EXISTENCE OF SOLUTIONS FOR P-KIRCHHOFF TYPE PROBLEMS WITH CRITICAL EXPONENT

Ahmed Hamdy, Mohammed Massar, Najib Tsouli

Abstract. We study the existence of solutions for the p-Kirchhoff type problem involving the critical Sobolev exponent,

\[-\left[ g \left( \int_{\Omega} |\nabla u|^p dx \right) \right] \Delta_p u = \lambda f(x, u) + |u|^{p^* - 2} u \quad \text{in } \Omega,
\]

\[ u = 0 \quad \text{on } \partial \Omega, \]

where \( \Omega \) is a bounded smooth domain of \( \mathbb{R}^N \), \( 1 < p < N \), \( p^* = \frac{Np}{N-p} \) is the critical Sobolev exponent, \( \lambda \) is a positive parameter, \( f \) and \( g \) are continuous functions. The main results of this paper establish, via the variational method, the concentration-compactness principle allows to prove that the Palais-Smale condition is satisfied below a certain level.

1. Introduction and main results

We are concerned with the existence of solutions for the p-Kirchhoff type problem

\[-\left[ g \left( \int_{\Omega} |\nabla u|^p dx \right) \right] \Delta_p u = \lambda f(x, u) + |u|^{p^* - 2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \tag{1.1} \]

where \( \Omega \) is a bounded smooth domain of \( \mathbb{R}^N \), \( 1 < p < N \), \( p^* = \frac{Np}{N-p} \) is the critical Sobolev exponent, and \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \), \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) are continuous functions that satisfy the following conditions:

(F1) \( f(x, t) = o(|t|^{p-1}) \) as \( t \to 0 \), uniformly for \( x \in \Omega \);

(F2) There exists \( q \in (p, p^*) \) such that

\[ \lim_{|t| \to +\infty} \frac{f(x, t)}{|t|^{p^* - 2} t} = 0, \quad \text{uniformly for } x \in \Omega. \]

(F3) There exists \( \theta \in (p/\sigma, p^*) \) such that \( 0 < \theta F(x, t) \leq tf(x, t) \) for all \( x \in \Omega \) and \( t \neq 0 \), where \( F(x, t) = \int_0^t f(x, s) ds \) and \( \sigma \) is given by (G2) below.

(G1) There exists \( \alpha_0 > 0 \) such that \( g(t) \geq \alpha_0 \) for all \( t \geq 0 \);

(G2) There exists \( \sigma > p/p^* \) such that \( G(t) \geq \sigma g(t)t \) for all \( t \geq 0 \), where \( G(t) = \int_0^t g(s) ds \);

2000 Mathematics Subject Classification. 35A15, 35B33, 35J62.
Key words and phrases. p-Kirchhoff; critical exponent; parameter; Lions principle.
©2011 Texas State University - San Marcos.
Much interest has grown on problems involving critical exponents, starting from the celebrated paper by Brezis and Nirenberg [5], where the case $p = 2$ is considered. We refer the reader to [1, 9, 10] and reference therein for the study of problems with critical exponent.

Problem (1.1) is a general version of a model presented by Kirchhoff [11]. More precisely, Kirchhoff introduced a model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where $\rho, \rho_0, h, E, L$ are constants, which extends the classical D’Alembert’s wave equation by considering the effects of the changes in the length of the strings during the vibrations. The problem

$$- \left( a + b \int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x, u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega$$

received much attention, mainly after the article by Lions [12]. Problems like (1.3) are also introduced as models for other physical phenomena as, for example, biological systems where $u$ describes a process which depends on the average of itself (for example, population density). See [3] and its references therein. For a more detailed reference on this subject we refer the interested reader to [4, 6, 7, 8, 14, 15].

Motivated by the ideas in [2], our approach for studying problem (1.1) is variational and uses minimax critical point theorems. The difficulty is due to the lack of compactness of the imbedding $W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and the Palais-Smale condition for the corresponding energy functional could not be checked directly. So the concentration-compact principle of Lions [13] is applied to deal with this difficulty.

The main result of this paper is the following theorem.

**Theorem 1.1.** Suppose that (G1)–(G2), (F1)–(F3) hold. Then, there exists $\lambda_*>0$, such that (1.1) has a nontrivial solution for all $\lambda \geq \lambda_*$.

2. **Preliminary results**

We consider the energy functional $I : W^{1,p}_0(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p} G(||u||^p) - \lambda \int_\Omega F(x, u) dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} dx,$$

where $W^{1,p}_0(\Omega)$ is the Sobolev space endowed with the norm $||u||^p = \int_\Omega |\nabla u|^p dx$. It is well known that a critical point of $I$ is a weak solution of problem (1.1).

