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COMPARISON AND EXISTENCE THEOREMS FOR BACKWARDS STOCHASTIC DE'S WITH DISCONTINUOUS GENERATORS

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Abstract. An existence result is proved for backwards stochastic differential equations (BSDEs) with a generator $f(t,x,z)$ which is possibly discontinuous in the x variable. For this comparison results are first established for BSDEs with the generator satisfying a generalized Lipschitz condition in its x variable.

1. Introduction

Let W_t be a standard one-dimensional Wiener process defined on the complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $\{\mathcal{F}_t^W\}$ be the natural filtration generated by the Wiener process and let $\{\mathcal{F}_t\}$ be the augmentation under $\mathbb{P}$ of this natural filtration. In addition, let $\mathcal{P}$ denote the σ-algebra of $\mathcal{F}_t$ progressively measurable subsets of $[0,T]\times\Omega$ and let $H^p(\mathbb{R})$ be the space of $\mathcal{P}$-measurable $X:[0,T]\times\Omega\to\mathbb{R}$ with $\|X\|^p := \mathbb{E}\int_0^T |X_s|^p\,ds < \infty$.

Suppose that the mapping $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is $\mathcal{B}\otimes\mathcal{B}\otimes\mathcal{B}$-measurable and consider the scalar backward stochastic differential equation (BSDE) with

$$x_t = \xi + \int_t^T f(s,x_s,z_s)\,ds - \int_t^T z_s\,dW_s. \tag{1.1}$$

The classical existence theorem for BSDE states that if the generator is globally Lipschitz in both variables then there exists a strong solution. Lepeltier and San Martin [7] prove an existence result for the case where the generator is only continuous in both variables. To do that they use the classical comparison theorem and monotonicity arguments. The comparison theorem proved in this note allows the conditions on the generator to be further relaxed. In particular, an existence theorem is established in the case where $x\to f(t,x,z)$ is left continuous. A similar result appears in the paper by Jia [3], however our assumptions here are more general because we use a comparison theorem which holds for generators having super-linear growth in the x variable.

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It is well known that BSDEs are related to PDEs, see for example [8]. Benth et al [2] showed that the nonlinearity in the semilinear Black and Scholes equation depends discontinuously on the American option value. Moreover, this discontinuity then appears in the generator of the associated BSDE, see also Karoui et al [4].

2. Comparison theorems

Consider the following scalar BSDEs:

$$y_t = \xi_1 + \int_t^T f(s, y_s, z_s^1)\, ds - \int_t^T z_s^1\, dW_s, \tag{2.1}$$

$$x_t = \xi_2 + \int_t^T g(s, x_s, z_s^2)\, ds - \int_t^T z_s^2\, dW_s, \tag{2.2}$$

and suppose that each admits a unique solution, which is denoted by (y_t, z_t^1) and (x_t, z_t^2), respectively. (Note that y_t and x_t have continuous modifications). The generator g satisfies the assumption

(A1) There exists a constant K such that

$$|g(t, x_1, z_1) - g(t, x_2, z_2)|^2 \le \kappa(|x_1 - x_2|^2) + K|z_1 - z_2|^2, \quad \text{a.s.},$$

for all t, x_1, x_2, z_1, z_2, where $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave increasing function with $\kappa(0) = 0$ and $\kappa(u) > 0$ for $u > 0$ such that

$$\int_{0^+} \frac{du}{\kappa(u)} = \infty.$$

The following comparison theorem will be used later to prove the existence of a solution to (1.1) with the generator f being discontinuous in its second variable.

Theorem 2.1. *Let (y_s, z_s^1) and (x_s, z_s^2) be the unique solutions of* (2.1) *and* (2.2)*, respectively. Suppose* (A1) *holds and, in addition, that*

$$f(t, y_s, z_s^1) \le g(t, y_s, z_s^1) \quad \text{for all } t \in [0, T], \text{ a.s.}.$$

Finally, suppose that $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ satisfy $\xi_1 \le \xi_2$. Then, $y_t \le x_t$, a.s., for all $t \in [0, T]$.

Proof. Consider the auxiliary problem

$$h_t = \xi_2 + \int_t^T g(s, \max\{h_s, y_s\}, z_s^h) ds - \int_t^T z_s^h dW_s. \tag{2.3}$$

The generator of this BSDE is the random function

$$\hat{g}(s, \omega, x, z) := g(s, \max\{x, y_s(\omega)\}, z).$$

Since the function $x \mapsto \max\{x, y_s(\omega)\}$ satisfies the Lipschitz condition with constant one and a growth condition $|\max\{x, y_s(\omega)\}| \le |x| + |y_s(\omega)|$, it follows that $\hat{g}$ is Lipschitz in x and has linear growth. Hence by [8, Theorem 7.4.1] this auxiliary BSDE (2.3) has a unique solution.

