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ALMOST AUTOMORPHIC SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article, we prove the existence and uniqueness of almost automorphic solutions to the non-autonomous evolution equation

$$\frac{d}{dt}(u(t) - F_1(t, B_1u(t))) = A(t)(u(t) - F_1(t, Bu(t))) + F_2(t, u(t), B_2u(t)), \quad t \in \mathbb{R}$$

where $A(t)$ generates a hyperbolic evolution family $U(t, s)$ (not necessarily periodic) in a Banach space, and B_1, B_2 are bounded linear operators. The results are obtained by means of fixed point methods.

1. INTRODUCTION

In [16], the author studied the existence and uniqueness of almost automorphic mild solution to the equation

$$\frac{d}{dt}u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}$$

where A is the generator of an exponentially stable semigroup of operators in a Banach space $\mathbb{X}$ and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is an almost automorphic function with respect to $t \in \mathbb{R}$ (in Bochner's sense [5]). In [6], the authors extended this result to the hyperbolic case in intermediate Banach spaces. Goldstein and N'Guérékata [11] have also studied this problem in the very original multi almost automorphic situation, i.e. when f takes the form $f(t, x) = P(t)Q(x)$. Most of the contributions to this problem deal with an operator A which is time independent, or $A(t)$ being periodic. Recently Ding, N'Guérékata and Wei [9] have studied the case where $A(t)$ is not necessarily periodic.

Our aim in this paper is to continue this study for the more general case of the following functional differential equation

$$\frac{d}{dt}(u(t) - F_1(t, B_1u(t))) = A(t)(u(t) - F_1(t, B_1u(t)) + F_2(t, u(t), B_2u(t)), \quad t \in \mathbb{R} \tag{1.1}$$

where the family $\{A(t) : t \in \mathbb{R}\}$ of operators in $\mathbb{X}$ generates a hyperbolic evolution family $\{U(t, s), t \geq s\}$, $F_1 : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ and $F_2 : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are two almost automorphic functions satisfying a suitable Lipschitz condition.

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In the particular case where there exists $\tau \in \mathbb{R}$ such that $B_i : BC(\mathbb{R}, X) \to X$ are shift operators $B_i u(t) := u(t-\tau)$ for all $t \in \mathbb{R}$, $i = 1, 2$, Eq. (1.1) turns out to be a neutral functional differential equation with delay. Such equations arise as models in several physical phenomena (see [12, 13, 19] and the reference therein). In [15], the author studied the existence of periodic solutions of (1.1) assuming that $A(t)$ is a nonsingular $n \times n$ matrix with continuous real-valued functions as its elements. The same equation is considered in [1] where the existence and uniqueness of a mild almost periodic solution is established when $A(t)$ generates an exponentially stable evolution family and F_2' is bounded. This last condition on F_2 is too restrictive when it comes to applications. So in this paper, we drop it. We will show the existence and the uniqueness of a mild almost automorphic solution of equation (1.1) under much broader conditions. We use the Krasnoselskii's fixed point theorem and the contraction mapping principle. To the best of our knowledge, the results here are new even in the context of almost periodicity.

In this article, we denote by $(\mathbb{X}, \|\cdot\|)$ a real Banach space and by $L(\mathbb{X})$ the Banach space of all bounded linear operators from $\mathbb{X}$ to itself endowed with the norm

$$\|T\|_{L(\mathbb{X})} = \sup\{\|Tx\| : x \in \mathbb{X},\ \|x\| \leq 1\}.$$

The work is organized as follows. In Section 2, we recall some definitions and facts on almost automorphic functions and evolutionary process and present our assumptions. Section 3 is devoted to the results.

2. Preliminaries

Let us first recall some properties of almost automorphic functions. Detailed presentations can be found in [17, 18].

Definition 2.1 (S. Bochner)**.** Let $f : \mathbb{R} \to \mathbb{X}$ be a bounded continuous function. We say that f is almost automorphic if for every sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f(t + \tau_n)$$

is well-defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - \tau_n) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{R}, \mathbb{X})$ the set of all such functions.

Definition 2.2. A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in B$, where B is any bounded subset of $\mathbb{X}$.

Definition 2.3. A continuous function $f : \mathbb{R} \times \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x, y)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in B$, where B is any bounded subset of $\mathbb{X} \times \mathbb{Y}$.