To use variational methods, we give some results related to the Palais-Smale compactness condition. Recall that a sequence $(u_n)$ is a Palais-Smale sequence of $I$ at the level $c$, if $I(u_n) \to c$ and $I'(u_n) \to 0$.

In the sequel, we show that the functional $I$ has the mountain pass geometry. This purpose is proved in the next lemmas.

**Lemma 2.1.** Suppose that (F1), (F2), (G1) hold. Then, there exist $r, \rho > 0$ such that $\inf_{||u||=r} I(u) \geq \rho > 0$.

**Proof.** It follows from (F1) and (F2) that for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$.

$$F(x, t) \leq \frac{1}{p} \varepsilon |t|^p + C(\varepsilon) |t|^p$$

for all $t$. (2.2)
By (G1) and the Sobolev embedding, we have
\begin{equation}
I(u) \geq \frac{\alpha_0}{p} \|u\|^p - \lambda C_1 \varepsilon \|u\|^p - \lambda C_2 \varepsilon \|u\|^q - C_3 \|u\|^{p^*} \\
= \|u\|^p \left( \left( \frac{\alpha_0}{p} - \lambda C_1 \varepsilon \right) \|u\|^{p-1} - \lambda C_2 \varepsilon \|u\|^{q-1} - C_3 \|u\|^{p^*-1} \right). \tag{2.3}
\end{equation}
Taking \( \varepsilon = \frac{\alpha_0}{(2p \lambda C_1)} \) and setting
\[ \xi(t) = \frac{\alpha_0}{2p} t^{p-1} - \lambda C_2 t^{q-1} - C_3 t^{p^*-1}. \]
Since \( p < q < p^* \), we see that there exist \( r > 0 \) such that \( \max_{t \geq 0} \xi(t) = \xi(r) \). Then, by (2.3), there exists \( \rho > 0 \) such that \( I(u) \geq \rho \) for all \( \|u\| = r. \)

**Lemma 2.2.** Suppose that \( (G2), (F3) \) hold. Then for all \( \lambda > 0 \), there exists a nonnegative function \( e \in W^{1,p}_0(\Omega) \) independent of \( \lambda \), such that \( \|e\| > r \) and \( I(e) < 0 \).

**Proof.** Choose a nonnegative function \( \phi_0 \in C^\infty_0(\Omega) \) with \( \|\phi_0\| = 1 \). By integrating (G2), we obtain
\[ G(t) \leq \frac{G(t_0)}{t_0^{1/\sigma}} t^{1/\sigma} = C_0 t^{1/\sigma} \quad \text{for all } t \geq t_0 > 0. \tag{2.4}\]
By (F3), \( \int_\Omega F(x,t\phi_0)dx \geq 0. \) Hence
\[ I(t\phi_0) \leq \frac{C_0}{p} t^{p/\sigma} - \frac{t^{p^*}}{p^*} \int_\Omega \phi_0^{p^*} dx \quad \text{for all } t \geq t_0. \]
Since \( p/\sigma < p^* \), the lemma is proved by choosing \( e = t_\ast \phi_0 \) with \( t_\ast > 0 \) large enough.

In view of Lemmas 2.1 and 2.2, we may apply a version of the Mountain Pass theorem without Palais-Smale condition to obtain a sequence \( (u_n) \subset W^{1,p}_0(\Omega) \) such that
\[ I(u_n) \to c_\ast \quad \text{and} \quad I'(u_n) \to 0, \]
where
\[ c_\ast = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0, \tag{2.5}\]
with
\[ \Gamma = \{ \gamma \in C([0,1], W^{1,p}_0(\Omega)) : \gamma(0) = 0, I(\gamma(1)) < 0 \}. \]

Denoted by \( S_\ast \) the best positive constant of the Sobolev embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega) \) given by
\[ S_\ast = \inf \left\{ \int_\Omega \|\nabla u\|^p dx : u \in W^{1,p}_0(\Omega), \int_\Omega |u|^{p^*} dx = 1 \right\}. \tag{2.6}\]

**Lemma 2.3.** Suppose that \( (G1)-(G2), (F1)-(F3) \) hold. Then there exists \( \lambda_\ast > 0 \) such that \( c_\ast \in (0, (\frac{1}{\theta} - \frac{1}{p^*})(\alpha_0 S_\ast)^{\frac{1}{p^*}}) \) for all \( \lambda \geq \lambda_\ast \), where \( c_\ast \) is given by (2.5).

