GORDON TYPE THEOREM FOR MEASURE PERTURBATION

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Abstract. Generalizing the concept of Gordon potentials to measures we prove a version of Gordon’s theorem for measures as potentials and show absence of eigenvalues for these one-dimensional Schrödinger operators.

1. Introduction

According to [2], the one-dimensional Schrödinger operator \( H = -\Delta + V \) has no eigenvalues if the potential \( V \in L_{1,\text{loc}}(\mathbb{R}) \) can be approximated by periodic potentials (in a suitable sense). The aim of this paper is to generalize this result to measures \( \mu \) instead of potential functions \( V \); i.e., to more singular potentials.

Although all statements remain valid for complex measures we only focus on real (but signed) measures \( \mu \), since we are interested in self-adjoint operators.

In the remaining part of this section we explain the situation and define the operator in question. We also describe the class of measures we are concerned with. Section 2 provides all the tools we need to prove the main theorem: \( H = -\Delta + \mu \) has no eigenvalues for suitable \( \mu \). In section 3 we show some examples for Schrödinger operators with measures as potentials.

We consider a Schrödinger operator of the form \( H = -\Delta + \mu \) on \( L_2(\mathbb{R}) \). Here, \( \mu = \mu_+ - \mu_- \) is a signed Borel measure on \( \mathbb{R} \) with locally finite total variation \( |\mu| \).

We define \( H \) via form methods. To this end, we need to establish form boundedness of \( \mu_- \). Therefore, we restrict the class of measures we want to consider.

Definition 1.1. A signed Borel measure \( \mu \) on \( \mathbb{R} \) is called uniformly locally bounded, if

\[
\|\mu\|_{\text{loc}} := \sup_{x \in \mathbb{R}} |\mu|([x, x+1]) < \infty.
\]

We call \( \mu \) a Gordon measure if \( \mu \) is uniformly locally bounded and if there exists a sequence \( (\mu^m)_{m \in \mathbb{N}} \) of uniformly locally bounded period \( p \)-periodic Borel measures such that \( p_m \to \infty \) and for all \( C \in \mathbb{R} \) we have

\[
\lim_{m \to \infty} e^{Cp_m} |\mu - \mu^m|([-p_m, 2p_m]) = 0;
\]

i.e., \( (\mu^m) \) approximates \( \mu \) on increasing intervals. Here, a Borel measure is \( p \)-periodic, if \( \mu = \mu(\cdot + p) \).

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Clearly, every generalized Gordon potential $V \in L_{1,\text{loc}}$ as defined in [2] induces a Gordon measure $\mu = V\lambda$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}$. Therefore, also every Gordon potential (see the original work [4]) induces a Gordon measure.

**Lemma 1.2.** Let $\mu$ be a uniformly locally bounded measure. Then $|\mu|$ is $-\Delta$-form bounded, and for all $0 < c < 1$ there is $\gamma \geq 0$ such that

$$\int_\mathbb{R} |u|^2 d|\mu| \leq c \|u'\|^2_2 + \gamma \|u\|^2_2 \quad (u \in W^1_2(\mathbb{R})).$$

**Proof.** For $\delta \in (0, 1)$ and $n \in \mathbb{Z}$ we have

$$\|u\|^2_{\infty, [n\delta, (n+1)\delta]} \leq 4\delta \|u'\|^2_{L^2_2(n\delta, (n+1)\delta)} + \frac{4}{\delta} \|u\|^2_{L^2_2(n\delta, (n+1)\delta)}$$

by Sobolev’s inequality.

