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EXISTENCE AND UNIQUENESS OF CLASSICAL SOLUTIONS TO SECOND-ORDER QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. This article studies the existence of solutions to the second-order quasilinear elliptic equation

$$-\nabla \cdot (a(u)\nabla u) + \mathbf{v} \cdot \nabla u = f$$

with the condition $u(\mathbf{x}_0) = u_0$ at a certain point in the domain, which is the 2 or the 3 dimensional torus. We prove that if the functions a, f, $\mathbf{v}$ satisfy certain conditions, then there exists a unique classical solution. Applications of our results include stationary heat/diffusion problems with convection and with a source/sink, when the value of the solution is known at a certain location.

1. Introduction

In this article, we consider the following quasilinear elliptic equation for $u(\mathbf{x})$ under periodic boundary conditions:

$$-\nabla \cdot (a(u)\nabla u) + \mathbf{v} \cdot \nabla u = f, \tag{1.1}$$

$$u(\mathbf{x}_0) = u_0, \tag{1.2}$$

where u_0 is a given constant and $\mathbf{x}_0$ a given point in the domain Ω. Here, $f(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ are given smooth functions for $\mathbf{x} \in \Omega$, where the domain $\Omega = \mathbb{T}^N$, the N-dimensional torus, with $N = 2, 3$. We assume that $a(u)$ is a smooth, positive function of u for $u \in \bar{G}$, where $G \subset \mathbb{R}$ is a bounded interval.

The purpose of this article is to prove the existence of a unique classical solution $u(\mathbf{x})$ to (1.1)-(1.2). What is new in this paper is the requirement that condition (1.2) holds for a quasilinear elliptic equation of the form (1.1) which includes a convection term $\mathbf{v} \cdot \nabla u$. The proof of the existence theorem uses the method of successive approximations in which an iteration scheme, based on solving a linearized version of the equation, will be defined and then convergence of the sequence of approximating solutions to a unique solution satisfying the quasilinear equation will be proven. It will be shown that there exist positive constants δ_0, δ_1, and δ_2 such that if $\big|\frac{da}{du}\big|^2_{s,\bar{G}_1}\|f\|^2_{s-1} \le \delta_0$, and $|\nabla \cdot \mathbf{v}|_{L^\infty} \le \delta_1$, and $\max\{1, |\mathbf{v}|^2_{L^\infty}\}\|f\|^2_{s-1} \le \delta_2$, and $\|D\mathbf{v}\|_s \le \frac{1}{2}$, where $s > \frac{N}{2} + 1$, and where $G_1 \subset G$, then there exists a classical solution $u(\mathbf{x})$ to (1.1)-(1.2). Here we define $\big|\frac{da}{du}\big|_{s,\bar{G}_1} = \max\{\big|\frac{d^{j+1}a}{du^{j+1}}(u_*)\big| : u_* \in \bar{G}_1, 0 \le j \le s\}$. And $u(\mathbf{x}) \in \bar{G}_1$ for all $\mathbf{x} \in \Omega$.

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The solution $u(\mathbf{x}) \in \bar{G}_1$ will be unique if $a''(u_*) \leq \frac{1}{a(u_*)}(a'(u_*))^2$ for all $u_* \in \bar{G}_1$. The key to the proof lies in obtaining a priori estimates for u.

Applications of the existence of a unique solution to (1.1)-(1.2) include stationary heat/diffusion problems with convection and with a source/sink. Solutions could be obtained for problems in which, for example, the temperature or the concentration of a substance in a fluid is monitored at a given spatial location $\mathbf{x}_0 \in \Omega$.

This article is organized as follows. First, the main result is presented and proved as Theorem 2.1 in the next section. Then lemmas supporting the proof of the theorem are proven in Appendix A (which proves the existence of a solution to the linearized equation used in the iteration scheme) and in Appendix B (which presents proofs of the a priori estimates used in the proof of the theorem).

2. Existence theorem

We use the Sobolev space $H^s(\Omega)$ (where s is a non-negative integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$, with norm given by $\|g\|_s^2 = \sum_{0\leq|\alpha|\leq s}\int_\Omega |D^\alpha g|^2 d\mathbf{x}$ and inner product $(g,h)_s = \sum_{0\leq|\alpha|\leq s}\int_\Omega (D^\alpha g)\cdot(D^\alpha h)d\mathbf{x}$. We use the notation $\|g\|_s^2 = \sum_{0\leq r\leq s}\int_\Omega |D^r g|^2 d\mathbf{x}$, where $D^r g$ is the set of all space derivatives $D^\alpha g$ with $|\alpha| = r$, and $|D^r g|^2 = \sum_{|\alpha|=r}|D^\alpha g|^2$, where $r \geq 0$ is an integer. Also, $C(\Omega)$ is the space of real-valued, continuous functions with domain Ω. Here, we are using the standard multi-index notation. Also, we let both ∇g and Dg denote the gradient of g.

Theorem 2.1. *Let $f(\mathbf{x}) \in C(\Omega)\cap H^{s-1}(\Omega)$, $\mathbf{v}(\mathbf{x}) \in C^2(\Omega)\cap H^{s+1}(\Omega)$, and let $a(u)$ be a smooth, positive function of u for $u \in \bar{G}$, where $G \subset \mathbb{R}$ is a bounded interval. We require that the given data $u(\mathbf{x}_0)$ satisfy $u(\mathbf{x}_0) \in G$, where $\mathbf{x}_0 \in \Omega$ and where $\Omega = \mathbb{T}^N$, the N-dimensional torus, with $N = 2,3$. There exist positive constants δ_0, δ_1, and δ_2, and an interval $G_1 \subset G$, such that if $\big|\frac{da}{du}\big|^2_{s,\bar{G}_1}\|f\|^2_{s-1} \leq \delta_0$, and $|\nabla\cdot\mathbf{v}|_{L^\infty} \leq \delta_1$, and $\max\{1, |\mathbf{v}|^2_{L^\infty}\}\|f\|^2_{s-1} \leq \delta_2$, and $\|D\mathbf{v}\|_s \leq 1/2$, then there exists a classical solution $u(\mathbf{x})$ to (1.1)-(1.2). And $u(\mathbf{x}) \in \bar{G}_1$ for all $\mathbf{x} \in \Omega$. Here, we define $|\frac{da}{du}|_{s,\bar{G}_1} = \max\{\big|\frac{d^{j+1}a}{du^{j+1}}(u_*)\big| : u_* \in \bar{G}_1, 0 \leq j \leq s\}$, where $s > \frac{N}{2}+1$. The solution $u(\mathbf{x}) \in \bar{G}_1$ will be unique if $a''(u_*) \leq \frac{1}{a(u_*)}(a'(u_*))^2$ for all $u_* \in \bar{G}_1$. The regularity of the solution is $u \in C^2(\Omega)\cap H^{s+1}(\Omega)$.*

Proof. We will construct the solution of the problem for (1.1)-(1.2) through an iteration scheme. To define the iteration scheme, we will let the sequence of approximate solutions be $\{u_k\}_{k=1}^\infty$. Set $u_0 = u(\mathbf{x}_0)$. For $k = 0,1,2,\dots$, construct u_{k+1} from the previous iterate u_k by solving

$$-\nabla\cdot(a(u_k)\nabla u_{k+1}) + \mathbf{v}\cdot\nabla u^{k+1} = f, \tag{2.1}$$

$$u_{k+1}(\mathbf{x}_0) = u(\mathbf{x}_0), \tag{2.2}$$

Existence of a sufficiently smooth solution to (2.1), (2.2) for fixed k is proven in Appendix A. The a priori estimates used in the proof are proven in Appendix B. We proceed now to prove convergence of the iterates as $k \to \infty$ to a unique classical solution of (1.1)-(1.2).

We fix an interval $G_1 \subset G$ by defining $G_1 = \{u_* \in G : |u_* - u_0|_{L^\infty} < R\}$, where $R = \operatorname{dist}(u_0, \partial G)$. We fix a positive constant c_1 such that $a(u_*) > c_1$ for all $u_* \in \bar{G}_1$. Using a proof by induction on k, we assume that $u_k(\mathbf{x}) \in \bar{G}_1$ for all $\mathbf{x} \in \Omega$, and then later we will show that $u_{k+1}(\mathbf{x}) \in \bar{G}_1$ for all $\mathbf{x} \in \Omega$. ∎

Proposition 2.2. *Assume that the hypotheses of Theorem 2.1 hold. Assume that* $|\nabla\cdot\mathbf{v}|_{L^\infty} \le \frac{c_1}{C_*}$, *where* C_* *is the constant from Poincaré's inequality* $\|\bar{u}\|_0^2 \le C_*\|\nabla u\|_0^2$, *and where* $\bar{u}(\mathbf{x}) = u(\mathbf{x}) - \frac{1}{|\Omega|}\int_\Omega u(\mathbf{x})d\mathbf{x}$. *There exist constants* C_4, C_5, C_1, L *such that if* $\big|\frac{da}{du}\big|^2_{s,\bar{G}_1}\|f\|^2_{s-1} \le \frac{1}{C_4}$, *and if* $\max\{1,|\mathbf{v}|^2_{L^\infty}\}\|f\|^2_{s-1} \le \frac{R^2}{C_5^2}$, *and if* $\|D\mathbf{v}\|_s \le \frac{1}{2}$, *where* $s > \frac{N}{2}+1$, *then the following hold for* $k = 1,2,3\dots$

$$\|\nabla u_k\|_s^2 \le 2C_1\|f\|^2_{s-1}, \tag{2.3}$$

$$|u_k - u_0|_{L^\infty} \le R, \tag{2.4}$$

$$\|u_k\|^2_{s+1} \le L, \tag{2.5}$$

$$\sum_{k=0}^{\infty} ||u_{k+1} - u_k\|^2_{s+1} < \infty \tag{2.6}$$

Here, $R = \operatorname{dist}(u_0, \partial G)$ *and* C_1 *is the constant in* (B.9) *from Lemma B.2 in Appendix B .*

Proof. The proof is done by induction on k. We show only the inductive step. We will derive estimates for u_{k+1}, and then use these estimates to show that if u_k satisfies the estimates (2.3), (2.4), (2.5) then u_{k+1} also satisfies the same estimates. We will prescribe L a priori, independent of k so that (2.5) holds for all $k \ge 1$. We assume by the induction hypothesis that $u_k(\mathbf{x}) \in \bar{G}_1$, and then we will show that $u_{k+1}(\mathbf{x}) \in \bar{G}_1$, for all $\mathbf{x} \in \mathbb{T}^N$. In the estimates below, we use C to denote a generic constant whose value may change from one relation to the next. Recall that we let both ∇g and Dg denote the gradient of g.

Estimate for $\|\nabla u_{k+1}\|_s^2$**:** We begin by applying estimate (B.9) from Lemma B.2 in Appendix B to equation (2.1), which yields

$$\|\nabla u_{k+1}\|_s^2 \le C_1\Big[\sum_{j=0}^{s}(\max\{\|D(a(u_k))\|^2_{s_1}, \|D\mathbf{v}\|_{s_1}\})^j\Big]\|f\|^2_{s-1} \tag{2.7}$$

where $s_1 = \max\{s-1, s_0\}$, and $s_0 = [\frac{N}{2}]+1 = 2$, and $s > \frac{N}{2}+1$, for $N = 2,3$, so $s \ge 3$ and $s_1 = s-1$.

We consider two cases: when $\max\{\|D(a(u_k))\|^2_{s_1}, \|D\mathbf{v}\|_{s_1}\} = \|D(a(u_k))\|^2_{s_1}$, and when $\max\{\|D(a(u_k))\|^2_{s_1}, \|D\mathbf{v}\|_{s_1}\} = \|D\mathbf{v}\|_{s_1}$.

Case 1: Suppose that $\max\{\|D(a(u_k))\|^2_{s_1}, \|D\mathbf{v}\|_{s_1}\} = \|D(a(u_k))\|^2_{s_1}$ in (2.7).