We want to compare this solution h_t with y_t. First, define

$$H_t := \int_t^T [f(s, y_s, z_s^1) - g(s, \max\{h_s, y_s\}, z_s^1)]\, ds + \int_t^T b_s \widehat{Z}_s\, ds - \int_t^T \widehat{Z}\, dW_s,$$

where

$$b_s := \frac{g(s, \max\{h_s, y_s\}, z_s^1) - g(s, \max\{h_s, y_s\}, z_s^2)}{z_s^1 - z_s^2}, \quad \widehat{Z}_s := z_s^1 - z_s^2.$$

Note that b_s is uniformly bounded by Assumption (A1). Then write

$$M(t) = \exp\Big(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t |b_s|^2 ds\Big), \quad t \in [0,T],$$

and define a new probability measure $\hat{\mathbb{P}}$ by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = M(T).$$

By Girsanov's theorem, $\widehat{W}_t := W_t - \int_0^t b_s\, ds$ is a $\widehat{\mathbb{P}}$-Wiener process. Hence, under $\widehat{\mathbb{P}}$, the difference $y_t - h_t$ satisfies the equation

$$y_t - h_t = \xi^1 - \xi^2 + \int_t^T \left[f(s, y_s, z_s^1) - g(s, \max\{h_s, y_s\}, z_s^1)\right] ds - \int_t^T \widehat{Z}_s\, d\widehat{W}_s. \tag{2.4}$$

It will now be shown that $y_t \le h_t$, a.s., for all $t \in [0,T]$. Suppose that this is not true. Then, there exists some t^* such that $y_{t^*} > h_{t^*}$ on an event A with $\widehat{\mathbb{P}}(A) > 0$. Note, that A is $\mathcal{F}_{t*}$ measurable, so the indicator function $\mathbb{I}_A$ of the event A is $\mathcal{F}_{t*}$ measurable. Then, by [8, Lemma 1.5.10],

$$\mathbb{I}_A \int_{t^*}^{\tau} \widehat{Z}_s\, d\hat{W}_s = \int_{t^*}^{\tau} \mathbb{I}_A \widehat{Z}_s\, d\widehat{W}_s.$$

Define the stopping time,

$$\tau := \inf\{t \in [t^*, T] : y_t \le h_t\}.$$

Since $y_\tau = h_\tau$ by continuity of y_t and h_t, it follows from (2.4) that

$$y_{t^*} - h_{t^*} = \int_{t^*}^{\tau} [f(s, y_s, z_s^1) - g(s, \max\{h_s, y_s\}, z_s^1)]\, ds - \int_{t^*}^{\tau} \widehat{Z}_s\, d\widehat{W}_s.$$

Multiplying this equation by $\mathbb{I}_A$ gives

$$\begin{aligned} &\mathbb{I}_A(y_{t^*} - h_{t^*}) \\ &= \int_{t^*}^{\tau} \mathbb{I}_A \left[f(s, y_s, z_s^1) - g(s, \max\{h_s, y_s\}, z_s^1)\right] ds - \int_{t^*}^{\tau} \mathbb{I}_A \widehat{Z}_s\, d\widehat{W}_s, \end{aligned} \tag{2.5}$$

Now $y_t \ge h_t$ on the stochastic interval $[t^*, \tau]$, so $\max\{h_t, y_t\} = y_t$. Finally, taking the expectation on both sides of (2.5) gives

$$\mathbb{E}(\mathbb{I}_A(y_{t^*} - x_{t^*})) \ge 0, \quad \mathbb{E}\Big(\int_{t^*}^{\tau} \mathbb{I}_A \widehat{Z}_s\, d\widehat{W}_s\Big) = 0,$$
$$\mathbb{E}\Big(\int_{t^*}^{\tau} \mathbb{I}_A [f(s, y_s, z_s^1) - g(s, y_s, z_s^1)]\, ds\Big) \le 0.$$

It follows that $\mathbb{E}\big(\mathbb{I}_A(y_{t^*} - h_{t^*})\big) = 0$ and hence that $\hat{\mathbb{P}}(A) = 0$, which is a contradiction. Thus, $h_t \ge y_t$ for all t, a.s. This means that $h_t = x_t$, where x_t is the unique solution of the BSDE (2.2). □

Remark 2.2. Theorem 2.1 is also valid for a random generator f; i.e., for a $\mathcal{P} \otimes \mathcal{B} \otimes \mathcal{B}$-measurable $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with values $f(t, \omega, x, z)$.