Now, we estimate

$$\int_\mathbb{R} |u|^2 d|\mu| = \sum_{n \in \mathbb{Z}} \int_{n\delta}^{(n+1)\delta} |u|^2 d\mu$$

$$\leq \sum_{n \in \mathbb{Z}} \|u\|^2_{\infty, [n\delta, (n+1)\delta]} \|\mu\|_{\text{loc}}$$

$$\leq \|\mu\|_{\text{loc}} \sum_{n \in \mathbb{Z}} \left(4\delta \|u'\|^2_{L^2_2(n\delta, (n+1)\delta)} + \frac{4}{\delta} \|u\|^2_{L^2_2(n\delta, (n+1)\delta)}\right)$$

$$= 4\delta \|\mu\|_{\text{loc}} \|u'\|^2_2 + \frac{4}{\delta} \|\mu\|_{\text{loc}} \|u\|^2_2.$$

\[\square\]

Let $\mu$ be a Gordon measure and define

$$D(\tau) := W^1_2(\mathbb{R}), \quad \tau(u, v) := \int u'v' + \int u\overline{v} \, d\mu.$$

Then $\tau$ is a closed symmetric semibounded form. Let $H$ be the associated self-adjoint operator.

In [1], Ben Amor and Remling introduced a direct approach for defining the Schrödinger operator $H = -\Delta + \mu$. Since we will use some of their results we sum up the main ideas: For $u \in W^1_{1,\text{loc}}(\mathbb{R})$ define $Au \in L_{1,\text{loc}}(\mathbb{R})$ by

$$Au(x) := u'(x) - \int_0^x u(t) \, d\mu(t),$$

where

$$\int_0^x u(t) \, d\mu(t) := \begin{cases} \int_{[0,x]} u(t) \, d\mu(t) & \text{if } x \geq 0, \\
- \int_{(x,0]} u(t) \, d\mu(t) & \text{if } x < 0. \end{cases}$$

Clearly, $Au$ is only defined as an $L^1_{1,\text{loc}}(\mathbb{R})$-element. We define the operator $T$ in $L^2_2(\mathbb{R})$ by

$$D(T) := \{u \in L^2_2(\mathbb{R}) \colon u, Au \in W^1_{1,\text{loc}}(\mathbb{R}), (Au)' \in L^1_2(\mathbb{R})\}, \quad Tu := -(Au)'.$$

**Lemma 1.3.** $H \subseteq T$. 

...
Proof. Let $u \in D(H)$. Then $u \in W^1_2(\mathbb{R}) \subseteq W^1_{1,\text{loc}}(\mathbb{R})$ and $Au \in L_{1,\text{loc}}(\mathbb{R})$. Let $\varphi \in C_c^\infty(\mathbb{R}) \subseteq D(\tau)$. Using Fubini’s Theorem, we compute
\[
\int_{\mathbb{R}} (Au)(x) \varphi'(x) \, dx = \int_{\mathbb{R}} \left( u'(x) - \int_{0}^{x} u(t) \, d\mu(t) \right) \varphi'(x) \, dx
\]
\[
= \int_{\mathbb{R}} u'(x) \varphi'(x) \, dx - \int_{\mathbb{R}} \int_{0}^{x} u(t) \, d\mu(t) \varphi'(x) \, dx
\]
\[
= \int_{\mathbb{R}} u'(x) \varphi'(x) \, dx + \int_{-\infty}^{0} \int_{-\infty}^{x} \varphi'(x) \, dx u(t) \, d\mu(t) - \int_{0}^{\infty} \int_{[t,\infty)} \varphi'(x) \, dx u(t) \, d\mu(t)
\]
\[
= \int_{\mathbb{R}} u'(x) \varphi'(x) \, dx + \int_{-\infty}^{0} u(t) \varphi(t) \, d\mu(t) + \int_{0}^{\infty} u(t) \varphi(t) \, d\mu(t)
\]
\[
= \int_{\mathbb{R}} u' \varphi' + \int_{\mathbb{R}} u(x) \varphi(x) \, dx
\]
\[
= \tau(u, \varphi) = (Hu \mid \varphi) = \int_{\mathbb{R}} Hu(x) \varphi(x) \, dx.
\]
Hence, $(Au)' = -Hu \in L_2(\mathbb{R})$. We conclude that $Au \in W^1_{1,\text{loc}}(\mathbb{R})$ and therefore $u \in D(T)$, $Tu = -(Au)' = Hu$. \hfill \Box

