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A NONLINEAR NEUTRAL PERIODIC DIFFERENTIAL EQUATION

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Abstract. In this article we consider the existence, uniqueness and positivity of a first order non-linear periodic differential equation. The main tool employed is the Krasnosel'skiĭ's fixed point theorem for the sum of a completely continuous operator and a contraction.

1. Introduction

Let $T > 0$ be fixed. We consider the existence, uniqueness and positivity of solutions for the nonlinear neutral periodic equation

$$\begin{gathered} x'(t) = -a(t)x(t) + c(t)x'\big(g(t)\big)g'(t) + q\big(t, x(t), x(g(t))\big), \\ x(t+T) = x(t). \end{gathered} \tag{1.1}$$

In recent years, there have been several papers written on the existence, uniqueness, stability and/or positivity of solutions for periodic equations of forms similar to equation (1.1); see [7, 8, 9, 10, 13, 14, 15, 16] and references therein. Neutral periodic equations such as (1.1) arise in blood cell models (see for example [1], [17] and [18]) and food-limited population models (see for example [2, 3, 4, 5, 6, 12]). In the above mentioned papers, the nonlinear term q and the function a are assumed to be continuous in all arguments. We impose much weaker conditions on the nonlinear term q and the argument function a.

The map $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy Carathéodory conditions with respect to $L^1[0, T]$ if the following conditions hold.

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t, z)$ is Lebesgue measurable.
(ii) For almost all $t \in [0, T]$, the mapping $z \mapsto f(t, z)$ is continuous on $\mathbb{R}^n$.
(iii) For each $r > 0$, there exists $\alpha_r \in L^1([0, T], \mathbb{R})$ such that for almost all $t \in [0, T]$ and for all z such that $|z| < r$, we have $|f(t, z)| \leq \alpha_r(t)$.

In Section 2 we present some preliminary material that we will employ to show the existence of a solution of (1.1). Also, we state a fixed point theorem due to Krasnosel'skiĭ. We present our main results in Section 3.

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2. Preliminaries

Define the set $P_T = \{\psi \in C(\mathbb{R},\mathbb{R}) : \psi(t+T) = \psi(t)\}$ and the norm $\|\psi\| = \sup_{t\in[0,T]} |\psi(t)|$. Then $(P_T, \|\cdot\|)$ is a Banach space. We will assume that the following conditions hold.

(A) $a \in L^1(\mathbb{R},\mathbb{R})$ is bounded, satisfies $a(t+T) = a(t)$ for all t and
$$1 - e^{-\int_{t-T}^{t} a(r)\,dr} \equiv \frac{1}{\eta} \neq 0.$$
(C) $c \in C^1(\mathbb{R},\mathbb{R})$ satisfies $c(t+T) = c(t)$ for all t.
(G) $g \in C^1(\mathbb{R},\mathbb{R})$ satisfies $g(t+T) = g(t)$ for all t.
(Q1) q satisfies Carathéodory conditions with respect to $L^1[0,T]$, and $q(t+T,x,y) = q(t,x,y)$.

In our first lemma, we state the integral equation equivalent to the periodic equation (1.1).

Lemma 2.1. *Suppose that conditions (A), (C), (G) and (Q_1) hold. Then $x \in P_T$ is a solution of equation* (1.1) *if, and only if, $x \in P_T$ satisfies*
$$x(t) = c(t)x(g(t)) + \eta \int_{t-T}^{t} \big[q\big(s, x(s), x(g(s))\big) - r(s)x(g(s))\big] e^{-\int_s^t a(r)\,dr}\,ds, \tag{2.1}$$
where
$$r(s) = a(s)c(s) + c'(s). \tag{2.2}$$