A similar argument gives a comparison theorem that applies to backward stochastic differential inequalities. Backward stochastic differential inequalities are closely related to self-financing super-strategies in mathematical finance, see for example [4, Definition 1.2].

Consider the following two backward stochastic differential inequalities:

$$y_t \le \xi_1 + \int_t^T f(s, y_s, z_s^1) ds - \int_t^T z_s^1 dW_s, \tag{2.6}$$

$$x_t \ge \xi_2 + \int_t^T g(s, x_s, z_s^2) ds - \int_t^T z_s^2 dW_s. \tag{2.7}$$

Theorem 2.3. *Suppose that $x \mapsto f(t,x,z)$ is non-increasing and that $x \mapsto g(t,x,z)$ is non-decreasing. In addition, suppose that $\xi_1 \le \xi_2$ with $\xi_1,\ \xi_2 \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and that*

$$f(t,x,z) \le g(t,x,z) \quad \text{for all } t \in [0,T],\ (x,z) \in \mathbb{R}^2.$$

Then $y_t \le x_t$, a.s., for all $t \in [0,T]$.

Proof. Define H_t by

$$H_t := \int_t^T [f(s, y_s, z_s^1) - g(s, x_s, z_s^1)]\, ds + \int_t^T b_s \widehat{Z}_s\, ds - \int_t^T \widehat{Z}\, dW_s,$$

where

$$b_s := \frac{g(s, x_s, z_s^1) - g(s, x_s, z_s^2)}{z_s^1 - z_s^2}, \quad \widehat{Z}_s := z_s^1 - z_s^2.$$

As before, using Girsanov's theorem, H_t can be rewritten as

$$H_t = \int_t^T \left[f(s, y_s, z_s^1) - g(s, x_s, z_s^1)\right]\, ds - \int_t^T \widehat{Z}_s\, d\widehat{W}_s \tag{2.8}$$

with respect to the new probability measure $\widehat{\mathbb{P}}$ and corresponding Wiener process $\widehat{W}_t$. Using the assumptions that $x \to f(t,x,z)$ is non-increasing and that $x \to g(t,x,z)$ is non-decreasing, it follows that

$$H_t \le \int_t^T \left[f(s, \min\{x_s, y_s\}, z_s^1) - g(s, \min\{x_s, y_s\}, z_s^1)\right]\, ds - \int_t^T \widehat{Z}_s\, d\widehat{W}_s.$$

Suppose now that there exists a t^* such that $H_{t^*} > 0$ on an event A with $\widehat{\mathbb{P}}(A) > 0$. Then, multiplying the above inequality by $\mathbb{I}_A$ and taking the expectation leads to a contradiction, since $\mathbb{E}(\int_{t^*}^T \mathbb{I}_A \widehat{Z}_s\, \widehat{W}_s) = 0$. Hence, $H_t \le 0$ for all $t \in [0,T]$ and it follows that

$$y_t - x_t \le \xi_1 - \xi_2 + H_t \le 0$$

for all $t \in [0,T]$, a.s. $\square$

3. An existence theorem

The first comparison theorem, Theorem 2.1, will now be applied to scalar BSDEs for which the generator f is not necessarily continuous in x. In particular, f is now assumed to satisfy the following assumptions:

(A2) The generator $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$-measurable and satisfies

- (i) The mapping f has the form $f(t,x,z) = f_1(t,x,z) + f_2(t,x,z)$, where $f_1(t,x,z)$ is continuous in all variables and satisfies (A1), while f_2 is continuous in t and z, is an increasing function of x, and satisfies a linear growth condition for both variables; i.e.,

$$|f_2(t,x,z)| \le C(|x| + |z| + 1),$$

and is possibly discontinuous in x, but is right or left continuous.

(ii) There exist a K such that

$$|f_2(t,x,z_1) - f_2(t,x,z_2)| \le K|z_1 - z_2|$$

for all $t, x, x_1, x_2, z, z_1, z_2$.

(A3) There exists functions $g_1(t,x,z)$ and $g_2(t,x,z)$ satisfying (A1) such that

$$g_1(t,x,z) \le f(t,x,z) \le g_2(t,x,z) \quad \text{for all } t,x,z.$$

(A4) $\mathbb{E}(|\xi|^2) < \infty$.