Clearly when the convergence above is uniform in $t \in \mathbb{R}$, f is almost periodic. The function g is measurable, but not continuous in general.

If the limit in the Definitions above is uniform on any compact subset $K \subset \mathbb{R}$, we say that f is compact almost automorphic.

Theorem 2.4. *Assume that f, f_1, and f_2 are almost automorphic and λ is any scalar, then the following hold:*

(i) *λf and $f_1 + f_2$ are almost automorphic,*
(ii) *$f_\tau(t) := f(t+\tau)$, $t \in \mathbb{R}$ is almost automorphic,*
(iii) *$\bar{f}(t) := f(-t)$, $t \in \mathbb{R}$ is almost automorphic,*
(iv) *The range R_f of f is precompact, so f is bounded.*

For the proof of the above theorem see [17, Theorems 2.1.3 and 2.1.4].

Theorem 2.5. *If $\{f_n\}$ is a sequence of almost automorphic $\mathbb{X}$-valued functions such that $f_n \to f$ uniformly on $\mathbb{R}$, then f is almost automorphic.*

For the proof of the above theorem, see [17, Theorem 2.1.10].

Remark 2.6. If we equip $AA(\mathbb{X})$, the space of almost automorphic functions with the sup norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$$

then it turns out to be a Banach space. If we denote $KAA(\mathbb{X})$, the space of compact almost automorphic $\mathbb{X}$-valued functions, then we have

$$AP(\mathbb{X}) \subset KAA(\mathbb{X}) \subset AA(\mathbb{X}) \subset BC(\mathbb{R},\mathbb{X}) \subset L^\infty(\mathbb{R},\mathbb{X}). \tag{2.1}$$

Theorem 2.7. *If $f \in AA(\mathbb{X})$ and its derivative f' exists and is uniformly continuous on $\mathbb{R}$, then $f' \in AA(\mathbb{X})$.*

For the proof of the above theorem, see [17, Theorem 2.4.1].

Theorem 2.8. *Let us define $F : \mathbb{R} \to \mathbb{X}$ by $F(t) = \int_0^t f(s)ds$ where $f \in AA(\mathbb{X})$. Then $F \in AA(\mathbb{X})$ iff $R_F = \{F(t)|\ t \in \mathbb{R}\}$ is precompact.*

For the proof of the above theorem, see [17, Theorem 2.4.4].

As a big difference between almost periodic functions and almost automorphic functions we remark that an almost automorphic function is not necessarily uniformly continuous, as shown in the following example due to Levitan (see also [4, Example 3.3])

Example 2.9. The function

$$f(t) := \sin\big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\big)$$

is almost automorphic, but not uniformly continuous. Therefore, it is not almost periodic.

We denote respectively by $AA(\mathbb{R},\mathbb{X})$, $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $AA(\mathbb{R}\times\mathbb{X}\times\mathbb{Y},\mathbb{X})$, the set of all almost automorphic functions $f : \mathbb{R} \to \mathbb{X}$, $f : \mathbb{R}\times X \to \mathbb{X}$ and $f : \mathbb{R}\times X\times\mathbb{Y} \to \mathbb{X}$. With the sup norm $\sup_{t\in\mathbb{R}} \|f(t)\|$, $\sup_{t\in\mathbb{R}} \|f(t,x)\|$ and $\sup_{t\in\mathbb{R}} \|f(t,x,y)\|$ these spaces turn out to be Banach spaces. We also need to recall some notation about evolution family.

Definition 2.10. A set $\{U(t,s) : t \geq s,\ t,s \in \mathbb{R}\}$ of bounded linear operator on $\mathbb{X}$ is called an evolution family (or evolutionary process) if

(i) $U(s,s) = I$, $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$ and $t,r,s \in \mathbb{R}$,
(ii) $(t,s) \in \{(\tau,\sigma) \in \mathbb{R}^2 : \tau \geq \sigma\} \to U(t,s)$ is strongly continuous.

Definition 2.11. An evolution family $U(t,s)$ is called hyperbolic (or has exponential dichotomy) if there are projections $P(t)$, $t \in \mathbb{R}$, being uniformly bounded and strongly continuous in t, and constants $N,\delta > 0$ such that

(i) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$,
(ii) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$ (and we set $U_Q(s,t) = U_Q(t,s)^{-1}$),
(iii) $\|U(t,s)P(s)\|_{L(\mathbb{X})} \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\|_{L(\mathbb{X})} \leq Ne^{-\delta(t-s)}$ for all $t \geq s$. Here and below $Q = I - P$.