Remark 1.4. For $u \in D(H)$ we obtain
\[
u' (x) = Au (x) + \int_{0}^{x} u(t) \, d\mu(t)
\]
for a.a. $x \in \mathbb{R}$. Since $Au \in W^1_{1,\text{loc}}(\mathbb{R})$ and $x \mapsto \int_{0}^{x} u(t) \, d\mu(t)$ is continuous at all $x \in \mathbb{R}$ with $\mu(\{x\}) = 0$, $u'$ is continuous at $x$ for all $x \in \mathbb{R} \setminus \text{spt} \mu_p$, where $\mu_p$ is the point measure part of $\mu$.

2. Absence of eigenvalues

We show that $H$ has no eigenvalues. The proof is based on two observations. The first one is a stability result and will be achieved in Lemma 2.6, the second one is an estimate of the solution for periodic measure perturbations, see Lemma 2.8.

As in [2] we start with a Gronwall Lemma, but in a more general version for locally finite measures. For the proof, see [3].

Lemma 2.1 (Gronwall). Let $\mu$ be a locally finite Borel measure on $[0, \infty)$, $u \in L_{1,\text{loc}}([0, \infty), \mu)$ and $\alpha : [0, \infty) \rightarrow [0, \infty)$ measurable. Suppose, that
\[
u(x) \leq \alpha(x) + \int_{[0,x]} u(s) \, d\mu(s) \quad (x \geq 0).
\]
Then
\[
u(x) \leq \alpha(x) + \int_{[0,x]} \alpha(s) \exp(\mu([s,x])) \, d\mu(s) \quad (x \geq 0).
\]

For $x \in \mathbb{R}$ we abbreviate
\[
I_x := [x \wedge 0, x \vee 0], \quad I_x(t) := I_x \cap ([t, x] \cup [x, t]) \quad (t \in \mathbb{R}).
\]
Let $\mu$ be uniformly locally bounded. Then
\[
|\mu|(I_x) \leq (|x| + 1)\|\mu\|_{\text{loc}} \quad (x \in \mathbb{R}).
\]
Furthermore, if $\mu$ is periodic and locally bounded, $\mu$ is uniformly locally bounded.

Let $H := -\Delta + \mu$ and $E \in \mathbb{R}$. Then $u \in W_1^{1, loc}(\mathbb{R}) (= D(A))$ is a solution of $Hu = Eu$, if $-(Au)' = Eu$ in the sense of distributions (i.e., $u$ satisfies the eigenvalue equation but without being an $L_2$-function).

**Lemma 2.2.** Let $\mu_1, \mu_2$ be two uniformly locally bounded measures, $E \in \mathbb{R}$ and $u_1$ and $u_2$ solutions of

$$H_1u_1 = Eu_1, \quad H_2u_2 = Eu_2$$

subject to

$$u_1(0) = u_2(0), \quad u_1'(0+) = u_2'(0+), \quad |u_1(0)|^2 + |u_1'(0+)|^2 = 1.$$ 

Then there are $C_0, C \geq 0$ such that for all $x \in \mathbb{R}$

$$\left\| \begin{pmatrix} u_1(x) \\ u_1'(x) \end{pmatrix} - \begin{pmatrix} u_2(x) \\ u_2'(x) \end{pmatrix} \right\|$$

$$\leq C_0 + \int_{I_x} |u_2(t)| \, d|\mu_1 - \mu_2|(t)$$

$$+ C \int_{I_x} \left( C_0 + \int_{I_x} |u_2| \, d|\mu_1 - \mu_2| \right) e^{C|\mu_1 - E\lambda|(I_x(t))} \, (d(\lambda + |\mu_1 - E\lambda|))(t).$$