Theorem 3.1. *Suppose that Assumptions* (A2), (A3), (A4) *hold. Then* (1.1) *has at least one solution.*

Proof. The solution will be obtained as the limit of an increasing or a decreasing sequence, which is constructed as follows.

Firstly, note that the BSDEs

$$\begin{aligned} L_t &= \xi + \int_t^T g_1(s, L_s, z_s^L)\, ds - \int_t^T z_s^L\, dW_s, \\ U_t &= \xi + \int_t^T g_2(s, U_s, z_s^U)\, ds - \int_t^T z_s^U\, dW_s \end{aligned}$$

admit unique solutions. Then consider a sequence (y_t^n, z_t^n) of stochastic processes obtained as the solutions of the BSDEs

$$y_t^n = \xi + \int_t^T [f_1(t, y^n, z^n) + f_2(t, y^{n-1}, z^n)]\, ds - \int_t^T z^n\, dW_s,$$

with $y_t^0 = L_t$. These solutions exist by [8, Theorem 7.4.1]. In particular, note that the Assumption (A4) on the final value ξ ensures that $y_t^n \in H^2(\mathbb{R})$.

We will now prove that $y_t^1 \ge L_t$. For this we have to compare a BSDE with generator $g_1(t,x,z)$ and a BSDE with random generator

$$\hat{f}(t,\omega,x,z) := f_1(t,x,z) + f_2(t, L_t(\omega), z).$$

It is clear that $\hat{f}$ satisfies the conditions of the comparison theorem, Theorem 2.1, and that $g_1(t, L_t, z_t^L) \le \hat{f}(t, L_t, z_t^L)$. Hence $L_t \le y_t^1$. It follows by the same argument that $y_t^n \ge y_t^{n-1}$, a.s., for each $n \in \mathbb{N}$.

It also follows similarly that $y_t^1 \le U_t$ and hence that $y_t^n \le U_t$ for each $n \in \mathbb{N}$.

Now it is easy to show that $y_t^n \to y_t^*$ in $H^2(\mathbb{R})$, where $y_t^n \le y_t^* \le y_t$, using the Lebesgue Dominated Convergence Theorem and the fact that y^n is an increasing and bounded sequence. To show that $z^n \to z$ in $H^2(\mathbb{R})$ we apply the Itô formula to $|y_n - y_m|^2$ and obtain

$$\begin{aligned} &\mathbb{E}|y_n - y_m|^2 + \mathbb{E}\int_t^T |z_n - z_m|^2\, ds \\ &= 2\mathbb{E}\int_t^T (y_n - y_m)\big(f_1(s, y_n, z_n) - f_1(s, y_m, z_m)\big)\, ds \\ &\quad + 2\mathbb{E}\int_t^T (y_n - y_m)\big(f_2(s, y_n, z_n) - f_2(s, y_m, z_m)\big)\, ds. \end{aligned}$$

Using the inequality $2|yz| \le \frac{y^2}{\varepsilon} + \varepsilon z^2$, the first term on right-hand side can be estimated by

$$
\begin{aligned}
&2\mathbb{E}\int_t^T (y_n - y_m)\Big(f_1(s, y_n, z_n) - f_1(s, y_m, z_m)\Big)\,ds \\
&\le \varepsilon\mathbb{E}\int_t^T |y_n - y_m|^2 ds + \frac{1}{\varepsilon}\mathbb{E}\int_t^T |f_1(s, y_n, z_n) - f_1(s, y_m, z_m)|^2\,ds \\
&\le \varepsilon\mathbb{E}\int_t^T |y_n - y_m|^2\,ds + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \kappa(|y_n - y_m|^2)\,ds + \frac{1}{\varepsilon}\mathbb{E}\int_t^T |z_n - z_m|^2\,ds.
\end{aligned}
$$

The same arguments can be used to estimate the second term on right-hand side of (3.1). Finally, choosing a suitable ε, it follows that $\{z^n\}$ is a Cauchy sequence in $H^2(\mathbb{R})$. Hence, $(y_n, z_n) \to (y^*, z^*)$, which is a solution of (1.1). $\square$

Remark 3.2. The above results can be applied for BSDEs with a generator of the form

$$
f(t, x, z) = f_1(t, x, z) + H(x - 1)x + z;
$$

where $H(x)$ is the Heaviside function and $f_1(t, x, z)$ satisfies assumption (**A1**). Here one can take

$$
g_1(t, x, z) = f_1(t, x, z) + z, \quad g_2(t, x, z) = f_1(t, x, z) + H(x)x + z.
$$

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