To estimate the term $\|D(a(u_k))\|^2_{s_1}$, we apply the Sobolev space inequality (B.1) from Lemma B.1 in Appendix B, which yields the following:

$$\begin{aligned}
\|D(a(u_k))\|^2_{s_1} &= \sum_{0\le r\le s_1}\|D^r(D(a(u_k)))\|_0^2 = \sum_{0\le r\le s_1}\|D^{r+1}(a(u_k))\|_0^2 \\
&\le \sum_{0\le r\le s_1}\Big[C\Big|\frac{da}{du}\Big|^2_{r,\bar{G}_1}(1+|u_k|_{L^\infty})^{2r}\|\nabla u_k\|_r^2\Big] \\
&\le C\Big|\frac{da}{du}\Big|^2_{s_1,\bar{G}_1}(1+|u_k|_{L^\infty})^{2s_1}\|\nabla u_k\|^2_{s_1} \\
&\le C\Big|\frac{da}{du}\Big|^2_{s_1,\bar{G}_1}(1+|u_k-u_0|_{L^\infty}+|u_0|_{L^\infty})^{2s_1}\|\nabla u_k\|^2_{s_1} \\
&\le C\Big|\frac{da}{du}\Big|^2_{s,\bar{G}_1}(1+R+|u(\mathbf{x}_0)|)^{2s}\|\nabla u_k\|^2_{s_1}
\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&\le C\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}(1+R+(1+|\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}\|\nabla u_k\|^2_{s_1}\\
&= C_2\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}\|\nabla u_k\|^2_{s_1}\\
&\le C_3\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}\|\nabla u_k\|^2_{s_1}
\end{aligned}
$$

where $|\frac{da}{du}|_{s,\bar G_1} = \max\{\big|\frac{d^{j+1}a}{du^{j+1}}(u_*)\big| : u_* \in \bar G_1, 0 \le j \le s\}$, from (B.1) in Lemma B.1. Here $C_2 = C(1+R+(1+|\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}$, and we define $C_3 = MC_2$, where M is a constant to be defined later and $M \ge 1$. We can assume that $C_2 \ge 1$, so that $C_3 \ge 1$. And we used the fact that $\big|\frac{da}{du}\big|_{r,\bar G_1} \le \big|\frac{da}{du}\big|_{s_1,\bar G_1} \le \big|\frac{da}{du}\big|_{s,\bar G_1}$ for $r \le s_1$ and $s_1 \le s$. We also used the fact that $|u_k - u_0|_{L^\infty} \le R$ holds by (2.4), since $u_k(\mathbf{x}) \in \bar G_1$ for all $\mathbf{x} \in \mathbb{T}^N$ by the induction hypothesis.

We now define the constant C_4 to be $C_4 = 4C_3^2C_1^2$, where C_1 is the constant in (2.3) and in estimate (B.9) from Lemma B.2 in Appendix B, and where we may assume that $C_1 \ge 1$. We assume that $\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}\|f\|^2_{s-1} \le \frac{1}{C_4}$. Substituting (2.8) into (2.7), and using estimate (2.3), namely $\|\nabla u_k\|^2_s \le 2C_1\|f\|^2_{s-1}$, which holds by the induction hypothesis for u_k, and using the fact that $s_1 \le s$, yields

$$
\begin{aligned}
\|\nabla u_{k+1}\|^2_s &\le C_1\Big[\sum_{j=0}^s C_3^j\Big|\frac{da}{du}\Big|^{2j}_{s,\bar G_1}\|\nabla u_k\|^{2j}_{s_1}\Big]\|f\|^2_{s-1}\\
&\le C_1\Big[\sum_{j=0}^s C_3^j(2C_1)^j\Big|\frac{da}{du}\Big|^{2j}_{s,\bar G_1}\|f\|^{2j}_{s-1}\Big]\|f\|^2_{s-1}\\
&\le C_1\Big[\sum_{j=0}^s C_3^j(2C_1)^j\Big(\frac{1}{C_4}\Big)^j\Big]\|f\|^2_{s-1}\\
&\le C_1\Big[\sum_{j=0}^s \Big(\frac{1}{2}\Big)^j\Big]\|f\|^2_{s-1}\\
&\le 2C_1\|f\|^2_{s-1}
\end{aligned}
\tag{2.9}
$$

where we used the fact that $|\frac{da}{du}|^2_{s,\bar G_1}\|f\|^2_{s-1} \le \frac{1}{C_4}$. And we used the fact that $\frac{2C_3C_1}{C_4} \le \frac12$ and $\frac{1}{C_3C_1} \le 1$, since $C_4 = 4C_3^2C_1^2$ and $C_3C_1 \ge 1$. Therefore (2.3) holds for $\|\nabla u_{k+1}\|^2_s$ when $\max\{\|D(a(u_k))\|^2_{s_1}, \|D\mathbf{v}\|_{s_1}\} = \|D(a(u_k))\|^2_{s_1}$.

Case 2: Suppose that $\max\{\|D(a(u_k))\|^2_{s_1}, \|D\mathbf{v}\|_{s_1}\} = \|D\mathbf{v}\|_{s_1}$ in (2.7). From (2.7), we obtain

$$
\|\nabla u_{k+1}\|^2_s \le C_1\Big[\sum_{j=0}^s \|D\mathbf{v}\|^j_{s_1}\Big]\|f\|^2_{s-1} \le C_1\Big[\sum_{j=0}^s \Big(\frac12\Big)^j\Big]\|f\|^2_{s-1} \le 2C_1\|f\|^2_{s-1} \tag{2.10}
$$

where we used the fact that $\|D\mathbf{v}\|_{s_1} \le \|D\mathbf{v}\|_s \le 1/2$. This is the same result as inequality (2.9), and therefore (2.3) holds for $\|\nabla u_{k+1}\|^2_s$.

Estimate for $|u_{k+1} - u_0|_{L^\infty}$: To obtain an estimate for $|u_{k+1} - u_0|_{L^\infty}$, we will use Sobolev's inequality $|h|^2_{L^\infty} \le C\|h\|^2_{s_0}$ (see, e.g., [1], [3]), where $s_0 = [\frac{N}{2}] + 1 = 2$. We will also apply inequality (B.4) from Lemma B.1 in Appendix B, which yields the estimate $\|u_{k+1} - u_0\|^2_0 \le C\|\nabla(u_{k+1} - u_0)\|^2_2$. And we will use the estimate (2.9), (2.10) just proven for $\|\nabla u_{k+1}\|_s^2$. We then obtain the following inequality:

$$
\begin{aligned}
|u_{k+1}-u_0|_{L^\infty}^2 &\le C\|u_{k+1}-u_0\|_{s_0}^2 \le C\|u_{k+1}-u_0\|_{s+1}^2 \\
&= C\|u_{k+1}-u_0\|_0^2 + C\sum_{1\le|\alpha|\le s+1}\|D^\alpha(u_{k+1}-u_0)\|_0^2 \\
&\le C\|u_{k+1}-u_0\|_0^2 + C\|\nabla(u_{k+1}-u_0)\|_s^2 \\
&\le C\|\nabla(u_{k+1}-u_0)\|_2^2 + C\|\nabla(u_{k+1}-u_0)\|_s^2 \\
&\le C\|\nabla(u_{k+1}-u_0)\|_s^2 \\
&= C\|\nabla u_{k+1}\|_s^2 \\
&\le 2CC_1\|f\|_{s-1}^2
\end{aligned}
\tag{2.11}
$$

Therefore, from (2.11) we have $|u_{k+1}-u_0|_{L^\infty} \le C_5\|f\|_{s-1}$, where we define $C_5 = (2CC_1)^{1/2}$ from (2.11). We will assume that $\max\{1, |\mathbf{v}|_{L^\infty}^2\}\|f\|_{s-1}^2 \le \frac{R^2}{C_5^2}$. It follows that $\|f\|_{s-1} \le \frac{R}{C_5}$, and therefore $|u_{k+1}-u_0|_{L^\infty} \le R$. And so (2.4) holds for u_{k+1}, and $u_{k+1}(\mathbf{x}) \in \bar{G}_1$ for all $\mathbf{x} \in \mathbb{T}^N$.

Estimate for $\|u_{k+1}\|_0^2$ and $\|u_{k+1}\|_{s+1}^2$: To obtain an L^2 estimate for u_{k+1} we apply inequality (B.5) from Lemma B.1 in Appendix B, which yields

$$
\begin{aligned}
\|u_{k+1}\|_0^2 &\le C\|u_0\|_0^2 + C\|\nabla u_0\|_2^2 + C\|\nabla u_{k+1}\|_2^2 \\
&\le C\|u_0\|_0^2 + C\|\nabla u_{k+1}\|_s^2 \\
&\le C|\Omega||u(\mathbf{x}_0)|^2 + 2CC_1\|f\|_{s-1}^2
\end{aligned}
\tag{2.12}
$$

Here we used the estimate for $\|\nabla u_{k+1}\|_s^2$ from the result just proven in (2.9), (2.10). And we used the fact that u_0 is a constant. From the estimates (2.9), (2.10), (2.12) and using the fact that $\|f\|_{s-1}^2 \le \frac{R^2}{C_5^2}$ where $C_5^2 = 2CC_1$, yields

$$
\begin{aligned}
\|u_{k+1}\|_{s+1}^2 &= \|u_{k+1}\|_0^2 + \sum_{1\le|\alpha|\le s+1}\|D^\alpha u_{k+1}\|_0^2 \\
&\le \|u_{k+1}\|_0^2 + C\|\nabla u_{k+1}\|_s^2 \\
&\le C|\Omega||u(\mathbf{x}_0)|^2 + 2CC_1\|f\|_{s-1}^2 \\
&\le C|\Omega||u(\mathbf{x}_0)|^2 + CR^2
\end{aligned}
\tag{2.13}
$$

We now define the constant L to be $L = C|\Omega||u(\mathbf{x}_0)|^2 + CR^2$ from (2.13). Then we have $\|u_{k+1}\|_{s+1}^2 \le L$, and so (2.5) holds for $\|u_{k+1}\|_{s+1}^2$. Therefore (2.3), (2.4), (2.5) hold for all $k \ge 1$, and $u_k(\mathbf{x}) \in \bar{G}_1$ for all $\mathbf{x} \in \mathbb{T}^N$ and for all $k \ge 1$.

Estimate for $\|u_{k+1}-u_k\|_{s+1}^2$: Subtracting the equation (2.1) for u_k from the equation (2.1) for u_{k+1} yields the following equation

$$
-\nabla\cdot(a(u_k)\nabla(u_{k+1}-u_k)) + \mathbf{v}\cdot\nabla(u^{k+1}-u^k) = \nabla\cdot((a(u_k)-a(u_{k-1}))\nabla u_k) \tag{2.14}
$$

We consider two cases: when $\max\{\|D(a(u_k))\|_{s_1}^2, \|D\mathbf{v}\|_{s_1}\} = \|D(a(u_k))\|_{s_1}^2$, and when $\max\{\|D(a(u_k))\|_{s_1}^2, \|D\mathbf{v}\|_{s_1}\} = \|D\mathbf{v}\|_{s_1}$.

Case 1: Suppose that $\max\{\|D(a(u_k))\|_{s_1}^2, \|D\mathbf{v}\|_{s_1}\} = \|D(a(u_k))\|_{s_1}^2$. Applying estimate (B.9) from Lemma B.2 in Appendix B to equation (2.14), and using estimate (2.8) for $\|D(a(u_k))\|_{s_1}^2$, and using estimate (2.3) for $\|\nabla u_k\|_s^2$, yields the following:

$$
\begin{aligned}
&\|\nabla(u_{k+1}-u_k)\|_s^2 \\
&\le C_1\Big[\sum_{j=0}^{s}\|D(a(u_k))\|_{s_1}^{2j}\Big]\|\nabla\cdot((a(u_k)-a(u_{k-1}))\nabla u_k)\|_{s-1}^2 \\
&\le CC_1\Big[\sum_{j=0}^{s}C_3^j\Big|\frac{da}{du}\Big|_{s,\bar G_1}^{2j}\|\nabla u_k\|_{s_1}^{2j}\Big]\|(a(u_k)-a(u_{k-1}))\nabla u_k\|_s^2 \\
&\le CC_1\Big[\sum_{j=0}^{s}C_3^j\Big|\frac{da}{du}\Big|_{s,\bar G_1}^{2j}\|\nabla u_k\|_{s}^{2j}\Big]\|a(u_k)-a(u_{k-1})\|_s^2\|\nabla u_k\|_s^2 \\
&\le CC_1\Big[\sum_{j=0}^{s}C_3^j(2C_1)^j\Big|\frac{da}{du}\Big|_{s,\bar G_1}^{2j}\|f\|_{s-1}^{2j}\Big]\|a(u_k)-a(u_{k-1})\|_s^2(2C_1)\|f\|_{s-1}^2 \\
&\le CC_1\Big[\sum_{j=0}^{s}C_3^j(2C_1)^j\Big(\frac{1}{C_4}\Big)^j\Big]\|a(u_k)-a(u_{k-1})\|_s^2(2C_1)\|f\|_{s-1}^2 \\
&\le CC_1\Big[\sum_{j=0}^{s}\big(\frac{1}{2}\big)^j\Big]\|a(u_k)-a(u_{k-1})\|_s^2(2C_1)\|f\|_{s-1}^2 \\
&\le C(2C_1)^2\|a(u_k)-a(u_{k-1})\|_s^2\|f\|_{s-1}^2
\end{aligned}
\tag{2.15}
$$

where we used the Sobolev calculus inequality $\|gh\|_r^2 \le C\|g\|_r^2\|h\|_r^2$ for $r > \frac{N}{2}$, where C is a constant which depends on r (see, e.g., [3], [9]), and we let $r = s$ where $s > \frac{N}{2} + 1$. We also used the fact that $|\frac{da}{du}|_{s,\bar G_1}^2\|f\|_{s-1}^2 \le \frac{1}{C_4}$. And we used the fact that $\frac{2C_3C_1}{C_4} \le \frac{1}{2}$ and $\frac{1}{C_3C_1} \le 1$, since $C_4 = 4C_3^2C_1^2$ and $C_3C_1 \ge 1$.