**Proof.** Write

$$u_1(x) - u_2(x) = \int_0^x (u_1'(t) - u_2'(t)) \, dt$$

and

$$u_1'(x) - u_2'(x) = u_1'(0+) - u_2'(0+) - (u_1(0)\mu_1(\{0\}) - u_2(0)\mu_2(\{0\}))$$

$$+ \int_0^x u_1(t) \, d\mu_1(t) - \int_0^x u_2(t) \, d\mu_2(t) - \int_0^x E(u_1(t) - u_2(t)) \, dt$$

$$= u_2(0) \left( \mu_2(\{0\}) - \mu_1(\{0\}) \right)$$

$$+ \int_0^x u_2(t) \, d(\mu_1 - \mu_2)(t) + \int_0^x (u_1(t) - u_2(t)) \, d(\mu_1 - E\lambda)(t).$$

Hence,

$$\begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ u_2(0) \left( \mu_2(\{0\}) - \mu_1(\{0\}) \right) \end{pmatrix} + \int_0^x \begin{pmatrix} 0 \\ u_2(t) \end{pmatrix} \, d(\mu_1 - \mu_2)(t)$$

$$+ \int_0^x \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} u_1(t) - u_2(t) \\ u_1'(t) - u_2'(t) \end{pmatrix} \, d \left( \begin{pmatrix} \lambda \\ \mu_1 - E\lambda \end{pmatrix}(t) \right).$$

We conclude, that

$$\left\| \begin{pmatrix} u_1(x) \\ u_1'(x) \end{pmatrix} - \begin{pmatrix} u_2(x) \\ u_2'(x) \end{pmatrix} \right\| \leq C_0 + \int_{I_x} |u_2(t)| \, d|\mu_1 - \mu_2|(t)$$

$$+ C \int_{I_x} \left\| \begin{pmatrix} u_1(t) \\ u_1'(t) \end{pmatrix} - \begin{pmatrix} u_2(t) \\ u_2'(t) \end{pmatrix} \right\| \, (d(\lambda + |\mu_1 - E\lambda|))(t).$$

An application of Lemma 2.1 [with $\alpha(x) = C_0 + \int_{I_x} |u_2(t)| \, d|\mu_1 - \mu_2|(t)$ and $\mu = C(\lambda + |\mu_1 - E\lambda|)$] yields the assertion. \hfill $\square$

**Remark 2.3.** Regarding the proof of Lemma 2.2 we can further estimate

$$C_0 \leq |u_2(0)||\mu_1 - \mu_2|(I_x) \quad (x \in \mathbb{R}).$$
Lemma 2.4. Let $E \in \mathbb{R}$ and $u_0$ be a solution of $-\Delta u_0 = Eu_0$. Then there is $C \geq 0$ such that $|u_0(x)| \leq Ce^{C|x|}$ for all $x \in \mathbb{R}$.

In the following lemmas and proofs the constant $C$ may change from line to line, but we will always state the dependence on the important quantities.

Lemma 2.5. Let $\mu_1$ be a locally bounded $p$-periodic measure, $E \in \mathbb{R}$, $u_1$ a solution of $H_1u_1 = Eu_1$. Then there is $C \geq 0$ such that

$$|u_1(x)| \leq Ce^{C|x|} \quad (x \in \mathbb{R}).$$

Proof. Let $u_0$ be a solution of $-\Delta u_0 = Eu_0$ subject to the same boundary conditions at 0 as $u_1$. By Lemma 2.2 we have

$$|u_1(x) - u_0(x)|$$

$$\leq C + \int_{I_x} |u_0(t)| d|\mu_1|(t)$$

$$+ C \int_{I_x} \left( C + \int_{I_x} |u_0(s)| d|\mu_1|(s) \right) e^{C(\lambda + |\mu_1 - E\lambda|)(I_x(t))} d(\lambda + |\mu_1 - E\lambda|)(t)$$

$$\leq C + |\mu_1|(I_x)e^{C|x|}$$

$$+ \int_{I_x} \left( C + |\mu_1|(I_x)e^{C|x|} \right) e^{C(\lambda + |\mu_1 - E\lambda|)(I_x(t))} d(\lambda + |\mu_1 - E\lambda|)(t)$$

$$\leq \left( C + C|\mu_1|(I_x)e^{C|x|} \right) \left( 1 + e^{\lambda + |\mu_1 - E\lambda|}(I_x)(\lambda + |\mu_1 - E\lambda|)(I_x) \right).$$