Proof. Let $x \in P_T$ be a solution of (1.1). We first rewrite (1.1) in the form
$$x'(t) + a(t)x(t) = c(t)x'(g(t))g'(t) + q\big(t, x(t), x(g(t))\big).$$
Multiply both sides of the above equation by $e^{\int_0^t a(r)\,dr}$ and then integrate the resulting equation from $t-T$ to t.
$$\begin{aligned} &x(t)e^{\int_0^t a(r)\,dr} - x(t-T)e^{\int_0^{t-T} a(r)\,dr} \\ &= \int_{t-T}^{t} c(s)x'(g(s))g'(s)e^{\int_0^s a(r)\,dr} + q\big(s, x(s), x(g(s))\big)e^{\int_0^s a(r)\,dr}\,ds. \end{aligned} \tag{2.3}$$
Now divide both sides of (2.3) by $e^{\int_0^t a(r)\,dr}$. Since $x \in P_T$, then
$$x(t)\frac{1}{\eta} = \int_{t-T}^{t} c(s)x'(g(s))g'(s)e^{-\int_s^t a(r)\,dr} + q\big(s, x(s), x(g(s))\big)e^{-\int_s^t a(r)\,dr}\,ds. \tag{2.4}$$
Consider the first term on the right hand side of (2.4).
$$\int_{t-T}^{t} c(s)x'(g(s))g'(s)e^{-\int_s^t a(r)\,dr}ds.$$
Integrate this term by parts to get,
$$\begin{aligned} &\int_{t-T}^{t} c(s)x'(g(s))g'(s)e^{-\int_s^t a(r)\,dr}ds \\ &= c(t)x(g(t)) - e^{-\int_{t-T}^{t} a(s)\,ds}c(t-T)x(g(t-T)) \\ &\quad - \int_{t-T}^{t} \frac{d}{ds}\big[c(s)e^{-\int_s^t a(r)\,dr}\big]x(g(s))\,ds. \end{aligned}$$

Since $c(t) = c(t-T)$, $g(t) = g(t-T)$, and $x \in P_T$, then

$$\begin{aligned} &\int_{t-T}^{t} c(s)x'(g(s))g'(s)e^{-\int_s^t a(r)\,dr} ds \\ &= \frac{1}{\eta}c(t)x(g(t)) - \int_{t-T}^{t} \frac{d}{ds}\big[c(s)e^{-\int_s^t a(r)\,dr}\big]x(g(s))\,ds \end{aligned} \tag{2.5}$$

Finally, we put the right hand side of (2.5) into (2.4) and simplify. We obtain that if $x \in P_T$ is a solution of (1.1), then x satisfies

$$x(t) = c(t)x(g(t)) + \eta \int_{t-T}^{t} \big[q\big(s, x(s), x(g(s))\big) - r(s)x(g(s))\big]e^{-\int_s^t a(r)\,dr}\,ds,$$

where $r(s) = a(s)c(s) + c'(s)$.

The converse implication is easily obtained and the proof is complete. □

We end this section by stating the fixed point theorem that we employ to help us show the existence of solutions to equation (1.1); see [11].

Theorem 2.2 (Krasnosel'skiĭ). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $\big(\mathcal{B}, \|\cdot\|\big)$. Suppose that*

(i) *the mapping $A : \mathbb{M} \to \mathcal{B}$ is completely continuous,*
(ii) *the mapping $B : \mathbb{M} \to \mathcal{B}$ is a contraction, and*
(iii) *$x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$.*

Then the mapping $A + B$ has a fixed point in $\mathbb{M}$.

3. Existence Results

We present our existence results in this section. To this end, we first define the operator H by

$$H\psi(t) = c(t)\psi(g(t)) + \eta \int_{t-T}^{t} \big[q\big(s, \psi(s), \psi(g(s))\big) - r(s)\psi(g(s))\big]e^{-\int_s^t a(r)\,dr}\,ds, \tag{3.1}$$

where r is given in equation (2.2). From Lemma 2.1 we see that fixed points of H are solutions of (1.1) and vice versa.