To estimate the term $\|a(u_k) - a(u_{k-1})\|_s^2$, we will apply the Sobolev space inequality (B.2) from Lemma B.1 in Appendix B, which yields

$$
\begin{aligned}
&\|a(u_k)-a(u_{k-1})\|_s^2 \\
&\le C\Big|\frac{da}{du}\Big|_{s,\bar G_1}^2(1+|u_k|_{L^\infty}+|u_{k-1}|_{L^\infty})^2(\|u_k\|_s+\|u_{k-1}\|_s)^2\|u_k-u_{k-1}\|_s^2 \\
&\le C\Big|\frac{da}{du}\Big|_{s,\bar G_1}^2(1+|u_k-u_0|_{L^\infty}+|u_{k-1}-u_0|_{L^\infty}+2|u_0|_{L^\infty})^2 \\
&\quad\times(\|u_k\|_s^2+\|u_{k-1}\|_s^2)\|u_k-u_{k-1}\|_s^2 \\
&\le C\Big|\frac{da}{du}\Big|_{s,\bar G_1}^2(2+2R+2|u_0|_{L^\infty})^2(\|u_k\|_{s+1}^2+\|u_{k-1}\|_{s+1}^2)\|u_k-u_{k-1}\|_s^2 \\
&\le CL\Big|\frac{da}{du}\Big|_{s,\bar G_1}^2(1+R+(1+|\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}\|u_k-u_{k-1}\|_s^2 \\
&\le CLC_2\Big|\frac{da}{du}\Big|_{s,\bar G_1}^2\|u_k-u_{k-1}\|_s^2 \\
&\le CLC_3\Big|\frac{da}{du}\Big|_{s,\bar G_1}^2\|u_k-u_{k-1}\|_s^2
\end{aligned}
\tag{2.16}
$$

where C depends on s, and we used (2.5) to estimate $\|u_k\|_{s+1}^2 \le L$ and $\|u_{k-1}\|_{s+1}^2 \le L$. We also used Cauchy's inequality $gh \le \frac{1}{2}g^2 + \frac{1}{2}h^2$. Here, C_2, C_3 are the same constants as in (2.8). Then from (2.15) and (2.16) we obtain

$$\begin{aligned} \|\nabla(u_{k+1}-u_k)\|_s^2 &\le CLC_3(2C_1)^2\Big|\frac{da}{du}\Big|^2_{s,\bar{G}_1}\|f\|^2_{s-1}\|u_k-u_{k-1}\|_s^2 \\ &\le CLC_3(2C_1)^2\Big(\frac{1}{C_4}\Big)\|u_k-u_{k-1}\|_s^2 \\ &= \frac{CL}{C_3}\|u_k-u_{k-1}\|^2_{s+1} \end{aligned} \tag{2.17}$$

Here we used the fact that $C_4 = 4C_3^2C_1^2$, and that $\big|\frac{da}{du}\big|^2_{s,\bar{G}_1}\|f\|^2_{s-1} \le \frac{1}{C_4}$.

Case 2: Suppose that $\max\{\|D(a(u_k))\|^2_{s_1}, \|D\mathbf{v}\|_{s_1}\} = \|D\mathbf{v}\|_{s_1}$. Applying estimate (B.9) from Lemma B.2 in Appendix B to equation (2.14), and using (2.3), (2.16), and using the proof of (2.17), yields the inequality

$$\begin{aligned} &\|\nabla(u_{k+1}-u_k)\|_s^2 \\ &\le C_1\Big[\sum_{j=0}^{s}\|D\mathbf{v}\|^j_{s_1}\Big]\|\nabla\cdot((a(u_k)-a(u_{k-1}))\nabla u_k)\|^2_{s-1} \\ &\le CC_1\Big[\sum_{j=0}^{s}\big(\frac{1}{2}\big)^j\Big]\|a(u_k)-a(u_{k-1})\|_s^2\|\nabla u_k\|_s^2 \\ &\le C(2C_1)\|a(u_k)-a(u_{k-1})\|_s^2(2C_1)\|f\|^2_{s-1} \\ &\le CLC_3(2C_1)^2\Big|\frac{da}{du}\Big|^2_{s,\bar{G}_1}\|f\|^2_{s-1}\|u_k-u_{k-1}\|_s^2 \\ &\le CLC_3(2C_1)^2\Big(\frac{1}{C_4}\Big)\|u_k-u_{k-1}\|_s^2 \\ &\le \frac{CL}{C_3}\|u_k-u_{k-1}\|^2_{s+1} \end{aligned} \tag{2.18}$$

which is the same result as (2.17). Here, we used the facts that $\big|\frac{da}{du}\big|^2_{s,\bar{G}_1}\|f\|^2_{s-1} \le \frac{1}{C_4}$ where $C_4 = 4C_3^2C_1^2$, and that $\|D\mathbf{v}\|_{s_1} \le \|D\mathbf{v}\|_s \le \frac{1}{2}$.

To obtain an L^2 estimate for $u_{k+1}-u_k$, we apply inequality (B.4) from Lemma B.1 in Appendix B, which yields

$$\|u_{k+1}-u_k\|_0^2 \le C\|\nabla(u_{k+1}-u_k)\|_2^2 \le C\|\nabla(u_{k+1}-u_k)\|_s^2 \tag{2.19}$$

From (2.17)–(2.19), we obtain

$$\begin{aligned} \|u_{k+1}-u_k\|^2_{s+1} &= \|u_{k+1}-u_k\|_0^2 + \sum_{1\le|\alpha|\le s+1}\|D^\alpha(u_{k+1}-u_k)\|_0^2 \\ &\le \|u_{k+1}-u_k\|_0^2 + C\|\nabla(u_{k+1}-u_k)\|_s^2 \\ &\le C\|\nabla(u_{k+1}-u_k)\|_s^2 \\ &\le \frac{CL}{C_3}\|u_k-u_{k-1}\|^2_{s+1} \end{aligned} \tag{2.20}$$

where $L = C|\Omega||u(\mathbf{x}_0)|^2+CR^2$ from (2.13) , and $C_2 = C(1+R+(1+|\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}$ and $C_3 = MC_2$ from (2.8). It follows that $\frac{CL}{C_3} \le \frac{C}{M}$, where C depends on s and $s \ge 3$. We now define the constant M to be large enough so that $\frac{C}{M} < 1$. Then from (2.20), we have

$$\sum_{k=0}^{\infty} \|u_{k+1} - u_k\|_{s+1}^2 < \infty \tag{2.21}$$

which is the inequality (2.6) to be proven. This completes the proof of Proposition 2.2. $\square$

We now complete the proof of Theorem 2.1. From Lemma A.1 in Appendix A, we know that $u_k \in C^2(\Omega) \cap H^{s+1}(\Omega)$ for each $k \geq 1$, where $s > \frac{N}{2} + 1$. From (2.5) in Proposition 2.2 and from Sobolev's inequality $|h|_{L^\infty}^2 \leq C\|h\|_{s_0}^2$ (see, e.g., [1], [3]), where $s_0 = [\frac{N}{2}] + 1 = 2$, we know that $\{u_k\}_{k=1}^{\infty}$ is bounded in $C^2(\Omega) \cap H^{s+1}(\Omega)$. And from (2.6) in Proposition 2.2, it follows that $\|u_{k+1} - u_k\|_{s+1} \to 0$ as $k \to \infty$. We conclude that there exists $u \in C^2(\Omega) \cap H^{s+1}(\Omega)$ such that $\|u_k - u\|_{s+1} \to 0$ as $k \to \infty$. From Lemma A.1 in Appendix A, we know that u_{k+1} is a solution of the linear equation (2.1) for each $k \geq 0$, and $u_{k+1}(\mathbf{x}_0) = u(\mathbf{x}_0)$ for each $k \geq 0$, and so it follows that u is a solution of the quasilinear equation (1.1), and u satisfies (1.2).

To prove uniqueness of the solution, let us assume that $u_1(\mathbf{x})$, $u_2(\mathbf{x})$ are two solutions of (1.1)-(1.2), and $u_1(\mathbf{x}) \in \bar{G}_1$ and $u_2(\mathbf{x}) \in \bar{G}_1$ for all $\mathbf{x} \in \mathbb{T}^N$. We will show that there exists a constant C_7, such that if $\left|\frac{da}{du}\right|_{s,\bar{G}_1}^2 \|f\|_{s-1}^2 \leq \frac{1}{C_7}$, and if $\|D\mathbf{v}\|_{s_1} \leq \frac{1}{2}$, and if $a''(u_*) \leq \frac{1}{a(u_*)}(a'(u_*))^2$ for all $u_* \in \bar{G}_1$, and if $\max\{1, |\mathbf{v}|_{L^\infty}^2\}\|f\|_{s-1}^2 \leq \frac{R^2}{C_5^2}$, and if $|\nabla \cdot \mathbf{v}|_{L^\infty} \leq \frac{c_1}{C_*}$, where C_5, c_1, C_* are the constants from the proof of Proposition 2.2, then $u_1 = u_2$.

Note that since $u_1(\mathbf{x}) \in \bar{G}_1$ and $u_2(\mathbf{x}) \in \bar{G}_1$, it follows that $|u_1 - u_0|_{L^\infty} \leq R$ and $|u_2 - u_0|_{L^\infty} \leq R$, and $a(u_1) > c_1$ and $a(u_2) > c_1$, and $a''(u_1) \leq \frac{1}{a(u_1)}(a'(u_1))^2$ and $a''(u_2) \leq \frac{1}{a(u_2)}(a'(u_2))^2$. By Lemma B.3 from Appendix B applied to equation (1.1) for u_1 and u_2, there exist constants C_7, C_8, such that if $\left|\frac{da}{du}\right|_{s,\bar{G}_1}^2 \|f\|_{s-1}^2 \leq \frac{1}{C_7}$, then u_1, u_2 satisfy

$$\begin{aligned} \|\nabla u_1\|_s^2 &\leq 2C_8\|f\|_{s-1}^2, \\ \|\nabla u_2\|_s^2 &\leq 2C_8\|f\|_{s-1}^2 \end{aligned} \tag{2.22}$$

From Lemma B.3 in Appendix B, the constant $C_7 = 4C_0^2C_3^2C_1^2K_1^2$, and the constant $C_8 = C_0C_1K_1$ so that we have $C_7 = 4C_3^2C_8^2$, and C_0 is a constant which depends on s, c_1, and the constant $K_1 = \max\{1, |\mathbf{v}|_{L^\infty}^2\}$. We may assume that $C_0 \geq 1$, so that $C_1 \leq C_8$.

And we have $\|u_1\|_0^2 \leq |\Omega||u_1|_{L^\infty}^2 \leq 2|\Omega|(|u_1 - u_0|_{L^\infty}^2 + |u_0|_{L^\infty}^2) \leq 2|\Omega|(R^2 + |u(\mathbf{x}_0)|^2)$. So $\|u_1\|_{s+1}^2 \leq \|u_1\|_0^2 + C\|\nabla u_1\|_s^2 \leq 2|\Omega|(R^2 + |u(\mathbf{x}_0)|^2) + 2CC_8\|f\|_{s-1}^2$. It follows that $u_1 \in C^2(\Omega) \cap H^{s+1}(\Omega)$. Similarly, $u_2 \in C^2(\Omega) \cap H^{s+1}(\Omega)$. Here, we used Sobolev's inequality $|h|_{L^\infty}^2 \leq C\|h\|_{s_0}^2$, where $s_0 = [\frac{N}{2}] + 1 = 2$.

Subtracting (1.1) for u_1 from (1.1) for u_2 yields the equation

$$-\nabla \cdot (a(u_1)\nabla(u_2 - u_1)) + \mathbf{v} \cdot \nabla(u_2 - u_1) = \nabla \cdot ((a(u_2) - a(u_1))\nabla u_2) \tag{2.23}$$

To obtain an estimate for $\|u_2 - u_1\|_{s+1}^2$, we repeat the proof of the estimate for $\|u_{k+1} - u_k\|_{s+1}^2$ from (2.15)-(2.20), and apply this proof to (2.23). We use inequality (B.4) from Lemma B.1 in Appendix B, which yields $\|u_2 - u_1\|_0^2 \leq C\|\nabla(u_2 - u_1)\|_2^2$, and we use inequality (B.9) from Lemma B.2 in Appendix B to estimate $\|\nabla(u_2 - u_1)\|_s^2$, and we use inequality (2.8) to estimate $\|D(a(u_1))\|_{s_1}^2$ and we use inequality (2.22) to estimate $\|\nabla u_1\|_s^2$ and $\|\nabla u_2\|_s^2$. We also use the inequality $\left|\frac{da}{du}\right|_{s,\bar{G}_1}^2 \|f\|_{s-1}^2 \le \frac{1}{C_7}$ and the inequality $\|D\mathbf{v}\|_{s_1} \le \frac{1}{2}$.