Since $\mu_1$ is periodic and locally bounded it is uniformly locally bounded and we have $|\mu_1|(I_x) \leq (|x| + 1)\|\mu_1\|_{\text{loc}}$. Furthermore, $\mu_1 - E\lambda$ is periodic and uniformly locally bounded, so $|\mu_1 - E\lambda|(I_x) \leq (|x| + 1)\|\mu_1 - E\lambda\|_{\text{loc}}$. We conclude that

$$|u_1(x) - u_0(x)| \leq \left( C + C(|x| + 1)\|\mu_1\|_{\text{loc}}e^{C|x|} \right)$$

$$\times \left( 1 + e^{(|x| + 1)(1 + \|\mu_1 - E\lambda\|_{\text{loc}})|x| + 1}\left( 1 + \|\mu_1 - E\lambda\|_{\text{loc}} \right) \right)$$

$$\leq Ce^{C|x|},$$

where $C$ is depending on $E$, $\|\mu_1\|_{\text{loc}}$ and $\|\mu_1 - E\lambda\|_{\text{loc}}$. Hence,

$$|u_1(x)| \leq |u_1(x) - u_0(x)| + |u_0(x)| \leq Ce^{C|x|}.$$  

□

Lemma 2.6. Let $\mu$ be a Gordon measure and $(\mu^m)$ the $p_m$-periodic approximants, $E \in \mathbb{R}$. Let $u$ be a solution of $Hu = Eu$, $u_m$ a solution of $Hmu_m = Eu_m$ for $m \in \mathbb{N}$ (obeying the same boundary conditions at 0). Then there is $C \geq 0$ such that

$$\| \left( \begin{array}{c} u(x) \\ u'(x) \end{array} \right) - \left( \begin{array}{c} u_m(x) \\ u'_m(x) \end{array} \right) \| \leq Ce^{C|x|}\|\mu - \mu^m\|(I_x) \quad (x \in \mathbb{R}).$$

Proof. By Lemma 2.2 and Remark 2.3 we know that

$$\| \left( \begin{array}{c} u(x) \\ u'(x) \end{array} \right) - \left( \begin{array}{c} u_m(x) \\ u'_m(x) \end{array} \right) \| \leq |u_m(0)||\mu - \mu^m|(I_x) + \int_{I_x} |u_m(t)| d|\mu - \mu^m|(t)$$

$$+ C \int_{I_x} \left( |u_m(0)||\mu - \mu^m|(I_x) + \int_{I_x} |u_m| d|\mu - \mu^m| \right)$$

$$\times e^{C(\lambda + |\mu - E\lambda|)(I_x(t))} d(\lambda + |\mu - E\lambda|)(t).$$
We have
\[ M := \sup_{m \in \mathbb{N}} \| \mu^m \|_{\text{loc}} < \infty, \]

since \( (\mu^m) \) approximates \( \mu \). Hence, also
\[ \sup_{m \in \mathbb{N}} \| \mu^m - E\lambda \|_{\text{loc}} < \infty \]

and Lemma 2.5 yields
\[ |u_m(x)| \leq C e^{C|x|}, \]

where \( C \) can be chosen independently of \( m \). Therefore
\[ \left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x) \end{pmatrix} \right\| \leq \left( C e^{C|x|} |\mu - \mu^m|(I_x) + C e^{C|x|} |\mu - \mu^m|(I_x) \right) \]
\[ \times \left( 1 + e^{C(\lambda + |\mu - E\lambda|)(I_x)} (\lambda + |\mu - E\lambda|)(I_x) \right), \]

Since
\[ |\mu - E\lambda|(I_x) \leq (|x| + 1)||\mu - E\lambda||_{\text{loc}}, \]

we further estimate
\[ \left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x) \end{pmatrix} \right\| \leq C e^{C|x|} |\mu - \mu^m|(I_x) \]

where \( C \) is depending on \( ||\mu - E\lambda||_{\text{loc}} \) (and of course on \( M, ||\mu||_{\text{loc}} \) and \( E ) \). \( \Box \)

Lemma 2.6 can be regarded as a stability (or continuity) result: if the measures converge in total variation, the corresponding solutions converge as well.

