0)$ is also a solution of (2.6) and so $v_0 = f(U(\cdot + \zeta_0))$ is a solution of (1.4). We have

$$\begin{aligned}\|v_{\varepsilon_n}(\cdot + x_{\varepsilon_n}) - v_0\|_{H^1}^2 &\le 2\|u_{\varepsilon_n}(\cdot + x_{\varepsilon_n}) - U(\cdot + \zeta_0)\|_{H^1}^2 \\ &\quad + 2\int_{\mathbb{R}} |f'\left(u_{\varepsilon_n}(x + x_{\varepsilon_n})\right) - f'(U(x + \zeta_0))|^2 |U'(x + \zeta_0)|^2 \, dx\end{aligned}$$

and by (5.4) and properties of f we get

$$v_{\varepsilon_n}(\cdot + x_{\varepsilon_n}) \to v_0 \quad \text{in } H^1(\mathbb{R}) \quad \text{as } n \to \infty.$$

At this point we have proved that, for small ε, $\tilde{u}_\varepsilon(x) := v_\varepsilon(x/\varepsilon)$ is a solution for the quasilinear equation (1.1) and satisfies (i)-(ii) in Theorem 1.1 with maximum point $\tilde{x}_\varepsilon = \varepsilon x_\varepsilon$.

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