**Proof.** For \( e \) given by Lemma 2.2 we have \( \lim_{t \to +\infty} I(te) = -\infty \), then there exists \( t_\lambda > 0 \) such that \( I(t_\lambda e) = \max_{t \geq 0} I(te) \). Therefore,
\[ t_\lambda^{p^*-1} g(\|t_\lambda e\|^p) \|e\|^p = \lambda \int_\Omega f(x,t_\lambda e) e dx + t_\lambda^{p^*-1} \int_\Omega e^{p^*} dx; \]
thus
\[ g(||t \lambda e||^p)||t \lambda e||^p = \lambda t \lambda \int_\Omega f(x, t \lambda e) e dx + t \lambda \int_\Omega e^p dx. \] (2.7)

By (2.4), it follows that
\[ \frac{C_0}{\sigma} ||e||^{p/\sigma} p/\sigma \leq t \lambda \int_\Omega e^p dx, \quad \text{with } t_0 < t_\lambda. \]

Since \( p/\sigma < p^*, \) \((t_\lambda)\) is bounded. So, there exists a sequence \( \lambda_n \to +\infty \) and \( s_0 \geq 0 \) such that \( t \lambda_n \to s_0 \) as \( n \to \infty. \) Hence, there exists \( C > 0 \) such that
\[ g(||t \lambda_n e||^p)||t \lambda_n e||^p \leq C \quad \text{for all } n; \]
that is,
\[ \lambda_n t \lambda_n \int_\Omega f(x, t \lambda_n e) e dx + t \lambda_n \int_\Omega e^p dx \leq C \quad \text{for all } n. \]

If \( s_0 > 0, \) the above inequality implies that
\[ \lambda_n t \lambda_n \int_\Omega f(x, t \lambda_n e) e dx + t \lambda_n \int_\Omega e^p dx \to +\infty \leq C, \quad \text{as } n \to \infty, \]
which is impossible, and consequently \( s_0 = 0. \) Let \( \gamma_*(t) = t e. \) Clearly \( \gamma_* \in \Gamma, \) thus
\[ 0 < c_* \leq \max_{t \geq 0} I(\gamma_*(t)) = I(t \lambda e) \leq \frac{1}{p} G(||t \lambda e||^p). \]

Since \( t \lambda_n \to 0 \) and \( \frac{1}{\theta} - \frac{1}{p^*} > 0, \) for \( \lambda > 0 \) sufficiently large, we have
\[ \frac{1}{p} G(||t \lambda e||^p) < \left( \frac{1}{\theta} - \frac{1}{p^*} \right)(\alpha_0 S_\lambda)^{N/p}, \]
and hence
\[ 0 < c_* < \left( \frac{1}{\theta} - \frac{1}{p^*} \right)(\alpha_0 S_\lambda)^{N/p}. \]

This completes the proof. \( \Box \)

**Proof of Theorem 1.1.** From Lemmas 2.1, 2.2 and 2.3 there exists a sequence \((u_n) \subset W_0^{1,p}(\Omega)\) such that
\[ I(u_n) \to c_* \quad \text{and} \quad I'(u_n) \to 0, \] (2.8)
with \( c_* \in (0, \frac{1}{\theta} - \frac{1}{p^*}(\alpha_0 S_\lambda)^{N/p}) \) for \( \lambda \geq \lambda_* \). Then, there exists \( C > 0 \) such that \( |I(u_n)| \leq C, \) and by \((F3)\) for \( n \) large enough, it follows from \((G1)\) and \((G2)\) that
\[ C + ||u_n|| \geq I(u_n) - \frac{1}{\theta} (I'(u_n), u_n) \]
\[ \geq \frac{1}{p} G(||u||^p) - \frac{1}{p} g(||u||^p)||u_n||^p \]
\[ \geq \left( \frac{\sigma}{p} - \frac{1}{\theta} \right) \alpha_0 ||u_n||^p. \] (2.9)

Since \( \theta > p/\sigma, \) \((u_n)\) is bounded. Hence, up to a subsequence, we may assume that
\[ u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \]
\[ u_n \to u \quad \text{a.e. in } \Omega, \]
\[ u_n \to u \quad \text{in } L^s(\Omega), \ 1 \leq s < p^*, \] (2.10)
\[ |\nabla u_n|^p \rightharpoonup \mu \quad \text{(weak*-sense of measures)}, \]
\[ |u_n|^p \rightharpoonup \nu \quad \text{(weak*-sense of measures)}, \]
where \( \mu \) and \( \nu \) are nonnegative bounded measures on \( \overline{\Omega} \). Then, by concentration-compactness principle due to Lions [13], there exists some at most countable index set \( J \) such that
\[
\nu = |u|^p + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \nu_j > 0,
\]
\[
\mu \geq |\nabla u|^p + \sum_{j \in J} \mu_j \delta_{x_j}, \quad \mu_j > 0,
\]
\[
S_n \nu_j^{p^*/p^*} \leq \mu_j,
\]
where \( \delta_{x_j} \) is the Dirac measure mass at \( x_j \in \overline{\Omega} \).