Now, we focus on periodic measures and estimate the solutions. This will then be applied to the periodic approximations of our Gordon measure \( \mu \).

**Remark 2.7.**

(a) Let \( f, g \) be two solutions of the equation \( Hu = Eu \). Define their Wronskian by \( W(f, g)(x) := f(x)g'(x) - f'(x)g(x) \). By [1], Proposition 2.5, \( W(f, g) \) is constant.

(b) Let \( u \) be a solution of the equation \( Hu = Eu \). Define the transfer matrix \( T_E(x) \) mapping \((u(0), u'(0+))\) to \((u(x), u'(x+))\). Consider now the two solutions \( u_N, u_D \) subject to
\[
\begin{pmatrix} u_N(0) \\ u'_N(0+) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_D(0) \\ u'_D(0+) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then
\[ T_E(x) = \begin{pmatrix} u_N(x) & u_D(x) \\ u'_N(x+) & u'_D(x+) \end{pmatrix}. \]

We obtain \( \det T_E(x) = W(u_N, u_D)(x) \) and \( \det T_E \) is constant, hence equals 1 for all \( x \in \mathbb{R} \).

**Lemma 2.8.** Let \( \mu \) be \( p \)-periodic and \( E \in \mathbb{R} \). Let \( u \) be a solution of \( Hu = Eu \) subject to
\[ |u(0)|^2 + |u'(0+)|^2 = 1. \]

Then
\[ \max \left\{ \left\| \begin{pmatrix} u(-p) \\ u'(-p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(2p) \\ u'(2p+) \end{pmatrix} \right\| \right\} \geq \frac{1}{2}. \]

The proof of this lemma is completely analogous to the proof of [2] Lemma 2.2].
Lemma 2.9. Let \( v \in L_2 \cap BV_{\text{loc}}(\mathbb{R}) \) and assume that for all \( r > 0 \) we have
\[
|v(x) - v(x + r)| \to 0 \quad (|x| \to \infty).
\]
Then \( |v(x)| \to 0 \) as \( |x| \to \infty \).

Proof. Without restriction, we can assume that \( v \ge 0 \). We prove this lemma by contradiction. Assume that \( v(x) \to 0 \) does not hold for \( x \to \infty \). Then we can find \( \delta > 0 \) and \( \{q_k\} \in \mathbb{R} \) with \( q_k \to \infty \) such that \( v(q_k) \ge \delta \) for all \( k \in \mathbb{N} \). By square integrability of \( v \) we have \( \|v1_{[q_k,q_k+1]}\|^2_2 \to 0 \). Therefore, we can find a subsequence \( (r_n) \) of \( (q_k) \) satisfying
\[
\|v1_{[r_n,r_n+1]}\|^2_2 \leq 2^{-3n/2} \quad (n \in \mathbb{N}).
\]
Now, Chebyshev's inequality implies
\[
\lambda(\{x \in [r_n,r_n+1]: v(x) \ge 2^{-n}\}) \leq 2^{2n}\|v1_{[r_n,r_n+1]}\|^2_2 \leq 2^{-n} \quad (n \in \mathbb{N}).
\]
Denote \( A_n := \{x \in [r_n,r_n+1]: v(x) \ge 2^{-n}\} - r_n \subseteq [0,1] \). Then \( \lambda(A_n) \leq 2^{-n} \) and
\[
\lambda(\bigcup_{n \ge 3} A_n) \leq \sum_{n \ge 3} \lambda(A_n) \leq 2^{-2} < 1.
\]
Hence, \( G := [0,1] \setminus (\bigcup_{n \ge 3} A_n) \) has positive measure. For \( r \in G \), \( r > 0 \) it follows \( v(r_n + r) \leq 2^{-n} \quad (n \geq 3) \). Therefore,
\[
\liminf_{n \to \infty} |v(r_n) - v(r_n + r)| \geq \delta > 0,
\]
a contradiction. \(\square\)