Observe that if $U(t,s)$ is hyperbolic, then the Green's function $\Gamma(t,s)$, corresponding to $U(t,s)$ and $P(.)$ defined by:

$$\Gamma(t,s) = \begin{cases} U(t,s)P(s), & t \geq s,\ t,s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R} \end{cases}$$

satisfies

$$\|\Gamma(t,s)\|_{L(\mathbb{X})} = \begin{cases} Ne^{-\delta(t-s)}, & t \geq s,\ t,s \in \mathbb{R}, \\ Ne^{\delta(t-s)}, & t < s,\ t,s \in \mathbb{R}. \end{cases} \tag{2.2}$$

For more details on the exponential dichotomy concept, we refer to [7, 8, 10].

In this paper, $A(t)$, $t \in \mathbb{R}$, satisfy the 'Acquistapace-Terreni' conditions introduced in [3], that is

(H0) there exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $L, K \geq 0$, and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t-s|^\alpha |\lambda|^{-\beta}$$

for $t, s \in \mathbf{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

Remark 2.12. If (H0) holds, then there exists a unique evolution family $\{U(t,s)\}_{-\infty<s\leq t<\infty}$ on $\mathbb{X}$, which governs the linear equation

$$\frac{d}{dt}v(t) = A(t)v(t).$$

This follows from [2, Theorem 2.3]; see also [3, 20, 21].

Now we state the following assumptions:

(H1) The evolution family $U(t,s)$ generated by $A(t)$ has an exponential dichotomy with constants $N, \delta > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function $\Gamma(t,s)$.
(H2) For every real sequence (s_m), there exists a subsequence (s_n) such that

$$\Lambda(t,s)x = \lim_{n\to\infty} \Gamma(t + s_n, s + s_n)x$$

is well defined for each $x \in \mathbb{X}$ and $t, s \in \mathbb{R}$. Moreover,

$$\lim_{n\to\infty} \Lambda(t - s_n, s - s_n)x = \Gamma(t,s)x$$

for each $x \in \mathbb{X}$ and $t, s \in \mathbb{R}$.
(H3) $F_1 \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $F_2 \in AA(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$ and there exist positive constants μ_1, μ_2, μ_3 such that

$$\|F_1(t,u_1) - F_1(t,u_2)\| \leq \mu_1\|u_1 - u_2\|, u_1, u_2 \in \mathbb{X} \tag{2.3}$$

and

$$\|F_2(t,u_1,v_1)-F_2(t,u_2,v_2)\| \le \mu_2\|u_1-u_2\|+\mu_3\|v_1-v_2\|, \tag{2.4}$$

where $u_i, v_i \in \mathbb{X}$, $i=1,2$, and $t \in \mathbb{R}$.

We refer to [9] for more details on assumption (H2).

3. Main results

Definition 3.1. A continuous function $u:\mathbb{R}\to\mathbb{X}$ is called a mild solution of (1.1) if

$$u(t)-F_1(t,B_1u(t)) = U(t,a)(u(a)-F_1(a,B_1u(a)))+\int_a^t U(t,s)F_2(s,u(s),B_2u(s))ds \tag{3.1}$$

for any $t \ge a$, $t,a\in\mathbb{R}$.

Lemma 3.2. *Assume that* (H1)–(H3) *hold and $u \in AA(\mathbb{R},\mathbb{X})$. Then the functions defined by $\phi_1(\cdot) := F_1(\cdot, B_1u(\cdot))$ and $\phi_2(\cdot) := F_2(\cdot,u(\cdot),B_2u(\cdot))$ belong to $AA(\mathbb{R},\mathbb{X})$. Consequently F_1 and F_2 are bounded functions.*

Proof. First let us observe that if $u \in AA(\mathbb{R},\mathbb{X})$ then $B_iu(\cdot) \in AA(\mathbb{R},\mathbb{X})$ [17, Corollary 2.1.6]. Then in view of [17, Theorem 2.2.5], we deduce the results since (H3) holds. □