We consider two cases: when $\max\{\|D(a(u_1))\|_{s_1}^2, \|D\mathbf{v}\|_{s_1}\} = \|D(a(u_1))\|_{s_1}^2$, and when $\max\{\|D(a(u_1))\|_{s_1}^2, \|D\mathbf{v}\|_{s_1}\} = \|D\mathbf{v}\|_{s_1}$.

Case 1: Suppose that $\max\{\|D(a(u_1))\|_{s_1}^2, \|D\mathbf{v}\|_{s_1}\} = \|D(a(u_1))\|_{s_1}^2$. We obtain

$$
\begin{aligned}
&\|u_2 - u_1\|_{s+1}^2 \\
&= \|u_2 - u_1\|_0^2 + \sum_{1 \le |\alpha| \le s+1} \|D^\alpha (u_2 - u_1)\|_0^2 \\
&\le \|u_2 - u_1\|_0^2 + C\|\nabla(u_2 - u_1)\|_s^2 \\
&\le C\|\nabla(u_2 - u_1)\|_2^2 + C\|\nabla(u_2 - u_1)\|_s^2 \\
&\le C\|\nabla(u_2 - u_1)\|_s^2 \\
&\le CC_1\Big[\sum_{j=0}^{s} (\max\{\|D(a(u_1))\|_{s_1}^2, \|D\mathbf{v}\|_{s_1}\})^j\Big] \|\nabla\cdot((a(u_2) - a(u_1))\nabla u_2)\|_{s-1}^2 \\
&\le CC_8\Big[\sum_{j=0}^{s} \|D(a(u_1))\|_{s_1}^{2j}\Big] \|a(u_2) - a(u_1)\|_s^2 \|\nabla u_2\|_s^2 \\
&\le CC_8\Big[\sum_{j=0}^{s} C_3^j \Big|\frac{da}{du}\Big|_{s,\bar{G}_1}^{2j} \|\nabla u_1\|_s^{2j}\Big] \|a(u_2) - a(u_1)\|_s^2 \|\nabla u_2\|_s^2 \\
&\le CC_8\Big[\sum_{j=0}^{s} C_3^j (2C_8)^j \Big|\frac{da}{du}\Big|_{s,\bar{G}_1}^{2j} \|f\|_{s-1}^{2j}\Big] \|a(u_2) - a(u_1)\|_s^2 (2C_8)\|f\|_{s-1}^2 \\
&\le CC_8\Big[\sum_{j=0}^{s} C_3^j (2C_8)^j \Big(\frac{1}{C_7}\Big)^j\Big] \|a(u_2) - a(u_1)\|_s^2 (2C_8)\|f\|_{s-1}^2 \\
&\le CC_8\Big[\sum_{j=0}^{s} \big(\frac{1}{2}\big)^j\Big] \|a(u_2) - a(u_1)\|_s^2 (2C_8)\|f\|_{s-1}^2 \\
&\le C(2C_8)^2 \|a(u_2) - a(u_1)\|_s^2 \|f\|_{s-1}^2
\end{aligned}
\tag{2.24}
$$

where we used that $C_1 \le C_8$, and $C_7 = 4C_3^2 C_8^2$, and $\frac{2C_3C_8}{C_7} \le \frac{1}{2}$, since $C_3C_8 \ge 1$.

From the third line in the proof of estimate (2.16), we have the inequality

$$
\begin{aligned}
&\|a(u_2) - a(u_1)\|_s^2 \\
&\le C\Big|\frac{da}{du}\Big|_{s,\bar{G}_1}^2 (2 + 2R + 2|u(\mathbf{x}_0)|)^2 (\|u_2\|_{s+1}^2 + \|u_1\|_{s+1}^2)\|u_2 - u_1\|_s^2 \\
&\le C\Big|\frac{da}{du}\Big|_{s,\bar{G}_1}^2 (1 + R + |u(\mathbf{x}_0)|)^2 (\|u_2\|_0^2 + C\|\nabla u_2\|_s^2 + \|u_1\|_0^2 \\
&\quad + C\|\nabla u_1\|_s^2)\|u_2 - u_1\|_s^2
\end{aligned}
\tag{2.25}
$$

By inequality (B.5) from Lemma B.1 in Appendix B, we have the estimate $\|u_2\|_0^2 \le C\|u_0\|_0^2 + C\|\nabla u_0\|_2^2 + C\|\nabla u_2\|_2^2 \le C|\Omega||u(\mathbf{x}_0)|^2 + C\|\nabla u_2\|_s^2$. And a similar inequality holds for $\|u_1\|_0^2$. Substituting these L^2 estimates into (2.25) yields

$$
\begin{aligned}
&\|a(u_2)-a(u_1)\|_s^2\\
&\le C\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}(1+R+|u(\mathbf{x}_0)|)^2(|\Omega||u(\mathbf{x}_0)|^2+\|\nabla u_2\|_s^2+\|\nabla u_1\|_s^2)\|u_2-u_1\|_s^2\\
&\le C\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}(1+R+|u(\mathbf{x}_0)|)^2(|\Omega||u(\mathbf{x}_0)|^2+4C_8\|f\|_{s-1}^2)\|u_2-u_1\|_s^2\\
&\le C\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}(1+R+|u(\mathbf{x}_0)|)^2\Big(|\Omega||u(\mathbf{x}_0)|^2+4C_8\Big(\frac{R^2}{K_1C_5^2}\Big)\Big)\|u_2-u_1\|_s^2\\
&= C\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}(1+R+|u(\mathbf{x}_0)|)^2\Big(|\Omega||u(\mathbf{x}_0)|^2+\Big(\frac{4C_0C_1K_1R^2}{2K_1CC_1}\Big)\Big)\|u_2-u_1\|_s^2\\
&\le C\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}(1+R+(1+|\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}(|\Omega||u(\mathbf{x}_0)|^2+R^2)\|u_2-u_1\|_s^2\\
&\le CC_3L\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}\|u_2-u_1\|_s^2
\end{aligned}
\tag{2.26}
$$

where we used $\|f\|_{s-1}^2 \le \frac{R^2}{K_1C_5^2}$, where $K_1=\max\{1,|\mathbf{v}|_{L^\infty}^2\}$ and $C_5=(2CC_1)^{1/2}$. And we used the fact that $C_8=C_0C_1K_1$, where C_0 depends on s, c_1. And we used inequality (2.22) to estimate $\|\nabla u_1\|_s^2$ and $\|\nabla u_2\|_s^2$. Also, $L=C|\Omega||u(\mathbf{x}_0)|^2+CR^2$ from (2.13), and $C_2=C(1+R+(1+|\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}$ and $C_3=MC_2$ from (2.8), where $M\ge 1$. Substituting (2.26) into (2.24) yields

$$
\begin{aligned}
\|u_2-u_1\|_{s+1}^2 &\le CC_3L(2C_8)^2\Big|\frac{da}{du}\Big|^2_{s,\bar G_1}\|f\|_{s-1}^2\|u_2-u_1\|_s^2\\
&\le CC_3L(2C_8)^2\Big(\frac{1}{C_7}\Big)\|u_2-u_1\|_s^2\\
&= \Big(\frac{CL}{C_3}\Big)\|u_2-u_1\|_{s+1}^2
\end{aligned}
\tag{2.27}
$$

where we used the fact that $C_7=4C_3^2C_8^2$ and $\big|\frac{da}{du}\big|^2_{s,\bar G_1}\|f\|_{s-1}^2\le\frac{1}{C_7}$.

Case 2: Suppose that $\max\{\|D(a(u_1))\|_{s_1}^2,\|D\mathbf{v}\|_{s_1}\}=\|D\mathbf{v}\|_{s_1}$. Repeating the proof of (2.24)–(2.27) yields the following:

$$
\begin{aligned}
&\|u_2-u_1\|_{s+1}^2\\
&\le CC_1\Big[\sum_{j=0}^{s}(\max\{\|D(a(u_1))\|_{s_1}^2,\|D\mathbf{v}\|_{s_1}\})^j\Big]\|\nabla\cdot((a(u_2)-a(u_1))\nabla u_2)\|_{s-1}^2\\
&\le CC_8\Big[\sum_{j=0}^{s}\|D\mathbf{v}\|_{s_1}^j\Big]\|a(u_2)-a(u_1)\|_s^2\|\nabla u_2\|_s^2\\
&\le CC_8\Big[\sum_{j=0}^{s}\big(\frac{1}{2}\big)^j\Big]\|a(u_2)-a(u_1)\|_s^2(2C_8)\|f\|_{s-1}^2\\
&\le C(2C_8)^2\|a(u_2)-a(u_1)\|_s^2\|f\|_{s-1}^2\\
&\le \Big(\frac{CL}{C_3}\Big)\|u_2-u_1\|_{s+1}^2
\end{aligned}
$$

which is the same estimate as (2.27).

Recall that $L=C|\Omega||u(\mathbf{x}_0)|^2+CR^2$ from (2.13) , and $C_2=C(1+R+(1+|\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}$ and $C_3=MC_2$ from (2.8). It follows that $\frac{CL}{C_3}\le\frac{C}{M}$, where C

depends on s. As in the proof of (2.21), we define the constant M to be large enough so that $\frac{C}{M} < 1$. It follows that $||u_2 - u_1\|_{s+1}^2 = 0$, and therefore $u_1 = u_2$ and the solution is unique.

This completes the proof of Theorem 2.1. Note that $\delta_0 = \min\{\frac{1}{C_4}, \frac{1}{C_7}\}$, and $\delta_1 = \frac{c_1}{C_*}$, and $\delta_2 = \frac{R^2}{C_5^2}$ in the statement of Theorem 2.1 in which we assume that $\left|\frac{da}{du}\right|_{s,\bar{G}_1}^2 \|f\|_{s-1}^2 \leq \delta_0$, and $|\nabla \cdot \mathbf{v}|_{L^\infty} \leq \delta_1$, and $\max\{1, |\mathbf{v}|_{L^\infty}^2\}\|f\|_{s-1}^2 \leq \delta_2$, and $\|D\mathbf{v}\|_s \leq \frac{1}{2}$. And δ_0, δ_1, δ_2, C_4, C_5, C_7 depend on s, c_1, R, $|\Omega|$, and $|u(\mathbf{x}_0)|$.

Appendix A. Existence for the linear equation

In this section, we present the proof of the existence of a solution to the linear problem (2.1), (2.2).

Lemma A.1. *Let $a_1 \in C^1(\Omega)\cap H^s(\Omega)$, $f \in C(\Omega)\cap H^{s-1}(\Omega)$, $\mathbf{v} \in C^1(\Omega)\cap H^s(\Omega)$ be given functions, where $a_1(\mathbf{x}) > c_1$ for some positive constant c_1, for $\mathbf{x} \in \Omega$, $\Omega = \mathbb{T}^N$, $N = 2$ or $N = 3$. We assume that $|\nabla \cdot \mathbf{v}|_{L^\infty} \leq \frac{c_1}{C_*}$, where C_* is the constant from Poincaré's inequality $\|\bar{u}\|_0^2 \leq C_*\|\nabla u\|_0^2$, and where $\bar{u}(\mathbf{x}) = u(\mathbf{x}) - \frac{1}{|\Omega|}\int_\Omega u(\mathbf{x})d\mathbf{x}$. Then there is a unique classical solution $u \in C^2(\Omega) \cap H^{s+1}(\Omega)$ of*

$$-\nabla \cdot (a_1 \nabla u) + \mathbf{v} \cdot \nabla u = f, \tag{A.1}$$

$$u(\mathbf{x}_0) = u_0, \tag{A.2}$$

where u_0 is a given constant and $\mathbf{x}_0 \in \Omega$ is a given point, and where $s > \frac{N}{2} + 1$.

Proof. The operator in (A.1) is linear with $a_1(\mathbf{x}) > c_1$ for $\mathbf{x} \in \Omega$. The existence of a zero-mean solution $\bar{u}(\mathbf{x})$ of equation (A.1) follows from the standard theory for elliptic equations, specifically, the Lax-Milgram Lemma (see, e.g., [4]). We then define the chosen solution $u(\mathbf{x})$ to (A.1), (A.2) to be $u(\mathbf{x}) = \bar{u}(\mathbf{x}) - \bar{u}(\mathbf{x}_0) + u_0$.