In order to employ Theorem 2.2 we need to express the operator H as the sum of two operators, one of which is completely continuous and the other of which is a contraction. Let $H\psi(t) = \mathcal{A}\psi(t) + \mathcal{B}\psi(t)$ where

$$\mathcal{B}\psi(t) = c(t)\psi(g(t)) \tag{3.2}$$

and

$$\mathcal{A}\psi(t) = \eta \int_{t-T}^{t} \big[q\big(s, \psi(s), \psi(g(s))\big) - r(s)\psi(g(s))\big]e^{-\int_s^t a(r)\,dr}\,ds. \tag{3.3}$$

Our first lemma in this section shows that $\mathcal{A} : P_T \to P_T$ is completely continuous.

Lemma 3.1. *Suppose that conditions* (A), (C), (G), (Q1) *hold. Then $\mathcal{A} : P_T \to P_T$ is completely continuous.*

Proof. From (3.3) and conditions (A), (C), (G) and (Q1), it follows trivially that $r(\sigma + T) = r(\sigma)$ and $e^{-\int_{\sigma+T}^{t+T} a(r)\,dr} = e^{-\int_\sigma^t a(\rho)\,d\rho}$. Consequently, we have that

$$\mathcal{A}\psi(t + T) = \mathcal{A}\psi(t).$$

That is, if $\psi \in P_T$ then $\mathcal{A}\psi$ is periodic with period T.

To see that $\mathcal{A}$ is continuous let $\{\psi_i\} \subset P_T$ be such that $\psi_i \to \psi$. By the Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{i\to\infty} \big|\mathcal{A}\psi_i(t) - \mathcal{A}\psi(t)\big| \\
&\le \lim_{i\to\infty} \eta \int_{t-T}^{t} \Big\{ |r(s)|\, \big|\psi_i(g(s)) - \psi(g(s))\big| \\
&\quad + \Big| q\big(s, \psi_i(s), \psi_i(g(s))\big) - q\big(s, \psi(s), \psi(g(s))\big) \Big| \Big\} e^{-\int_s^t a(r)\,dr}\, ds \\
&= \eta \int_{t-T}^{t} \lim_{i\to\infty} \Big\{ |r(s)|\, \big|\psi_i(g(s)) - \psi(g(s))\big| \\
&\quad + \Big| q\big(s, \psi_i(s), \psi_i(g(s))\big) - q\big(s, \psi(s), \psi(g(s))\big) \Big| \Big\} e^{-\int_s^t a(r)\,dr}\, ds \to 0.
\end{aligned}$$

Hence $\mathcal{A} : P_T \to P_T$.

Finally, we show that $\mathcal{A}$ is completely continuous. Let $\mathcal{B} \subset P_T$ be a closed bounded subset and let C be such that $\|\psi\| \le C$ for all $\psi \in \mathcal{B}$. Then

$$\begin{aligned}
|\mathcal{A}\psi(t)| &\le \eta \int_{t-T}^{t} \Big\{ \big|q\big(s, \psi(s), \psi(g(s))\big)\big| + |r(s)|\big|\psi(g(s))\big| \Big\} e^{-\int_s^t a(r)\,dr}\, ds \\
&\le \eta N \Big\{ \int_{t-T}^{t} \alpha_C(s)\, ds + C \int_{t-T}^{t} |r(s)|\, ds \Big\} \equiv K,
\end{aligned}$$

where $N = \max_{s\in[t-T,t]} e^{-\int_s^t a(r)\,dr}$. And so, the family of functions $\mathcal{A}\psi$ is uniformly bounded.

Again, let $\psi \in \mathcal{B}$. Without loss of generality, we can pick $\tau < t$ such that $t - \tau < T$. Then

$$\begin{aligned}
&|\mathcal{A}\psi(t) - \mathcal{A}\psi(\tau)| \\
&= \eta \Big| \int_{t-T}^{t} \Big\{ q\big(s, \psi(s), \psi(g(s))\big) - r(s)\psi(g(s)) \Big\} e^{-\int_s^t a(r)\,dr}\, ds \\
&\quad - \eta \int_{\tau-T}^{\tau} \Big\{ q\big(s, \psi(s), \psi(g(s))\big) - r(s)\psi(g(s)) \Big\} e^{-\int_s^\tau a(r)\,dr}\, ds \Big|.
\end{aligned}$$

We can rewrite the left hand side as the sum of three integrals.