Let \( \psi(x) \in C_0^\infty \) such that \( 0 \leq \psi \leq 1 \),
\[
\psi(x) = \begin{cases} 
1 & \text{if } |x| < 1 \\
0 & \text{if } |x| \geq 2
\end{cases}
\]
and \( |\nabla \psi|_\infty \leq 2 \).

For \( \varepsilon > 0 \) and \( j \in J \), denote \( \psi_j^\varepsilon(x) = \psi((x - x_j)/\varepsilon) \). Since \( I'(u_n) \to 0 \) and \( (\psi_j^\varepsilon u_n) \) is bounded, \( \langle I'(u_n), \psi_j^\varepsilon u_n \rangle \to 0 \) as \( n \to \infty \); that is,
\[
g(||u_n||^p) \int_\Omega |\nabla u_n|^p \psi_j^\varepsilon dx
= -g(||u_n||^p) \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_j^\varepsilon dx + \lambda \int_\Omega f(x,u_n) u_n \psi_j^\varepsilon dx + \int_\Omega |u_n|^{p^*} \psi_j^\varepsilon dx + o_n(1) .
\]
By (2.10) and Vitali’s theorem, we see that
\[
\lim_{n \to \infty} \int_\Omega |u_n \nabla \psi_j^\varepsilon|^p dx = \int_\Omega |u \nabla \psi_j^\varepsilon|^p dx
\]
Hence, by Hölder’s inequality we obtain
\[
\limsup_{n \to \infty} \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_j^\varepsilon dx
\leq \limsup_{n \to \infty} \left( \int_\Omega |\nabla u_n|^p dx \right)^{(p-1)/p} \left( \int_\Omega |u_n \nabla \psi_j^\varepsilon|^p dx \right)^{1/p}
\leq C_1 \left( \int_{B(x_j,2\varepsilon)} |u|^p |\nabla \psi_j^\varepsilon|^p dx \right)^{1/p}
\leq C_1 \left( \int_{B(x_j,2\varepsilon)} |\nabla \psi_j^\varepsilon|^N dx \right)^{1/N} \left( \int_{B(x_j,2\varepsilon)} |u|^{p^*} dx \right)^{1/p^*}
\leq C_2 \left( \int_{B(x_j,2\varepsilon)} |u|^p dx \right)^{1/p^*} \to 0 \quad \text{as } \varepsilon \to 0 .
\]