We remark that the condition for the Lax-Milgram Lemma that $\|\bar{u}\|_1^2 \leq CB[\bar{u}, \bar{u}]$, where $B[\bar{u}, \bar{u}] = (a_1\nabla\bar{u}, \nabla\bar{u}) + (\mathbf{v} \cdot \nabla\bar{u}, \bar{u})$, and where $\bar{u}(\mathbf{x}) = u(\mathbf{x}) - \frac{1}{|\Omega|}\int_\Omega u(\mathbf{x})d\mathbf{x}$, follows from the following inequality:

$$\begin{aligned}
(c_1\nabla u, \nabla u) &\leq (a_1\nabla u, \nabla u)\\
&= -(\mathbf{v} \cdot \nabla\bar{u}, \bar{u}) + B[\bar{u}, \bar{u}]\\
&= \frac{1}{2}(\nabla \cdot \mathbf{v} \cdot \bar{u}, \bar{u}) + B[\bar{u}, \bar{u}]\\
&\leq \frac{1}{2}|\nabla \cdot \mathbf{v}|_{L^\infty}\|\bar{u}\|_0^2 + B[\bar{u}, \bar{u}]\\
&\leq \frac{1}{2}C_*|\nabla \cdot \mathbf{v}|_{L^\infty}\|\nabla u\|_0^2 + B[\bar{u}, \bar{u}]\\
&\leq \frac{c_1}{2}\|\nabla u\|_0^2 + B[\bar{u}, \bar{u}]
\end{aligned}$$

where we used the fact that $|\nabla \cdot \mathbf{v}|_{L^\infty} \leq \frac{c_1}{C_*}$. And so $\frac{1}{2}(c_1\nabla u, \nabla u) \leq B[\bar{u}, \bar{u}]$. From Poincaré's inequality $\|\bar{u}\|_0^2 \leq C_*\|\nabla u\|_0^2$, we obtain the desired inequality $\|\bar{u}\|_1^2 = \|\bar{u}\|_0^2 + \|\nabla u\|_0^2 \leq (C_* + 1)\|\nabla u\|_0^2 \leq \frac{2(C_*+1)}{c_1}B[\bar{u}, \bar{u}]$.

The regularity of the chosen solution $u(\mathbf{x})$ follows from the estimates (B.5) from Lemma B.1 and (B.9) from Lemma B.2 in Appendix B, applied to equation (A.1), which yield:

$$\|u\|_0^2 \leq C\|u(\mathbf{x}_0)\|_0^2 + C\|\nabla(u(\mathbf{x}_0))\|_2^2 + C\|\nabla u\|_2^2 \leq C|\Omega||u(\mathbf{x}_0)|^2 + C\|\nabla u\|_s^2$$

$$\|\nabla u\|_s^2 \le C_1\Big[\sum_{j=0}^{s}(\max\{\|Da_1\|_{s_1}^2, \|D\mathbf{v}\|_{s_1}\})^j\Big]\|f\|_{s-1}^2$$

where $s_1 = \max\{s-1, s_0\} = s-1$, and $s_0 = [\frac{N}{2}] + 1 = 2$, and $s > \frac{N}{2} + 1$, so $s \ge 3$. It follows that $u \in C^2(\Omega) \cap H^{s+1}(\Omega)$ by the above estimates and by Sobolev's inequality $|h|_{L^\infty}^2 \le C\|h\|_{s_0}^2$ (see, e.g., [1, 3]). $\square$

Appendix B. A priori estimates

Recall that we will be using the Sobolev space $H^s(\Omega)$ (where $s \ge 0$ is an integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$, with norm given by $\|g\|_s^2 = \sum_{|\alpha|\le s}\int_\Omega |D^\alpha g|^2 d\mathbf{x}$ and inner product $(g,h)_s = \sum_{|\alpha|\le s}\int_\Omega (D^\alpha g)\cdot(D^\alpha h)d\mathbf{x}$. The domain Ω is the N-dimensional torus $\mathbb{T}^N$, where $N = 2$ or $N = 3$. Here, we are using the standard multi-index notation. For convenience, we are going to denote derivatives by $g_\alpha = D^\alpha g$. And we will denote the L^2 inner product by $(g,h) = \int_\Omega g\cdot h\ d\mathbf{x}$. We will use C to denote a generic constant whose value may change from one relation to the next. Recall that we let both ∇g and Dg denote the gradient of g.

We begin by listing several standard Sobolev space inequalities.

Lemma B.1 (Calculus Inequalities)**.**

(a) Let $g(u)$ be a smooth function on G, where $u(\mathbf{x})$ is a continuous function and where $u(\mathbf{x}) \in G_1$ for $\mathbf{x} \in \Omega$ and $G_1 \subset G$ and $u \in H^r(\Omega) \cap L^\infty(\Omega)$. Then for $r \ge 1$,

$$\|D^r(g(u))\|_0 \le C\Big|\frac{dg}{du}\Big|_{r-1,\bar{G}_1}(1 + |u|_{L^\infty})^{r-1}\|Du\|_{r-1}, \tag{B.1}$$

where $|h|_{r,\bar{G}_1} = \max\{\big|\frac{d^jh}{du^j}(u_*)\big| : u_* \in \bar{G}_1, 0 \le j \le r\}$, and where C depends on r, Ω.

(b) And

$$\|g(u) - g(v)\|_r \le C\Big|\frac{dg}{du}\Big|_{r,\bar{G}_1}(1 + |u|_{L^\infty} + |v|_{L^\infty})(\|u\|_r + \|v\|_r)\|u - v\|_r, \tag{B.2}$$

where C depends on r, Ω.

(c) If $Dg \in H^{r_1}(\Omega)$, $h \in H^{r-1}(\Omega)$, where $r_1 = \max\{r-1, s_0\}$, $s_0 = [\frac{N}{2}] + 1$, then for any $r \ge 1$, g, h satisfy the estimate

$$\|D^\alpha(gh) - gD^\alpha h\|_0 \le C\|Dg\|_{r_1}\|h\|_{r-1}, \tag{B.3}$$

where $r = |\alpha|$, and the constant C depends on r, Ω.

(d) Let v, w be $C^1(\Omega) \cap H^3(\Omega)$ functions on a bounded, open, connected, convex domain Ω. And let $v(\mathbf{x}_0) = w(\mathbf{x}_0)$ at a point $\mathbf{x}_0 \in \Omega$. Then $v - w$ and v satisfy the estimates

$$\|v - w\|_0^2 \le C\|\nabla(v - w)\|_2^2, \tag{B.4}$$

$$\|v\|_0^2 \le C\|w\|_0^2 + C\|\nabla w\|_2^2 + C\|\nabla v\|_2^2 \tag{B.5}$$

Here C is a constant which depends on Ω.

Proofs of the inequalities (a), (b) may be found, for example, in [8], [10]. Proof of inequalities (c), (d) may be found in [2]. Inequalities (a), (b) also appear in [3].

Lemmas B.2 and B.3 provide the key a priori estimates used in the proof of the theorem.

Lemma B.2. *Let $a_1(\mathbf{x})$, $\mathbf{v}(\mathbf{x})$, and $f(\mathbf{x})$ be sufficiently smooth functions in the following equation*

$$-\nabla \cdot (a_1 \nabla u) + \mathbf{v} \cdot \nabla u = f, \tag{B.6}$$

where $a_1(\mathbf{x}) > c_1$, for some positive constant c_1, and for all $\mathbf{x} \in \Omega$, with $\Omega = \mathbb{T}^N$, and $N = 2$ or $N = 3$. We assume that $|\nabla \cdot \mathbf{v}|_{L^\infty} \le \frac{c_1}{C_*}$, where C_* is the constant from Poincaré's inequality $\|\bar{u}\|_0^2 \le C_* \|\nabla u\|_0^2$, and where $\bar{u}(\mathbf{x}) = u(\mathbf{x}) - \frac{1}{|\Omega|}\int_\Omega u(\mathbf{x}) d\mathbf{x}$. Then we obtain the inequalities:

$$\|\nabla u\|_0^2 \le C \|f\|_0^2, \tag{B.7}$$

$$\|\nabla u\|_r^2 \le C \max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\} \|\nabla u\|_{r-1}^2 + C\|f\|_{r-1}^2, \tag{B.8}$$

$$\|\nabla u\|_r^2 \le C_1 \Big[\sum_{j=0}^{r} (\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^j \Big] \|f\|_{r-1}^2 \tag{B.9}$$

where $r \ge 1$, where $r_1 = \max\{r-1, s_0\}$, and where $s_0 = [\frac{N}{2}] + 1 = 2$. Here constant C in (B.7) *depends on c_1, and the constant C in* (B.8) *depends on r, c_1, and the constant C_1 in* (B.9) *depends on r, c_1.*

Proof. First, we obtain an L^2 estimate. Integrating equation (B.6) by parts with $\bar{u}$, where $\bar{u}(\mathbf{x}) = u(\mathbf{x}) - \frac{1}{|\Omega|}\int_\Omega u(\mathbf{x}) d\mathbf{x}$, yields

$$\begin{aligned}(c_1 \nabla u, \nabla u) &\le (a_1 \nabla u, \nabla u) = -(\nabla \cdot (a_1 \nabla u), \bar{u}) = -(\mathbf{v} \cdot \nabla u, \bar{u}) + (f, \bar{u}) \\ &= \frac{1}{2}(\nabla \cdot \mathbf{v} \cdot \bar{u}, \bar{u}) + (f, \bar{u}) \\ &\le \frac{1}{2}|\nabla \cdot \mathbf{v}|_{L^\infty} \|\bar{u}\|_0^2 + \frac{1}{4\epsilon}\|f\|_0^2 + \epsilon \|\bar{u}\|_0^2 \\ &\le \frac{1}{2} C_* |\nabla \cdot \mathbf{v}|_{L^\infty} \|\nabla u\|_0^2 + \frac{1}{4\epsilon}\|f\|_0^2 + \epsilon C_* \|\nabla u\|_0^2\end{aligned} \tag{B.10}$$

where we used Cauchy's inequality with ϵ, namely $gh \le \frac{1}{4\epsilon} g^2 + \epsilon h^2$, and where we used the fact that $a_1(\mathbf{x}) > c_1$. We also used Poincaré's inequality (see, e.g., [3], [4]) to estimate $\|\bar{u}\|_0^2 \le C_* \|\nabla u\|_0^2$, where C_* is a constant. We assume that $|\nabla \cdot \mathbf{v}|_{L^\infty} \le \frac{c_1}{C_*}$. And we let $\epsilon = \frac{c_1}{4C_*}$. Then from (B.10), we obtain

$$\|\nabla u\|_0^2 \le C \|f\|_0^2 \tag{B.11}$$

where C depends on c_1. This is the desired inequality (B.7).

Next, after applying D^α to the equation (B.6), we obtain the equation:

$$-\nabla \cdot (a_1 \nabla u_\alpha) + \mathbf{v} \cdot \nabla u_\alpha = F_\alpha \tag{B.12}$$

where $F_\alpha = f_\alpha + [\nabla \cdot (a_1 \nabla u)_\alpha - \nabla \cdot (a_1 \nabla u_\alpha)] - [(\mathbf{v} \cdot \nabla u)_\alpha - \mathbf{v} \cdot \nabla u_\alpha]$.

From (B.12) we obtain

$$\begin{aligned}c_1(\nabla u_\alpha, \nabla u_\alpha) &\le (a_1 \nabla u_\alpha, \nabla u_\alpha) \\ &= -(\nabla \cdot (a_1 \nabla u_\alpha), u_\alpha) \\ &= -(\mathbf{v} \cdot \nabla u_\alpha, u_\alpha) + (F_\alpha, u_\alpha) \\ &\le \frac{1}{2}|\nabla \cdot \mathbf{v}|_{L^\infty} \|u_\alpha\|_0^2 + |(F_\alpha, u_\alpha)| \\ &\le C|D\mathbf{v}|_{L^\infty} \|\nabla u\|_{k-1}^2 + |(F_\alpha, u_\alpha)|\end{aligned} \tag{B.13}$$

where $|\alpha| = k$.