We obtain the following.

$$\begin{aligned}
&|\mathcal{A}\psi(t) - \mathcal{A}\psi(\tau)| \\
&\le \eta \int_{\tau}^{t} \Big\{ \Big| q\big(s, \psi(s), \psi(g(s))\big) \Big| + |r(s)|\big|\psi(g(s))\big| \Big\}\, e^{-\int_s^t a(r)\,dr}\, ds \\
&\quad + \eta \int_{\tau-T}^{\tau} \Big\{ \Big| q\big(s, \psi(s), \psi(g(s))\big) \Big| + |r(s)|\big|\psi(g(s))\big| \Big\} \\
&\quad \times \Big| e^{-\int_s^t a(r)\,dr} - e^{-\int_s^\tau a(r)\,dr} \Big|\, ds \\
&\quad + \eta \int_{\tau-T}^{t-T} \Big\{ \Big| q\big(s, \psi(s), \psi(g(s))\big) \Big| + |r(s)|\big|\psi(g(s))\big| \Big\}\, e^{-\int_s^\tau a(r)\,dr}\, ds \\
&\le 2\eta N \Big\{ \int_{\tau}^{t} a_C(s) + C|r(s)|\, ds \Big\}
\end{aligned}$$

$$+ \eta \int_{t-T}^{\tau} \big[a_C(s) + C|r(s)|\big] \Big| e^{-\int_s^t a(r)\,dr} - e^{-\int_s^{\tau} a(r)\,dr} \Big| \, ds.$$

Now $\int_\tau^t a_C(s) + |r(s)|\, ds \to 0$ as $(t-\tau) \to 0$. Also, since

$$\begin{aligned} &\int_{t-T}^{\tau} \big[a_c(s) + |r(s)|\big] \Big| e^{-\int_s^t a(r)\,dr} - e^{-\int_s^{\tau} a(r)\,dr} \Big| \, ds \\ &\le \int_0^T \big[a_c(s) + |r(s)|\big] \Big| e^{-\int_s^t a(r)\,dr} - e^{-\int_s^{\tau} a(r)\,dr} \Big| \, ds, \end{aligned}$$

and $|e^{-\int_s^t a(r)\,dr} - e^{-\int_s^{\tau} a(r)\,dr}| \to 0$ as $(t-\tau) \to 0$, then by the Dominated Convergence Theorem,

$$\int_{t-T}^{\tau} \big[a_c(s) + |r(s)|\big] \big| e^{-\int_s^t a(r)\,dr} - e^{-\int_s^{\tau} a(r)\,dr} \big| \, ds \to 0$$

as $(t-\tau) \to 0$. Thus $|\mathcal{A}\psi(t) - \mathcal{A}\psi(\tau)| \to 0$ as $(t-\tau) \to 0$ independently of $\psi \in \mathcal{B}$. As such, the family of functions $\mathcal{A}\psi$ is equicontinuous on $\mathcal{B}$.

By the Arzelà-Ascoli Theorem, $\mathcal{A}$ is completely continuous and the proof is complete. □

Our next lemma gives a sufficient condition under which $\mathcal{B} : P_T \to P_T$ is a contraction.

Lemma 3.2. *Suppose*

$$\|c\| \le \zeta < 1. \tag{3.4}$$

Then $\mathcal{B} : P_T \to P_T$ is a contraction.

The proof of the above lemma is trivial and hence is omitted. We now define some quantities that will be used in the following theorem. Let $\delta = \max_{t\in[0,T]} e^{-\int_0^t a(r)\,dr}$, $R = \sup_{t\in[0,T]} |r(t)|$, $A = \int_0^T |\alpha(s)|\, ds$, $B = \int_0^T |\beta(s)|\, ds$, $\Gamma = \int_0^T |\gamma(s)|\, ds$. Also, we need the following condition on the nonlinear term q.