Now, if u is a mild solution of (1.1), then following [7] it can be shown that it satisfies the representation

$$u(t)-F_1(t,B_1u(t)) = \int_{\mathbb{R}} \Gamma(t,s)F_2(s,u(s),B_2u(s))ds \tag{3.2}$$

Lemma 3.3. *Assume that* (H1)–(H3) *hold and $u \in AA(\mathbb{X},\mathbb{R})$. Then the function Φ defined by*

$$\Phi(t) = \int_{-\infty}^{+\infty} \Gamma(t,s)F_2(s,u(s),B_2u(s))ds$$

is in $AA(\mathbb{R},\mathbb{X})$

Proof. Based on Lemma 3.2 it suffices to apply Theorem 2.2 in [9] with $f(\cdot) = F_2(\cdot,u(\cdot),B_2u(\cdot))$. □

Theorem 3.4. *Assume that* (H1)–(H3) *hold, and*

$$\mu_1\|B\| + \frac{2N}{\delta}(\mu_2+\mu_3) < 1. \tag{3.3}$$

Then (1.1) *has a unique almost automorphic mild solution which is given by* (3.2).

Proof. Note that the operator $Q: AA(\mathbb{R},\mathbb{X}) \to AA(\mathbb{R},\mathbb{X})$ given by

$$(Qu)(t) = F_1(t,B_1u(t)) + \int_{-\infty}^{+\infty} \Gamma(t,s)F_2(s,u(s),B_2u(s))ds$$

is well defined.

Let $u \in AA(\mathbb{R},\mathbb{X})$; then in view of Lemma 3.2, the function $t \to F_1(t,B_1u(t))$ belongs to $AA(\mathbb{R},\mathbb{X})$. Also the function $t \to \int_{-\infty}^{+\infty}\Gamma(t,s)F_2(s,u(s),B_2u(s))ds$ being in $AA(\mathbb{R},\mathbb{X})$ according to Lemma 3.3; so we deduce that $(Qu) \in AA(\mathbb{R},\mathbb{X})$.

Now we choose r such that

$$r > \sup_{t\in\mathbb{R}} \|F_1(t,0)\| + \frac{2N}{\delta} \sup_{s\in\mathbb{R}} \|F_2(s,0,0)\| + \Big(\mu_1\|B_1\|_{L(\mathbb{X})} + \frac{2N}{\delta}(\mu_2 + \mu_3\|B_2\|_{L(\mathbb{X})})\Big)r$$

and we set

$$B_r = \{u \in AA(\mathbb{R},\mathbb{X}) : \|u\|_{AA(\mathbb{R},\mathbb{X})} = \sup_{t\in\mathbb{R}} \|u(t)\| \leq r\}.$$

For $t \in \mathbb{R}$, we obtain

$$\begin{aligned}
\|(Qu)(t)\| &\leq \|F_1(t, B_1u(t))\| + \int_{-\infty}^{+\infty} \|\Gamma(t,s)F_2(s,u(s),B_2u(s))\|ds \\
&\leq \|F_1(t, B_1u(t))\| + \int_{-\infty}^{t} Ne^{-\delta(t-s)}\|F_2(s,u(s),B_2u(s))\|ds \\
&\quad + \int_{t}^{+\infty} Ne^{-\delta(s-t)}\|F_2(s,u(s),B_2u(s))\|ds \\
&\leq \|F_1(t,0)\| + \mu_1\|B_1u(t))\| \\
&\quad + \int_{-\infty}^{t} Ne^{-\delta(t-s)}(\|F_2(s,0,0)\| + \mu_2\|u(s)\| + \mu_3\|B_2u(s))\|)ds \\
&\quad + \int_{t}^{+\infty} Ne^{-\delta(s-t)}(\|F_2(s,0,0)\| + \mu_2\|u(s)\| + \mu_3\|B_2u(s))\|)ds \\
&\leq \sup_{t\in\mathbb{R}} \|F_1(t,0)\| + \mu_1 r\|B_1\|_{L(\mathbb{X})} \\
&\quad + \Big[\int_{-\infty}^{t} Ne^{-\delta(t-s)}ds\Big]\big[\sup_{s\in\mathbb{R}} \|F_2(s,0,0)\| + (\mu_2 + \mu_3\|B_2\|_{L(\mathbb{X})})r\big] \\
&\quad + \Big[\int_{t}^{+\infty} Ne^{-\delta(s-t)}ds\Big]\big[\sup_{s\in\mathbb{R}} \|F_2(s,0,0)\| + (\mu_2 + \mu_3\|B_1\|_{L(\mathbb{X})})r\big]
\end{aligned}$$