Next, we estimate $|(F_\alpha, u_\alpha)|$. We use integration by parts, and then apply inequality (B.3) from Lemma B.1, to obtain the following inequality:

$$
\begin{aligned}
&|(F_\alpha, u_\alpha)| \\
&\le |(f_\alpha, u_\alpha)| + |([\nabla\cdot(a_1\nabla u)_\alpha - \nabla\cdot(a_1\nabla u_\alpha)], u_\alpha)| + |((\mathbf{v}\cdot\nabla u)_\alpha - \mathbf{v}\cdot\nabla u_\alpha, u_\alpha)| \\
&= |(f_{\alpha-\beta}, u_{\alpha+\beta})| + |([(a_1\nabla u)_\alpha - a_1\nabla u_\alpha], \nabla u_\alpha)| + |((\mathbf{v}\cdot\nabla u)_\alpha - \mathbf{v}\cdot\nabla u_\alpha, u_\alpha)| \\
&\le \|f_{\alpha-\beta}\|_0\|u_{\alpha+\beta}\|_0 + \|(a_1\nabla u)_\alpha - a_1\nabla u_\alpha\|_0\|\nabla u_\alpha\|_0 \\
&\quad + \|(\mathbf{v}\cdot\nabla u)_\alpha - \mathbf{v}\cdot\nabla u_\alpha\|_0\|u_\alpha\|_0 \\
&\le C\|f\|_{k-1}\|\nabla u\|_k + C\|Da_1\|_{k_1}\|\nabla u\|_{k-1}\|\nabla u\|_k + C\|D\mathbf{v}\|_{k_1}\|\nabla u\|_{k-1}^2 \\
&\le \frac{C}{4\epsilon}\|f\|_{k-1}^2 + \epsilon\|\nabla u\|_k^2 + \frac{C}{4\epsilon}\|Da_1\|_{k_1}^2\|\nabla u\|_{k-1}^2 + \epsilon\|\nabla u\|_k^2 + C\|D\mathbf{v}\|_{k_1}\|\nabla u\|_{k-1}^2
\end{aligned} \tag{B.14}
$$

where $|\beta| = 1$, $k = |\alpha|$, and $k_1 = \max\{k-1, s_0\}$, with $s_0 = [\frac{N}{2}]+1$. Again, we used Cauchy's inequality with ϵ. Substituting (B.14) into (B.13), and adding (B.13) over $|\alpha| = k \le r$, including the estimate (B.10), we obtain for $r \ge 1$ the estimate

$$
\|\nabla u\|_r^2 \le \frac{C}{4\epsilon}(\|Da_1\|_{r_1}^2 + \|D\mathbf{v}\|_{r_1})\|\nabla u\|_{r-1}^2 + \frac{C}{4\epsilon}\|f\|_{r-1}^2 + \epsilon C\|\nabla u\|_r^2 \tag{B.15}
$$

where $r_1 = \max\{r-1, s_0\}$, with $s_0 = [\frac{N}{2}]+1$, and where C depends on r, c_1. Here we used Sobolev's lemma to obtain $|D\mathbf{v}|_{L^\infty} \le C\|D\mathbf{v}\|_{s_0}$. Choosing ϵ sufficiently small yields

$$
\|\nabla u\|_r^2 \le C\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\}\|\nabla u\|_{r-1}^2 + C\|f\|_{r-1}^2 \tag{B.16}
$$

where C depends on r, c_1. This is the desired inequality (B.8).

Applying the inequality (B.16) to $\|\nabla u\|_{r-1}^2$ which appears on the right-hand side of (B.16) yields

$$
\begin{aligned}
\|\nabla u\|_r^2 &\le C(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\}) \\
&\qquad \times \Big[C(\max\{\|Da_1\|_{r_2}^2, \|D\mathbf{v}\|_{r_2}\})\|\nabla u\|_{r-2}^2 + C\|f\|_{r-2}^2\Big] + C\|f\|_{r-1}^2 \\
&\le C(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^2\|\nabla u\|_{r-2}^2 \\
&\quad + C(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})\|f\|_{r-2}^2 + C\|f\|_{r-1}^2
\end{aligned} \tag{B.17}
$$

where $r_1 = \max\{r-1, s_0\}$, $r_2 = \max\{r-2, s_0\}$, $r_2 \le r_1$, with $s_0 = [\frac{N}{2}]+1 = 2$ for $N = 2, 3$.

Similarly, by applying the estimate (B.16) to $\|\nabla u\|_{r-j}^2$ for $j = 2, 3, \dots, r-1$, which will appear in the term $C(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^j\|\nabla u\|_{r-j}^2$ on the right-hand side of (B.17), we obtain

$$
\begin{aligned}
\|\nabla u\|_r^2 &\le C\sum_{j=1}^{r-1}(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^j\|f\|_{r-1-j}^2 \\
&\quad + C(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^r\|\nabla u\|_0^2 + C\|f\|_{r-1}^2 \\
&\le C\Big[\sum_{j=0}^{r-1}(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^j\Big]\|f\|_{r-1}^2 \\
&\quad + C(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^r\|\nabla u\|_0^2
\end{aligned} \tag{B.18}
$$

Substituting the estimate $\|\nabla u\|_0^2 \le C\|f\|_0^2$ into (B.18) yields

$$
\begin{aligned}
&\|\nabla u\|_r^2 \\
&\le C\Big[\sum_{j=0}^{r-1}(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^j\Big]\|f\|_{r-1}^2 + C(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^r\|f\|_0^2 \\
&\le C_1\Big[\sum_{j=0}^{r}(\max\{\|Da_1\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^j\Big]\|f\|_{r-1}^2
\end{aligned}
$$

where C_1 depends on r, c_1. This completes the proof. □

Lemma B.3. *Let $a(u)$ be a smooth function of u, and let $\mathbf{v}(\mathbf{x})$ and $f(\mathbf{x})$ be sufficiently smooth functions in the equation*

$$-\nabla\cdot(a(u)\nabla u) + \mathbf{v}\cdot\nabla u = f \tag{B.19}$$

for $\mathbf{x}\in\Omega$, where $\Omega = \mathbb{T}^N$, $N = 2, 3$, where $a(u) > c_1$, for some positive constant c_1, and where $|u-u_0|_{L^\infty} \le R$, where u_0, R are given constants. We assume that $|\nabla\cdot\mathbf{v}|_{L^\infty} \le \frac{c_1}{C_*}$, where C_* is the constant from Poincaré's inequality $\|\bar{u}\|_0^2 \le C_*\|\nabla u\|_0^2$, and where $\bar{u}(\mathbf{x}) = u(\mathbf{x}) - \frac{1}{|\Omega|}\int_\Omega u(\mathbf{x})d\mathbf{x}$. Then there exist constants C_7, C_8, such that if $\left|\frac{da}{du}\right|_{s,\bar{G}_1}^2\|f\|_{s-1}^2 \le \frac{1}{C_7}$, and if $\|D\mathbf{v}\|_s \le \frac{1}{2}$, and if $a''(u) \le \frac{1}{a(u)}(a'(u))^2$, then u satisfies the inequality

$$\|\nabla u\|_s^2 \le 2C_8\|f\|_{s-1}^2 \tag{B.20}$$

We define $C_7 = 4C_3^2C_8^2$ and $C_8 = C_0C_1K_1$, where C_1 is the constant from estimate (B.9) *in Lemma B.2, and where $C_2 = C(1+R+(1+|\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}$ and $C_3 = MC_2$ are the same constants as in* (2.8) *from Proposition 2.2, and C_0 is a constant which depends on s, c_1, where $s > \frac{N}{2}+1$. We define the constant $K_1 = \max\{1, |\mathbf{v}|_{L^\infty}^2\}$. And we define $|\frac{da}{du}|_{s,\bar{G}_1} = \max\{\left|\frac{d^{j+1}a}{du^{j+1}}(u_*)\right| : u_*\in\bar{G}_1, 0\le j\le s\}$.*

Proof. First we obtain estimates for $\|\nabla u\|_0^2$, $\|\nabla u\|_1^2$, $\|\nabla u\|_2^2$, and $\|\nabla u\|_r^2$, where $3 \le r \le s$. It is necessary to have an estimate for $\|\nabla u\|_j^2$ in order to obtain an estimate for $\|\nabla u\|_{j+1}^2$, for $j = 0, 1, 2, \dots, s-1$. We will apply estimate (B.9) from Lemma B.2 to obtain an estimate for $\|\nabla u\|_r^2$, when $3 \le r \le s$.

From inequality (B.7) in Lemma B.2 applied to equation (B.19), we obtain

$$\|\nabla u\|_0^2 \le C\|f\|_0^2 \tag{B.21}$$

where C depends on c_1. We now obtain an estimate for $\|\nabla u\|_1^2$. Applying D^α to equation (B.19), with $|\alpha| = 1$, yields

$$-\nabla\cdot(a(u)\nabla u_\alpha) = \nabla\cdot((a(u))_\alpha\nabla u) - (\mathbf{v}\cdot\nabla u)_\alpha + f_\alpha \tag{B.22}$$

Integrating (B.22) by parts with u_α, where $|\alpha| = 1$, and using the fact from equation (B.19) that $\nabla a(u) \cdot \nabla u = -a(u)\Delta u + \mathbf{v} \cdot \nabla u - f$, yields

$$
\begin{aligned}
(a(u)\nabla u_\alpha, \nabla u_\alpha) &= -(\nabla \cdot (a(u)\nabla u_\alpha), u_\alpha) \\
&= (\nabla \cdot ((a(u))_\alpha \nabla u), u_\alpha) - ((\mathbf{v} \cdot \nabla u)_\alpha, u_\alpha) + (f_\alpha, u_\alpha) \\
&= -((a(u))_\alpha \nabla u, \nabla u_\alpha) - ((\mathbf{v} \cdot \nabla u)_\alpha, u_\alpha) + (f_\alpha, u_\alpha) \\
&= -\frac{1}{2}((a(u))_\alpha, (\nabla u \cdot \nabla u)_\alpha) - ((\mathbf{v} \cdot \nabla u)_\alpha, u_\alpha) + (f_\alpha, u_\alpha) \\
&= -\frac{1}{2}(a'(u)u_\alpha, (\nabla u \cdot \nabla u)_\alpha) - ((\mathbf{v} \cdot \nabla u)_\alpha, u_\alpha) + (f_\alpha, u_\alpha) \\
&= -\frac{1}{2}(u_\alpha, (a'(u)\nabla u \cdot \nabla u)_\alpha) + \frac{1}{2}(u_\alpha, (a'(u))_\alpha(\nabla u \cdot \nabla u)) \\
&\quad - ((\mathbf{v} \cdot \nabla u)_\alpha, u_\alpha) + (f_\alpha, u_\alpha) \\
&= -\frac{1}{2}(u_\alpha, (\nabla a(u) \cdot \nabla u)_\alpha) + \frac{1}{2}(u_\alpha, a''(u)u_\alpha(\nabla u \cdot \nabla u)) \\
&\quad - ((\mathbf{v} \cdot \nabla u)_\alpha, u_\alpha) + (f_\alpha, u_\alpha) \\
&= \frac{1}{2}(u_\alpha, (a(u)\Delta u - \mathbf{v} \cdot \nabla u + f)_\alpha) + \frac{1}{2}((u_\alpha)^2, a''(u)(\nabla u \cdot \nabla u)) \\
&\quad - ((\mathbf{v} \cdot \nabla u)_\alpha, u_\alpha) + (f_\alpha, u_\alpha) \\
&= -\frac{1}{2}(u_{\alpha+\alpha}, a(u)\Delta u) + \frac{3}{2}((\mathbf{v} \cdot \nabla u), u_{\alpha+\alpha}) - \frac{3}{2}(f, u_{\alpha+\alpha}) \\
&\quad + \frac{1}{2}((u_\alpha)^2, a''(u)(\nabla u \cdot \nabla u))
\end{aligned}
\tag{B.23}
$$

Adding (B.23) over $|\alpha| = 1$ yields

$$
\begin{aligned}
\sum_{|\alpha|=1} (a(u)\nabla u_\alpha, \nabla u_\alpha) &= -\frac{1}{2}\sum_{|\alpha|=1} (u_{\alpha+\alpha}, a(u)\Delta u) + \frac{3}{2}\sum_{|\alpha|=1} ((\mathbf{v} \cdot \nabla u), u_{\alpha+\alpha}) \\
&\quad - \frac{3}{2}\sum_{|\alpha|=1} (f, u_{\alpha+\alpha}) + \frac{1}{2}\sum_{|\alpha|=1} ((u_\alpha)^2, a''(u)(\nabla u \cdot \nabla u)) \\
&= -\frac{1}{2}(\Delta u, a(u)\Delta u) + \frac{3}{2}((\mathbf{v} \cdot \nabla u), \Delta u) - \frac{3}{2}(f, \Delta u) \\
&\quad + \frac{1}{2}((\nabla u \cdot \nabla u), a''(u)(\nabla u \cdot \nabla u))
\end{aligned}
\tag{B.24}
$$