(Q2) There exists periodic functions $\alpha, \beta, \gamma \in L^1[0,T]$, with period T, such that

$$|q(t,x,y)| \le \alpha(t)|x| + \beta(t)|y| + \gamma(t),$$

for all $x, y \in \mathbb{R}$.

Theorem 3.3. *Suppose that conditions* (A), (C), (G), (Q1), (Q2) *hold. Let $\zeta > 0$ be such that $\|c\| \le \zeta < 1$. Suppose there exists a positive constant J satisfying the inequality*

$$\Gamma\delta\eta + \big(\zeta + \delta\eta(RT + A + B)\big)J \le J.$$

Then (1.1) *has a solution $\psi \in P_T$ such that $\|\psi\| \le J$.*

Proof. Define $\mathbb{M} = \{\psi \in P_T : \|\psi\| \le J\}$. By Lemma 3.1, the operator $\mathcal{A} : \mathbb{M} \to P_T$ is completely continuous. Since $\|c\| \le \zeta < 1$, then by Lemma 3.2, the operator $\mathcal{B} : \mathbb{M} \to P_T$ is a contraction. Conditions, (i) and (ii) of Theorem 2.2 are satisfied. We need to show that condition (iii) is fulfilled. To this end, let $\psi, \varphi \in \mathbb{M}$. Then

$$|\mathcal{A}\psi(t) + \mathcal{B}\varphi(t)| \le |c(t)|\big|\varphi(g(t))\big| + \eta \int_{t-T}^{t} |r(s)|\big|\psi(g(s))\big| e^{-\int_s^t a(r)\,dr}\, ds$$

$$
\begin{aligned}
&+ \eta \int_{t-T}^{t} \big|q\big(s, \psi(s), \psi(g(s))\big)\big| e^{-\int_s^t a(r)\,dr}\,ds \\
&\le \zeta J + \eta\big(R\delta J + \Gamma\delta + A\delta J + B\delta J\big) \\
&= \Gamma\delta\eta + \big(\zeta + \delta\eta(R + A + B)\big)J \ \le\ J.
\end{aligned}
$$

Thus $\|A\psi + B\varphi\| \le J$ and so $A\psi + B\varphi \in \mathbb{M}$. All the conditions of Theorem 2.2 are satisfied and consequently the operator H defined in (3.1) has a fixed point in $\mathbb{M}$. By Lemma 2.1 this fixed point is a solution of (1.1) and the proof is complete. □

The condition (Q2) is a global condition on the function q. In the next theorem we replace this condition with the following local condition.

(Q2*) There exists periodic functions $\alpha^*, \beta^*, \gamma^* \in L^1[0, T]$, with period T, such that $|q(t, x, y)| \le \alpha^*(t)|x| + \beta^*(t)|y| + \gamma^*(t)$, for all x, y with $|x| < J$ and $|y| < J$.

The constants A^*, B^* and Γ^* are defined as before with the understanding that the functions α^*, β^* and γ^* are those from condition (Q2*).

Theorem 3.4. *Suppose that conditions* (A), (C), (G), (Q1) *hold. Suppose there exists a positive constant* J *such that* (Q2*) *holds and such that the inequality*

$$
\Gamma^*\delta\eta + \big(\zeta + \delta\eta(RT + A^* + B^*)\big)J \le J
$$

is satisfied. Then equation (1.1) *has a solution* $\psi \in P_T$ *such that* $\|\psi\| \le J$.

The proof of the above theorem parallels that of Theorem 3.3. For our next result, we give a condition for which there exists a unique solution of (1.1). We replace condition (Q2) with the following condition.