Lemma 2.10. Let \( \mu \) be a Gordon measure, \( E \in \mathbb{R} \), \( u \in D(H) \) a solution of \( Hu = Eu \). Then \( u(x) \to 0 \) as \( x \to \infty \) and \( u'(x) \to 0 \) as \( x \to \infty \).

Proof. Since \( u \in D(H) \subseteq D(\tau) \subseteq W_2^1(\mathbb{R}) \) we have \( u(x) \to 0 \) as \( |x| \to \infty \). Lemma \ref{1.3} yields \( u \in D(T) \) and \( -(Au)' = Hu = Eu \). Let \( r > 0 \). Then, for almost all \( x \in \mathbb{R} \),
\[
\begin{align*}
  u'(x + r) - u'(x) &= Au(x + r) - Au(x) + \int_{(x,x+r]} u(t) \, d\mu(t) \\
  &= \int_{x}^{x+r} (Au)'(y) \, dy + \int_{(x,x+r]} u(t) \, d\mu(t).
\end{align*}
\]
Hence,
\[
|u'(x + r) - u'(x)| \leq |E| \int_{x}^{x+r} |u(y)| \, dy + \int_{(x,x+r]} |u(t)| \, d\mu(t) \\
\leq |E|r \|u\|_{\infty,[x,x+r]} + \|u\|_{\infty,[x,x+r]} \|\mu\|([x,x+r]) \\
\leq \|u\|_{\infty,[x,x+r]} (|E|r + (r + 1)) \|\mu\|_{\text{loc}}.
\]
By Sobolev's inequality, there is \( C \in \mathbb{R} \) (depending on \( r \), but \( r \) is fixed anyway) such that
\[
\|u\|_{\infty,[x,x+r]} \leq C \|u\|_{W_2^1([x,x+r])} \to 0 \quad (|x| \to \infty).
\]
Thus,
\[
|u'(x + r) - u'(x)| \to 0 \quad (|x| \to \infty).
\]
An application of Lemma \ref{2.9} with \( v := u' \) yields \( u'(x) \to 0 \) as \( |x| \to \infty \). \(\square\)

Now, we can state the main result of this paper.
Theorem 2.11. Let $\mu$ be a Gordon measure. Then $H$ has no eigenvalues.

Proof. Let $(\mu^m)$ be the periodic approximants of $\mu$. Let $E \in \mathbb{R}$ and $u$ be a solution of $Hu = Eu$. Let $(u_m)$ be the solutions for the measures $(\mu^m)$. By Lemma 2.6 we find $m_0 \in \mathbb{N}$ such that

$$
\left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u'_m(x) \end{pmatrix} \right\| \leq \frac{1}{4}
$$

for $m \geq m_0$ and almost all $x \in [-p_m, 2p_m]$. By Lemma 2.8 we have

$$
\limsup_{|x| \to \infty} (|u(x)|^2 + |u'(x)|^2) \geq \frac{1}{4} > 0.
$$

Hence, $u$ cannot be in $D(H)$ by Lemma 2.10. $\square$

3. Examples

Remark 3.1 (periodic measures). Every locally bounded periodic measure on $\mathbb{R}$ is a Gordon measure. Thus, for $\mu := \sum_{n \in \mathbb{Z}} \delta_{n+\frac{1}{2}}$ the operator $H := -\Delta + \mu$ has no eigenvalues.