Next, we estimate the term $\frac{1}{2}((\nabla u \cdot \nabla u), a''(u)(\nabla u \cdot \nabla u))$ in (B.24). We assume that $a''(u) \le \frac{1}{a(u)}(a'(u))^2$. We then obtain the inequality

$$
\begin{aligned}
&\frac{1}{2}((\nabla u \cdot \nabla u), a''(u)(\nabla u \cdot \nabla u)) \\
&\le \frac{1}{2}(\frac{1}{a(u)}(a'(u))^2(\nabla u \cdot \nabla u), (\nabla u \cdot \nabla u)) \\
&= \frac{1}{2}((\nabla a(u) \cdot \nabla u), \frac{1}{a(u)}(\nabla a(u) \cdot \nabla u)) \\
&= \frac{1}{2}((a(u)\Delta u - \mathbf{v} \cdot \nabla u + f), \frac{1}{a(u)}(a(u)\Delta u - \mathbf{v} \cdot \nabla u + f))
\end{aligned}
\tag{B.25}
$$

$$
\begin{aligned}
&= \frac{1}{2}(\Delta u, a(u)\Delta u) + \frac{1}{2}(f, \frac{1}{a(u)}f) + \frac{1}{2}(\mathbf{v}\cdot\nabla u, \frac{1}{a(u)}\mathbf{v}\cdot\nabla u) \\
&\quad + (\Delta u, f) - (\mathbf{v}\cdot\nabla u, \Delta u) - (\mathbf{v}\cdot\nabla u, \frac{1}{a(u)}f)
\end{aligned}
\tag{B.26}
$$

Substituting (B.26) into (B.24) yields

$$
\begin{aligned}
\sum_{|\alpha|=1}(a(u)\nabla u_\alpha, \nabla u_\alpha) &= \frac{1}{2}(f, \frac{1}{a(u)}f) + \frac{1}{2}(\mathbf{v}\cdot\nabla u, \frac{1}{a(u)}\mathbf{v}\cdot\nabla u) \\
&\quad - \frac{1}{2}(\Delta u, f) + \frac{1}{2}(\mathbf{v}\cdot\nabla u, \Delta u) - (\mathbf{v}\cdot\nabla u, \frac{1}{a(u)}f) \\
&\le \frac{1}{c_1}(f,f) + \frac{1}{c_1}|\mathbf{v}|_{L^\infty}^2\|\nabla u\|_0^2 + \epsilon(\Delta u, \Delta u) \\
&\quad + \frac{1}{16\epsilon}(f,f) + \frac{1}{16\epsilon}|\mathbf{v}|_{L^\infty}^2\|\nabla u\|_0^2 + \epsilon(\Delta u, \Delta u)
\end{aligned}
\tag{B.27}
$$

where we used Cauchy's inequality with ϵ, and we used the fact that $a(u) > c_1$.

We now use the fact that $\sum_{|\alpha|=1}(\nabla u_\alpha, \nabla u_\alpha) = \sum_{|\alpha|=1}((u_{\alpha+\alpha}, \Delta u) = (\Delta u, \Delta u)$. We also use the fact that $a(u) > c_1$, and we define $\epsilon = \frac{c_1}{4}$. Then (B.27) becomes

$$
\begin{aligned}
&\sum_{|\alpha|=1}(c_1\nabla u_\alpha, \nabla u_\alpha) \le \sum_{|\alpha|=1}(a(u)\nabla u_\alpha, \nabla u_\alpha) \\
&\le \frac{c_1}{2}(\Delta u, \Delta u) + \frac{5}{4c_1}(f,f) + \frac{5}{4c_1}|\mathbf{v}|_{L^\infty}^2\|\nabla u\|_0^2 \\
&= \frac{c_1}{2}\sum_{|\alpha|=1}(\nabla u_\alpha, \nabla u_\alpha) + \frac{5}{4c_1}(f,f) + \frac{5}{4c_1}|\mathbf{v}|_{L^\infty}^2\|\nabla u\|_0^2
\end{aligned}
\tag{B.28}
$$

Subtracting the term $\frac{c_1}{2}\sum_{|\alpha|=1}(\nabla u_\alpha, \nabla u_\alpha)$ on both sides of (B.28), and using inequality (B.21), namely $\|\nabla u\|_0^2 \le C\|f\|_0^2$, yields the estimate

$$
\sum_{|\alpha|=1}(\nabla u_\alpha, \nabla u_\alpha) \le C\max\{1, |\mathbf{v}|_{L^\infty}^2\}\|f\|_0^2 \tag{B.29}
$$

where C depends on c_1. Adding the inequalities (B.29), (B.21) yields

$$
\|\nabla u\|_1^2 = \|\nabla u\|_0^2 + \sum_{|\alpha|=1}(\nabla u_\alpha, \nabla u_\alpha) \le C\max\{1, |\mathbf{v}|_{L^\infty}^2\}\|f\|_0^2 = CK_1\|f\|_0^2 \tag{B.30}
$$

where C depends on c_1, and where we define the constant $K_1 = \max\{1, |\mathbf{v}|_{L^\infty}^2\}$.

We now obtain an estimate for $\|\nabla u\|_2^2$. Applying D^α to equation (B.19), with $|\alpha| = 2$, yields

$$
\begin{aligned}
-\nabla\cdot(a(u)\nabla u_\alpha) &= \nabla\cdot((a(u))_\alpha\nabla u) + \nabla\cdot((a(u))_{\alpha-\beta}\nabla u_\beta) + \nabla\cdot((a(u))_\beta\nabla u_{\alpha-\beta}) \\
&\quad - \mathbf{v}\cdot\nabla u_\alpha - \mathbf{v}_\alpha\cdot\nabla u - \mathbf{v}_\beta\cdot\nabla u_{\alpha-\beta} - \mathbf{v}_{\alpha-\beta}\cdot\nabla u_\beta + f_\alpha
\end{aligned}
\tag{B.31}
$$

where $|\beta| = 1$. Integrating by parts with u_α, where $|\alpha| = 2$ and $|\beta| = 1$, and using inequality (B.1) from Lemma B.1, yields

$$
\begin{aligned}
&((a(u)\nabla u_\alpha, \nabla u_\alpha) \\
&= -(\nabla\cdot(a(u)\nabla u_\alpha), u_\alpha) \\
&= (\nabla\cdot((a(u))_\alpha\nabla u), u_\alpha) + (\nabla\cdot((a(u))_{\alpha-\beta}\nabla u_\beta), u_\alpha) \\
&\quad + (\nabla\cdot((a(u))_\beta\nabla u_{\alpha-\beta}), u_\alpha) - (\mathbf{v}\cdot\nabla u_\alpha, u_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&- (\mathbf{v}_\alpha \cdot \nabla u, u_\alpha) - (\mathbf{v}_\beta \cdot \nabla u_{\alpha-\beta}, u_\alpha) - (\mathbf{v}_{\alpha-\beta} \cdot \nabla u_\beta, u_\alpha) + (f_\alpha, u_\alpha) \\
&= -((a(u))_\alpha \nabla u, \nabla u_\alpha) - ((a(u))_{\alpha-\beta} \nabla u_\beta, \nabla u_\alpha) \\
&\quad - ((a(u))_\beta \nabla u_{\alpha-\beta}, \nabla u_\alpha) + \frac{1}{2}((\nabla \cdot \mathbf{v}) u_\alpha, u_\alpha) - (\mathbf{v}_\alpha \cdot \nabla u, u_\alpha) \\
&\quad - (\mathbf{v}_\beta \cdot \nabla u_{\alpha-\beta}, u_\alpha) - (\mathbf{v}_{\alpha-\beta} \cdot \nabla u_\beta, u_\alpha) - (f_{\alpha-\beta}, u_{\alpha+\beta}) \\
&\le \|(a(u))_\alpha\|_0 |\nabla u|_{L^\infty} \|\nabla u_\alpha\|_0 + |(a(u))_{\alpha-\beta}|_{L^\infty} \|\nabla u_\beta\|_0 \|\nabla u_\alpha\|_0 \\
&\quad + |(a(u))_\beta|_{L^\infty} \|\nabla u_{\alpha-\beta}\|_0 \|\nabla u_\alpha\|_0 + \frac{1}{2} |\nabla \cdot \mathbf{v}|_{L^\infty} \|u_\alpha\|_0^2 + |\mathbf{v}_\alpha|_{L^\infty} \|\nabla u\|_0 \|u_\alpha\|_0 \\
&\quad + |\mathbf{v}_\beta|_{L^\infty} \|\nabla u_{\alpha-\beta}\|_0 \|u_\alpha\|_0 + |\mathbf{v}_{\alpha-\beta}|_{L^\infty} \|\nabla u_\beta\|_0 \|u_\alpha\|_0 + \|f_{\alpha-\beta}\|_0 \|u_{\alpha+\beta}\|_0 \\
&\le C\|D^2(a(u))\|_0 \|\nabla u\|_2 \|\nabla u_\alpha\|_0 + C\|D(a(u))\|_2 \|\nabla u\|_1 \|\nabla u_\alpha\|_0 + C|\nabla \cdot \mathbf{v}|_{L^\infty} \|\nabla u\|_1^2 \\
&\quad + C|D^2\mathbf{v}|_{L^\infty} \|\nabla u\|_0 \|\nabla u\|_1 + C|D\mathbf{v}|_{L^\infty} \|\nabla u\|_1^2 + C\|\nabla f\|_0 \|\nabla u_\alpha\|_0 \\
&\le \frac{C}{4\epsilon} \|D^2(a(u))\|_0^2 \|\nabla u\|_2^2 + \epsilon \|\nabla u_\alpha\|_0^2 + \frac{C}{4\epsilon} \|D(a(u))\|_2^2 \|\nabla u\|_1^2 + \epsilon \|\nabla u_\alpha\|_0^2 \\
&\quad + C|D\mathbf{v}|_{L^\infty} \|\nabla u\|_1^2 + C|D^2\mathbf{v}|_{L^\infty} \|\nabla u\|_0^2 + C|D^2\mathbf{v}|_{L^\infty} \|\nabla u\|_1^2 \\
&\quad + C|D\mathbf{v}|_{L^\infty} \|\nabla u\|_1^2 + \frac{C}{4\epsilon} \|\nabla f\|_0^2 + \epsilon \|\nabla u_\alpha\|_0^2 \\
&\le \frac{C}{4\epsilon} \|D^2(a(u))\|_0^2 \|\nabla u\|_2^2 + \frac{C}{4\epsilon} \Big[\sum_{0 \le r \le 2} \|D^{r+1}(a(u))\|_0^2 \Big] \|\nabla u\|_1^2 \\
&\quad + C(|D\mathbf{v}|_{L^\infty} + |D^2\mathbf{v}|_{L^\infty}) \|\nabla u\|_1^2 + C|D^2\mathbf{v}|_{L^\infty} \|\nabla u\|_0^2 + \frac{C}{4\epsilon} \|\nabla f\|_0^2 + 3\epsilon \|\nabla u_\alpha\|_0^2 \\
&\le \frac{C}{4\epsilon} \Big| \frac{da}{du} \Big|_{1,\bar{G}_1}^2 (1 + |u|_{L^\infty})^2 \|\nabla u\|_1^2 \|\nabla u\|_2^2 \\
&\quad + \frac{C}{4\epsilon} \Big[\sum_{0 \le r \le 2} \Big| \frac{da}{du} \Big|_{r,\bar{G}_1}^2 (1 + |u|_{L^\infty})^{2r} \|\nabla u\|_r^2 \Big] \|\nabla u\|_1^2 \\
&\quad + C(|D\mathbf{v}|_{L^\infty} + |D^2\mathbf{v}|_{L^\infty}) \|\nabla u\|_1^2 + C|D^2\mathbf{v}|_{L^\infty} \|\nabla u\|_0^2 \\
&\quad + \frac{C}{4\epsilon} \|\nabla f\|_0^2 + 3\epsilon \|\nabla u_\alpha\|_0^2 \qquad\qquad (B.32) \\
&\le \frac{C}{4\epsilon} \Big| \frac{da}{du} \Big|_{s,\bar{G}_1}^2 (1 + |u|_{L^\infty})^{2s} \|\nabla u\|_1^2 \|\nabla u\|_2^2 \\
&\quad + \frac{C}{4\epsilon} \Big| \frac{da}{du} \Big|_{s,\bar{G}_1}^2 (1 + |u|_{L^\infty})^{2s} \Big[\sum_{0 \le r \le 2} \|\nabla u\|_r^2 \Big] \|\nabla u\|_1^2 \\
&\quad + C(\|D\mathbf{v}\|_{s_0} + \|D^2\mathbf{v}\|_{s_0}) \|\nabla u\|_1^2 + C\|D^2\mathbf{v}\|_{s_0} \|\nabla u\|_0^2 + \frac{C}{4\epsilon} \|\nabla f\|_0^2 + 3\epsilon \|\nabla u_\alpha\|_0^2 \\
&\le \frac{C}{4\epsilon} \Big| \frac{da}{du} \Big|_{s,\bar{G}_1}^2 (1 + |u|_{L^\infty})^{2s} \|\nabla u\|_1^2 \|\nabla u\|_2^2 + \frac{C}{4\epsilon} \Big| \frac{da}{du} \Big|_{s,\bar{G}_1}^2 (1 + |u|_{L^\infty})^{2s} \|\nabla u\|_2^2 \|\nabla u\|_1^2 \\
&\quad + C\|D\mathbf{v}\|_s \|\nabla u\|_1^2 + C\|D\mathbf{v}\|_s \|\nabla u\|_0^2 + \frac{C}{4\epsilon} \|\nabla f\|_0^2 + 3\epsilon \|\nabla u_\alpha\|_0^2 \\
&\le \frac{C}{2\epsilon} \Big| \frac{da}{du} \Big|_{s,\bar{G}_1}^2 (1 + |u - u_0|_{L^\infty} + |u_0|_{L^\infty})^{2s} \|\nabla u\|_1^2 \|\nabla u\|_2^2 \\
&\quad + C\|D\mathbf{v}\|_s \|\nabla u\|_1^2 + C\|D\mathbf{v}\|_s \|\nabla u\|_0^2 + \frac{C}{4\epsilon} \|\nabla f\|_0^2 + 3\epsilon \|\nabla u_\alpha\|_0^2 \\
&\le \frac{C}{2\epsilon} \Big| \frac{da}{du} \Big|_{s,\bar{G}_1}^2 (1 + R + |u(\mathbf{x}_0)|)^{2s} \|\nabla u\|_1^2 \|\nabla u\|_2^2
\end{aligned}
$$