Hence we deduce hat

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \|(Qu)(t)\| &\leq \sup_{t\in\mathbb{R}} \|F_1(t,0)\| + \frac{2N}{\delta} \sup_{s\in\mathbb{R}} \|F_2(s,0,0)\| \\
&\quad + (\mu_1\|B_1\|_{L(\mathbb{X})} + \frac{2N}{\delta}(\mu_2 + \mu_3\|B_2\|_{L(\mathbb{X})}))r \leq r.
\end{aligned}$$

Thus $Qu \in B_r$. This implies that $Q(B_r) \subset B_r$.

For $u, v \in B_r$, we have

$$\begin{aligned}
&\|(Qu)(t) - (Qv)(t)\| \\
&\leq \|F_1(s, B_1u(s)) - F_1(s, B_1v(s))\| \\
&\quad + \int_{-\infty}^{+\infty} \|\Gamma(t,s)\|\|F_2(s,u(s),B_2u(s)) - F_2(s,v(s),B_2v(s))\|ds \\
&\leq \mu_1\|B_1u(s) - B_1v(s)\| \\
&\quad + \int_{-\infty}^{t} Ne^{-\delta(t-s)}\|F_2(s,u(s),B_2u(s)) - F_2(s,v(s),B_2v(s))\|ds \\
&\quad + \int_{t}^{+\infty} Ne^{-\delta(s-t)}\|F_2(s,u(s),B_2u(s)) - F_2(s,v(s),B_2v(s))\|ds \\
&\leq \mu_1\|B_1\|_{L(\mathbb{X})}\|u(s)) - v(s)\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2N}{\delta}(\mu_2\|u(s)-v(s)\|+\mu_3\|B_2u(s))-B_2v(s)\|)\\
&\le\Big[\mu_1\|B_1\|_{L(\mathbb{X})}+\frac{2N}{\delta}(\mu_2+\mu_3\|B_2\|_{L(\mathbb{X})})\Big]\|u-v\|_{AA(\mathbb{R},\mathbb{X})}.
\end{aligned}
$$

This implies

$$
\sup_{t\in\mathbb{R}}\|(Qu)(t)-(Qv)(t)\|\le\Big[\mu_1\|B_1\|_{L(\mathbb{X})}+\frac{2N}{\delta}(\mu_2+\mu_3)\|B_2\|_{L(\mathbb{X})}\Big]\|u-v\|_{AA(\mathbb{R},\mathbb{X})}
$$

and since (3.3) holds we deduce that Q is a contraction on B_r. Therefore Q has a unique fixed point u in B_r, which is the mild solution of (1.1). □

Corollary 3.5. *Consider the neutral functional differential equation*

$$
\frac{d}{dt}(u(t)-F_1(t,u(t-\tau)))=A(t)u(t)+F_2(t,u(t),u(t-\tau)),\quad \tau,t\in\mathbb{R}. \tag{3.4}
$$

Assume that (H1)–(H3) *hold, and*

$$
\mu_1+\frac{2N}{\delta}(\mu_2+\mu_3)<1.
$$

Then (3.4) *has a unique almost automorphic mild solution which is given by* (3.2)

Proof. It suffices to consider shift operators $B_iu(t):=u(t-\tau)$ for all $t\in\mathbb{R}$, $i=1,2$, thus $\|B_i\|_{L(\mathbb{X})}=1$, $i=1,2$. □

Compare the next corollary with [9, Theorem 3.1].

Corollary 3.6. *Consider the equation*

$$
\frac{d}{dt}u(t)=A(t)u(t)+f(t,u(t)) \tag{3.5}
$$

and assume that assumptions (H1), (H2) *hold. Assume also that* $f\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ *and*

$$
\|f(t,u)-f(t,v)\|<\mu\|u-v\|,\quad \forall u,v\in\mathbb{X},\ t\in\mathbb{R}
$$

then (3.5) *has a unique mild almost automorphic solution if we let* $\mu<\frac{\delta}{2N}$.

Proof. It suffices to apply Theorem 3.4 with $F_1=0$, $\tau=0$, and $\mu_2=\mu$ □

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