Some examples of quasi-periodic $L^1_{\text{loc}}$-potentials can be found in [2].

For a measure $\mu$ and $x \in \mathbb{R}$ let $T_x \mu := \mu(\cdot - x)$. If $\mu$ is periodic with period $p$, then $T_p \mu = \mu$.

Example 3.2. Let $\alpha \in (0, 1) \setminus \mathbb{Q}$. There is a unique continued fraction expansion

$$
\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \ldots}}}},
$$

with $a_n \in \mathbb{N}$. For $m \in \mathbb{N}$ we set $\alpha_m = \frac{p_m}{q_m}$, where

$$
p_0 = 0, \quad p_1 = 1, \quad p_m = a_m p_{m-1} + p_{m-2},
q_0 = 1, \quad q_1 = a_1, \quad q_m = a_m p_{m-1} + q_{m-2}.
$$

The number $\alpha$ is called Liouville number, if there is $B \geq 0$ such that

$$
|\alpha - \alpha_m| \leq B m^{-q_m}.
$$

The set of Liouville numbers is a dense $G_\delta$.

Let $\nu, \tilde{\nu}$ be 1-periodic measures and assume that there is $\gamma > 0$ such that

$$
|\nu(\cdot - x) - \nu|([0, 1]) \leq |x|^\gamma \quad (x \in \mathbb{R}).
$$

Define $\mu := \tilde{\nu} + \nu \circ \alpha$ and $\mu^m := \tilde{\nu} + \nu \circ \alpha_m$ for $m \in \mathbb{N}$. Then $\mu^m$ is $q_m$-periodic and

$$
|\mu - \mu^m|([-q_m, 2q_m]) = |\nu \circ \alpha - \nu \circ \alpha_m|([-q_m, 2q_m])
= \left\| \nu \circ \alpha \right\|_\alpha - \nu |([-p_m, 2p_m])
\leq \sum_{n=-p_m}^{2p_m-1} |\nu \circ \alpha \circ n - \nu|([n, n + 1]).
$$
Now, we have

\[ |\nu \circ \frac{\alpha}{\alpha_m} - \nu|([n, n+1]) = T_{-n} |\nu \circ \frac{\alpha}{\alpha_m} - \nu|([0, 1]) \]
\[ = |T_{-n}(\nu \circ \frac{\alpha}{\alpha_m}) - T_{-n}\nu |([0, 1]) \]
\[ = |T_{-n}(\nu \circ \frac{\alpha}{\alpha_m}) - \nu |([0, 1]). \]

With \(g_{m,n}(y) := y + (\frac{\alpha}{\alpha_m} - 1)(y + n)\) and using periodicity of \(\nu\) we obtain

\[ T_{-n}(\nu \circ \frac{\alpha}{\alpha_m}) = \nu \circ g_{m,n}.\]

Hence,

\[ |\nu \circ \frac{\alpha}{\alpha_m} - \nu|([n, n+1]) = |\nu \circ g_{m,n} - \nu|([0, 1]). \]

For \(y \in [0, 1]\) and \(n \in \{-p_m, \ldots, 2p_m - 1\}\) we have

\[ |(\frac{\alpha}{\alpha_m} - 1)(y + n)| \leq |\frac{\alpha}{\alpha_m} - 1| \leq 2q_mBm^{-qm}. \]

Thus,

\[ |\nu \circ g_{m,n} - \nu|([0, 1]) \leq (2q_mBm^{-qm})^\gamma = (2q_mB)^\gamma m^{-qm}. \]

We conclude that

\[ |\mu - \mu_m|([-q_m, 2q_m]) \leq 3p_m(2q_mB)^\gamma m^{-qm}. \]

and therefore for arbitrary \(C \geq 0\)

\[ e^{Cq_m} |\mu - \mu_m|([-q_m, 2q_m]) \to 0 \quad (m \to \infty). \]

Hence \(\mu\) is a Gordon potential and \(H := -\Delta + \mu\) does not have any eigenvalues.

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References


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