$$+ C\|D\mathbf{v}\|_s\|\nabla u\|_1^2 + C\|D\mathbf{v}\|_s\|\nabla u\|_0^2 + \frac{C}{4\epsilon}\|\nabla f\|_0^2 + 3\epsilon\|\nabla u_\alpha\|_0^2$$

where we used inequality (B.1) from Lemma B.1. We also used Sobolev's lemma to obtain $|D\mathbf{v}|_{L^\infty} \le C\|D\mathbf{v}\|_{s_0}$ and $|D^2\mathbf{v}|_{L^\infty} \le C\|D^2\mathbf{v}\|_{s_0}$, where $s_0 = [\frac{N}{2}] + 1 = 2$ and $s \ge 3$. We also used Cauchy's inequality with ϵ.

We assume that $\left|\frac{da}{du}\right|^2_{s,\bar{G}_1}\|f\|^2_{s-1} \le \frac{1}{C_7}$, where the constant C_7 will be defined later. And we assume that $\|D\mathbf{v}\|_s \le \frac{1}{2}$. Substituting estimates (B.30), (B.21) for $\|\nabla u\|_1^2$, $\|\nabla u\|_0^2$ into (B.32), and using the fact that $a(u) > c_1$, and letting $\epsilon = \frac{c_1}{6}$, yields

$$\begin{aligned}
&(c_1\nabla u_\alpha, \nabla u_\alpha)\\
&\le ((a(u)\nabla u_\alpha, \nabla u_\alpha)\\
&\le CK_1\Big|\frac{da}{du}\Big|^2_{s,\bar{G}_1}(1 + R + |u(\mathbf{x}_0)|)^{2s}\|f\|_0^2\|\nabla u\|_2^2 + CK_1\|D\mathbf{v}\|_s\|f\|_0^2\\
&\quad + C\|D\mathbf{v}\|_s\|f\|_0^2 + C\|f\|_1^2 + \frac{c_1}{2}\|\nabla u_\alpha\|_0^2\\
&\le CK_1\Big|\frac{da}{du}\Big|^2_{s,\bar{G}_1}(1 + R + |u(\mathbf{x}_0)|)^{2s}\|f\|^2_{s-1}\|\nabla u\|_2^2 + CK_1\|f\|_1^2 + \frac{c_1}{2}\|\nabla u_\alpha\|_0^2\\
&\le CK_1\Big(\frac{1}{C_7}\Big)(1 + R + |u(\mathbf{x}_0)|)^{2s}\|\nabla u\|_2^2 + CK_1\|f\|_1^2 + \frac{c_1}{2}\|\nabla u_\alpha\|_0^2
\end{aligned} \tag{B.33}$$

where C depends on s, c_1, and where we used the facts that $\|D\mathbf{v}\|_s < 1$, and that $K_1 = \max\{1, |\mathbf{v}|^2_{L^\infty}\}$.

Adding (B.33) over all $|\alpha| = 2$ after moving the term $\frac{c_1}{2}\|\nabla u_\alpha\|_0^2$ to the left-hand side, and adding the estimate (B.30) for $\|\nabla u\|_1^2$ yields

$$\begin{aligned}
\|\nabla u\|_2^2 &= \|\nabla u\|_1^2 + \sum_{|\alpha|=2}(\nabla u_\alpha, \nabla u_\alpha)\\
&\le CK_1\Big(\frac{1}{C_7}\Big)(1 + R + |u(\mathbf{x}_0)|)^{2s}\|\nabla u\|_2^2 + CK_1\|f\|_1^2\\
&\le CK_1\Big(\frac{1}{C_7}\Big)(1 + R + (1 + |\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}\|\nabla u\|_2^2 + CK_1\|f\|_1^2 \qquad \text{(B.34)}\\
&\le \Big(\frac{C_0C_2K_1}{C_7}\Big)\|\nabla u\|_2^2 + C_0K_1\|f\|_1^2\\
&\le \Big(\frac{C_0C_3K_1}{C_7}\Big)\|\nabla u\|_2^2 + C_0K_1\|f\|_1^2
\end{aligned}$$

where the constant C_0 depends on s, c_1, and $C_2 = C(1 + R + (1 + |\Omega|^{1/2})|u(\mathbf{x}_0)|)^{2s}$ and $C_3 = MC_2$ are the same constants as in (2.8) from Proposition 2.2, and $M \ge 1$.

We now define $C_7 = 4C_0^2C_3^2C_1^2K_1^2$, where C_0 is the constant from (B.34), and where C_1 is the constant from estimate (B.9) in Lemma B.2, and we may assume that $C_1 \ge 1$, $C_0 \ge 1$, and $C_3 \ge 1$. Substituting the definition of C_7 into (B.34) yields

$$\|\nabla u\|_2^2 \le \frac{1}{4C_0C_1^2C_3K_1}\|\nabla u\|_2^2 + C_0K_1\|f\|_1^2 \le \frac{1}{2}\|\nabla u\|_2^2 + C_0K_1\|f\|_1^2 \tag{B.35}$$

where we used that $K_1 \ge 1$. We define $C_8 = C_0C_1K_1$. It follows from (B.35) that

$$\|\nabla u\|_2^2 \le 2C_0K_1\|f\|_1^2 \le 2C_8\|f\|_1^2\,. \tag{B.36}$$

Note that since $C_8 = C_0C_1K_1$ we have $C_7 = 4C_3^2C_8^2$.

Next we estimate $\|\nabla u\|_r^2$, where $3 \le r \le s$. Using estimate (B.9) from Lemma B.2 in Appendix B applied to equation (B.19) yields

$$\begin{aligned} \|\nabla u\|_r^2 &\le C_1\Big[\sum_{j=0}^{r}(\max\{\|D(a(u))\|_{r_1}^2, \|D\mathbf{v}\|_{r_1}\})^j\Big]\|f\|_{r-1}^2 \\ &\le C_8\Big[\sum_{j=0}^{r}(\max\{\|D(a(u))\|_{r-1}^2, \|D\mathbf{v}\|_{r-1}\})^j\Big]\|f\|_{r-1}^2, \end{aligned} \tag{B.37}$$

where $r_1 = \max\{r-1, s_0\} = r-1$, and $s_0 = [\frac{N}{2}] + 1 = 2$.

We consider two cases: when $\max\{\|D(a(u))\|_{r-1}^2, \|D\mathbf{v}\|_{r-1}\} = \|D(a(u))\|_{r-1}^2$, and when $\max\{\|D(a(u))\|_{r-1}^2, \|D\mathbf{v}\|_{r-1}\} = \|D\mathbf{v}\|_{r-1}$.

Case 1: Suppose that $\max\{\|D(a(u))\|_{r-1}^2, \|D\mathbf{v}\|_{r-1}\} = \|D(a(u))\|_{r-1}^2$. From (2.8) in Proposition 2.2, we have $\|D(a(u))\|_{r-1}^2 \le C_3\big|\frac{da}{du}\big|_{s,\bar{G}_1}^2\|\nabla u\|_{r-1}^2$. Repeatedly applying estimate (B.37), letting $r = 3, 4, \dots, s$ and using the fact that $\|\nabla u\|_{r-1}^2 \le 2C_8\|f\|_{r-2}^2$, and using estimate (2.8) for $\|D(a(u))\|_{r-1}^2$, and using the fact that $r_1 = \max\{r-1, s_0\} = r-1$ when $r \ge 3$, we obtain

$$\begin{aligned} \|\nabla u\|_r^2 &\le C_8\Big[\sum_{j=0}^{r}\|D(a(u))\|_{r-1}^{2j}\Big]\|f\|_{r-1}^2 \\ &\le C_8\Big[\sum_{j=0}^{r}C_3^j\Big|\frac{da}{du}\Big|_{s,\bar{G}_1}^{2j}\|\nabla u\|_{r-1}^{2j}\Big]\|f\|_{r-1}^2 \\ &\le C_8\Big[\sum_{j=0}^{r}C_3^j(2C_8)^j\Big|\frac{da}{du}\Big|_{s,\bar{G}_1}^{2j}\|f\|_{r-2}^{2j}\Big]\|f\|_{r-1}^2 \\ &\le C_8\Big[\sum_{j=0}^{s}C_3^j(2C_8)^j\Big|\frac{da}{du}\Big|_{s,\bar{G}_1}^{2j}\|f\|_{s-1}^{2j}\Big]\|f\|_{r-1}^2 \\ &\le C_8\Big[\sum_{j=0}^{s}C_3^j(2C_8)^j\Big(\frac{1}{C_7}\Big)^j\Big]\|f\|_{r-1}^2 \\ &\le C_8\Big[\sum_{j=0}^{s}\big(\frac{1}{2}\big)^j\Big]\|f\|_{r-1}^2 \\ &\le 2C_8\|f\|_{r-1}^2 \end{aligned} \tag{B.38}$$

where we used the fact that $\big|\frac{da}{du}\big|_{s,\bar{G}_1}^2\|f\|_{s-1}^2 \le \frac{1}{C_7}$, and $C_7 = 4C_3^2C_8^2$, and $C_3C_8 \ge 1$.

Case 2: Suppose that $\max\{\|D(a(u))\|_{r-1}^2, \|D\mathbf{v}\|_{r-1}\} = \|D\mathbf{v}\|_{r-1}$. From (B.37), and using the fact that $\|D\mathbf{v}\|_s \le \frac{1}{2}$, we obtain the following:

$$\|\nabla u\|_r^2 \le C_8\Big[\sum_{j=0}^{r}\|D\mathbf{v}\|_{r-1}^j\Big]\|f\|_{r-1}^2 \le C_8\Big[\sum_{j=0}^{s}\big(\frac{1}{2}\big)^j\Big]\|f\|_{r-1}^2 \le 2C_8\|f\|_{r-1}^2$$

for $3 \le r \le s$, which is the same estimate as (B.38). Therefore we have $\|\nabla u\|_r^2 \le 2C_8\|f\|_{r-1}^2$ for $3 \le r \le s$. It follows that $\|\nabla u\|_s^2 \le 2C_8\|f\|_{s-1}^2$. This completes the proof. $\square$

References

[1] R. Adams and J. Fournier; *Sobolev Spaces*, Academic Press, 2003.

[2] D. L. Denny; *Existence and uniqueness of global solutions to a model for the flow of an incompressible, barotropic fluid with capillary effects*, Electronic Journal of Differential Equations 39 (2007), 1–23.

[3] P. Embid; *On the Reactive and Non-diffusive Equations for Zero Mach Number Combustion*, Comm. in Partial Differential Equations 14, nos. 8 and 9, (1989), 1249–1281.

[4] L. Evans; *Partial Differential Equations*, Graduate Studies in Mathematics 19, American Mathematical Society, Providence, Rhode Island, 1998.

[5] D. Gilbarg and N. S. Trudinger; *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag: Berlin, Heidelberg,,New York, Tokyo, 1983.

[6] H. Ishii; *On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic PDE's*, Comm. Pure Appl. Math. 42 (1989), 15–45.

[7] R. Jensen; *Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations*, Indiana Univ. Math. J., 38 (1989), 629-667.

[8] S. Klainerman and A. Majda; *Singular Limits of Quasilinear Hyperbolic Systems with Large Parameters and the Incompressible Limit of Compressible Fluids*, Comm. Pure Appl. Math. 34 (1981), 481–524.

[9] A. Majda; *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag: New York, 1984.

[10] J. Moser; *A Rapidly Convergent Iteration Method and Non-linear Differential Equations*, Ann. Scuola Norm. Sup., Pisa 20 (1966), 265–315.

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