(Q2†) There exists periodic functions $\alpha^\dagger, \beta^\dagger, \in L^1[0, T]$, with period T, such that

$$
|q(t, x_1, y_1) - q(t, x_2, y_2)| \le \alpha^\dagger(t)|x_1 - x_2| + \beta^\dagger(t)|y_1 - y_2|,
$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Theorem 3.5. *Suppose that conditions* (A), (C), (G), (Q1), (Q2†) *hold. If*

$$
\zeta + \delta\eta(RT + A^\dagger + B^\dagger) < 1,
$$

then (1.1) *has a unique* T*-periodic solution.*

Proof. Let $\varphi, \psi \in P_T$. By (3.1) we have for all t,

$$
\begin{aligned}
|H\varphi(t) - H\psi(t)| &\le |c(t)|\,\|\varphi - \psi\| + \delta\eta \int_{t-T}^{t} |r(s)|\,\|\varphi - \psi\|\,ds \\
&\quad + \delta\eta \int_{t-T}^{t} \Big|q\big(s, \varphi(s), \varphi(g(s))\big) - q\big(s, \psi(s), \psi(g(s))\big)\Big|\,ds \\
&\le \zeta\|\varphi - \psi\| + R\delta\eta T\|\varphi - \psi\| + \eta(A^\dagger + B^\dagger)\delta\|\varphi - \psi\|.
\end{aligned}
$$

Hence, $\|H\varphi - H\psi\| \le \big(\zeta + \eta\delta(RT + A^\dagger + B^\dagger)\big)\|\varphi - \psi\|$. By the contraction mapping principal, H has a fixed point in P_T and by Lemma 2.1, this fixed point is a solution of (1.1). The proof is complete. □

For our last result, we give sufficient conditions under which there exists positive solutions of equation (1.1). We begin by defining some new quantities. Let

$$
m \equiv \min_{s \in [t-T, t]} e^{-\int_s^t a(r)\,dr}, \quad M \equiv \max_{s \in [t-T, t]} e^{-\int_s^t a(r)\,dr}.
$$

Given constants $0 < L < K$, define the set $\mathbb{M}_2 = \{\psi \in P_T : L \le \psi(t) \le K, t \in [0,T]\}$.

Assume the following conditions hold.

(C2) $c \in C^1(\mathbb{R},\mathbb{R})$ satisfies $c(t+T) = c(t)$ for all t and there exists a $c^* > 0$ such that $c^* < c(t)$ for all $t \in [0,T]$.

(Q3) There exists constants $0 < L < K$ such that

$$\frac{(1-c^*)L}{\eta m T} \le q(s,\rho,\rho) - r(s)\rho \le \frac{(1-\zeta)K}{\eta M T}$$

for all $\rho \in \mathbb{M}$ and $s \in [t-T,t]$.

Theorem 3.6. *Suppose that conditions* (A), (C2), (G), (Q1), (Q3) *hold. Suppose that there exists ζ such that $\|c\| \le \zeta < 1$. Then there exists a positive solution of* (1.1).

Proof. As in the proof of Theorem 3.3, we just need to show that condition (iii) of Theorem 2.2 is satisfied. Let $\varphi, \psi \in \mathbb{M}$. Then

$$\begin{aligned}
&\mathcal{A}\psi(t) + \mathcal{B}\varphi(t)\\
&= c(t)\varphi(g(t)) + \eta \int_{t-T}^{t} \Big[q\big(s,\psi(s),\psi(g(s))\big) - r(s)\psi(g(s))\Big] e^{-\int_s^t a(r)\,dr}\,ds\\
&\ge c^* L + \eta m T \frac{(1-c^*)L}{\eta m T} = L.
\end{aligned}$$

Likewise,

$$\mathcal{A}\psi(t) + \mathcal{B}\varphi(t) \le \zeta K + \eta M T \frac{(1-\zeta)K}{\eta M T} = K.$$

By Theorem 2.2, the operator H has a fixed point in $\mathbb{M}_2$. This fixed point is a positive solution of (1.1) and the proof is complete. □

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