On the other hand, from (2.10) we have
\[
f(x,u_n)u_n \to f(x,u)u \quad \text{a.e. in } \Omega ,
\]
and \( u_n \to u \) strongly in \( L^p(\Omega) \) and in \( L^q(\Omega) \). By (F1)–(F3), for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[
|f(x,t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{q-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R} ;
\]
thus
\[
|f(x, u_n)u_n| \leq \varepsilon|u_n|^p + C\varepsilon|u_n|^q.
\]
This is what we need to apply Vitali’s theorem, which yields
\[
\lim_{n \to \infty} \int \Omega f(x, u_n)u_n dx = \int \Omega f(x, u) u dx.
\]
Since \(\psi_j^\varepsilon\) has compact support, letting \(n \to \infty\) in (2.13) we deduce from (2.10) and (2.14) that
\[
\alpha_0 \int \Omega \psi_j^\varepsilon \, d\mu \leq C (\int_{B(x_j, 2\varepsilon)} |u|^p \, dx)^{1/p^*} + \lambda \int_{B(x_j, 2\varepsilon)} f(x, u) u dx + \int \Omega \psi_j^\varepsilon \, d\nu.
\]
Letting \(\varepsilon \to 0\), we obtain \(\alpha_0 \mu_j \leq \nu_j\). Therefore,
\[
(\alpha_0 S_j)^{N/p} \leq \nu_j. \tag{2.16}
\]
We will prove that this inequality is not possible. Let us assume that \((\alpha_0 S_j)^{N/p} \leq \nu_{j_0}\) for some \(j_0 \in J\). From (G2) we see that
\[
\frac{1}{p} G(||u_n||^p) - \frac{1}{p^*} g(||u_n||^p) ||u_n||^p \geq 0 \quad \text{for all } n.
\]
Since
\[
c_* = I(u_n) - \frac{1}{\theta} \langle I'(u_n), u_n \rangle + o_n(1),
\]
it follows that
\[
c_* \geq \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \int \Omega |u_n|^{p^*} \, dx + o_n(1)
\]
\[
\geq \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \int \Omega \psi_j^0 |u_n|^{p^*} \, dx + o_n(1)
\]
Letting \(n \to \infty\), we obtain
\[
c_* \geq \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \sum_{j \in J} \psi_j^0(x_j) \nu_j \geq \left(\frac{1}{\theta} - \frac{1}{p^*}\right) (\alpha_0 S_j)^{N/p}.
\]
This contradicts Lemma 2.3. Then \(J = \emptyset\), and hence \(u_n \to u\) in \(L^{p^*}(\Omega)\). By (2.15) we have
\[
\int \Omega |f(x, u_n)(u_n - u)| \, dx \leq \int \Omega (\varepsilon|u_n|^{p-1} + C\varepsilon|u_n|^{q-1}) |u_n - u| \, dx
\]
\[
\leq \varepsilon \left(\int \Omega |u_n|^p \, dx\right)^{p-1/p} \left(\int \Omega |u_n - u|^p \, dx\right)^{1/p}
\]
\[
+ C\varepsilon \left(\int \Omega |u_n|^q \, dx\right)^{(q-1)/q} \left(\int \Omega |u_n - u|^q \, dx\right)^{1/q}.
\]
Then, using again (2.10), we obtain
\[
\lim_{n \to \infty} \int \Omega f(x, u_n)(u_n - u) \, dx = 0. \tag{2.17}
\]
Since \(u_n \to u\) in \(L^{p^*}(\Omega)\), we see that
\[
\lim_{n \to \infty} \int \Omega |u_n|^{p^*-2} u_n (u_n - u) \, dx = 0. \tag{2.18}
\]
From \( \langle I'(u_n), u_n - u \rangle = o_n(1) \), we deduce that
\[
\langle I'(u_n), u_n - u \rangle = g(\|u_n\|^p) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) dx
\]
\[
- \lambda \int_\Omega f(x, u_n)(u_n - u) dx - \int_\Omega |u_n|^{p^*-2} u_n(u_n - u) dx = o_n(1)
\]
This, (2.17) and (2.18) imply
\[
\lim_{n \to \infty} g(\|u_n\|^p) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) dx = 0.
\]
Since \( u_n \) is bounded and \( g \) is continuous, up to subsequence, there is \( t_0 \geq 0 \) such that
\[
g(\|u_n\|^p) \to g(t_0^p) \geq \alpha_0, \quad \text{as } n \to \infty,
\]
and so
\[
\lim_{n \to \infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) dx = 0.
\]
Thus by the \((S_+)\) property, \( u_n \to u \) strongly in \( W_0^{1,p}(\Omega) \), and hence \( I'(u) = 0 \). The proof is complete. □

3. A SPECIAL CASE

We consider the problem
\[
- \left( \alpha + \beta \int_\Omega |\nabla u|^p dx \right) \Delta_p u = \lambda f(x, u) + |u|^{p^*-2} u \quad \text{in } \Omega
\]
\[
u = 0 \quad \text{on } \partial \Omega,
\]
where \( \Omega \) is a bounded smooth domain of \( \mathbb{R}^N \), \( 1 < p < N < 2p \), \( \alpha \) and \( \beta \) are positive constants.

Set \( g(t) = \alpha + \beta t \). Then, \( g(t) \geq \alpha \) and
\[
G(t) = \int_0^1 g(s) ds = \alpha t + \frac{1}{2} \beta t^2 \geq \frac{1}{2} (\alpha + \beta) t = \sigma g(t) t
\]
where \( \sigma = 1/2 \). Hence the conditions (G1) and (G2) are satisfied.

For this case, a typical example of a function satisfying the conditions (F1)–(F3) is given by
\[
f(x, t) = \sum_{i=1}^k a_i(x)|t|^{q_i-2} t,
\]
where \( k \geq 1, 2p < q_i < p^* \) and \( a_i(x) \in C(\overline{\Omega}) \). In view of Theorem 1.1, we have the following corollary.

**Corollary 3.1.** Suppose that (F1)–(F3) hold. Then, there exists \( \lambda_* > 0 \), such that problem (3.1) has a nontrivial solution for all \( \lambda \geq \lambda_* \).
References


Ahmed Hamydy
University Mohamed I, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: a_hamydy@yahoo.fr

Mohammed Massar
University Mohamed I, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: massarmed@hotmail.com

Najib Tsouli
University Mohamed I, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: tsouli@hotmail.com