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STABILITY OF ENTROPY SOLUTIONS FOR LÉVY MIXED HYPERBOLIC-PARABOLIC EQUATIONS

KENNETH H. KARLSEN, SÜLEYMAN ULUSOY

Abstract. We analyze entropy solutions for a class of Lévy mixed hyperbolic-parabolic equations containing a non-local (or fractional) diffusion operator originating from a pure jump Lévy process. For these solutions we establish uniqueness (L^1 contraction property) and continuous dependence results.

1. Introduction

The subject of this paper is uniqueness and stability results for properly defined entropy solutions of mixed hyperbolic-parabolic quasilinear equations appended with a nonlocal (fractional) diffusion operator. These equations take the form

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}(a(u)\nabla u) + \mathcal{L}[u], \tag{1.1}$$

where $u = u(t,x)$ is the unknown, $(t,x) \in Q_T := (0,T) \times \mathbb{R}^d$, $d \geq 1$, $T > 0$ is a fixed final time, and $\mathcal{L}$ is a pure jump Lévy operator.

Equation (1.1) is subject to initial data

$$u(0,x) = u_0(x) \in (L^1 \cap L^\infty)(\mathbb{R}^d). \tag{1.2}$$

In (1.1),

$$f = (f_1, \dots, f_d) \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^d) \tag{1.3}$$

is a given vector-valued flux function, $a = (a_{ij}) \geq 0$ is a given symmetric matrix-valued diffusion function of the form

$$a = \sigma^a (\sigma^a)^{\mathrm{tr}}, \quad \sigma^a \in \mathbb{R}^{d \times K}, \quad 1 \leq K \leq d. \tag{1.4}$$

More precisely, the components of a are $a_{ij} = \sum_{k=1}^K \sigma^a_{ik}\sigma^a_{jk}$ for $i,j = 1,\dots,d$. We assume that the matrix-valued function $\sigma^a = (\sigma^a_{ik}) : \mathbb{R} \to \mathbb{R}^{d \times K}$ satisfies

$$\sigma^a \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^{d \times K}). \tag{1.5}$$

Observe that we do not assume the matrix $a(\cdot)$ to be strictly positive definite, so the operator $\operatorname{div}(a(u)\nabla u)$ may be strongly degenerate, and hence the phrase "mixed hyperbolic-parabolic" is justified.

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In terms of its singular integral representation, the nonlocal operator $\mathcal{L}$ in (1.1) takes the form

$$\mathcal{L}[u](t,x) = \int_{\mathbb{R}^d\setminus\{0\}} \left[u(t,x+z) - u(t,x) - z\cdot\nabla u\,\mathbf{1}_{|z|<1}\right]\pi(dz), \tag{1.6}$$

where the singular Lévy measure $\pi(dz)$ is a positive, σ-finite Borel measure on $\mathbb{R}^d\setminus\{0\}$ satisfying $\pi(\{0\})=0$, $\pi(d(-z)) = -\pi(dz)$, and

$$\int_{\mathbb{R}^d\setminus\{0\}} \left(|z|^2\mathbf{1}_{|z|<1} + |z|\mathbf{1}_{|z|\geq 1}\right)\pi(dz) < \infty, \tag{1.7}$$

where we note that z can be replaced by a certain regular jump function $j(z)$ easily throughout the analysis. A typical example is provided by taking

$$\pi(z) = \frac{1}{|z|^{d+\alpha}}\mathbf{1}_{|z|<1}\,dz, \quad \alpha\in(0,2).$$

This example is related to to the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\frac{\alpha}{2}}$ on $\mathbb{R}^d$, which can also be defined in terms of the Fourier transform as

$$\widehat{\Delta_\alpha v}(\omega) = |\omega|^\alpha \widehat{v}(\omega), \quad \omega\in\mathbb{R}^d.$$

This definition is employed in [28] to prove (1.6) in this case.

Nonlocal operators like Δ_α are examples of a Fourier multiplier operator $\mathcal{P}$ with a symbol $a(\omega)\geq 0$ such that $\widehat{\mathcal{P}v}(\omega) = a(\omega)\widehat{v}(\omega)$. The function $e^{-ta(\omega)}$ is positive definite, and thus, by the Lévy-Khintchine formula, it can be represented as

$$a(\omega) = ib\cdot\omega + q(\omega) + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{-iz\cdot\omega} - iz\cdot\omega\,\mathbf{1}_{|z|<1}(z)\right)\pi(dz),$$

where $b\in\mathbb{R}^d$ represents the drift term, $q(\omega) = \sum_{i,j=1}^d q_{ij}\omega_i\omega_j$ is a positive definite quadratic function representing the pure diffusion part ($q(\omega) = |\omega|^2$ gives raise to the usual Laplacian $-\Delta$), and the Lévy measure $\pi(dz)$ accounts for the jump (nonlocal) part. In our setting of $\mathcal{L}$, cf. (1.6), we assume $b\equiv 0$ and $q\equiv 0$, i.e, we are dealing with a pure jump operator. For more details about the Lévy-Khintchine formula and Lévy processes in general, we refer to [13, 31, 32, 33, 47].

Integro-partial differential equations, also known as nonlocal, fractional or Lévy partial differential equations, appear frequently in many different areas of research and find many applications in engineering and finance, including nonlinear acoustics, statistical mechanics, biology, fluid flow, pricing of financial instruments, and portfolio optimization. Many authors have recently contributed to advancing the mathematical theory for quasilinear and fully nonlinear partial differential equations that are supplemented with a fractional diffusion operator arising as the generator of a Lévy semigroup, addressing questions like existence, uniqueness, regularity, formation of singularities, and asymptotic behavior of solutions.

Another very popular subject recently, where non-local operators appear, is the so-called quasi-geostrophic equation. This equation can be written in divergence form and the variational techniques are useful. Interested readers can consult [21, 41, 53, 54] and the references therein for further discussion of this subject.

For results with reference to fully nonlinear equations, such as the Hamilton-Jacobi-Bellman equation, and the (in this context relevant) theory of viscosity solutions, we refer to [2, 5, 7, 8, 9, 18, 19, 20, 30, 34, 35, 46, 49, 48, 50, 51, 52], see also [11, 12, 26] for some concrete applications to finance.

More recently, a number of authors [1, 4, 14, 15, 16, 17, 28, 36] have studied questions regarding existence, uniqueness, regularity, and temporal asymptotics for quasilinear equations, such as the fractal Burgers equation

$$\partial_t u + \partial_x(u^2/2) = -(-\partial^2_{xx})^{\frac{\alpha}{2}} u, \tag{1.8}$$

and more generally multi-dimensional fractional conservation laws

$$\partial_t u + \operatorname{div} f(u) = \Delta_\alpha u, \tag{1.9}$$

where the parameter α is assumed lie in the interval $(0, 2)$. Of course, the excluded case $\alpha = 2$ corresponds to the already fully understood viscous conservation law $\partial_t u + \operatorname{div} f(u) = \Delta u$, solutions of which are always smooth in $t > 0$. Regarding the less studied case $\alpha \in [1, 2)$, it was recently proved in [27, 40] that solutions of the fractional Burgers equation (1.8) are also smooth in $t > 0$. In the case $\alpha < 1$ for the fractional conservation law (1.9) the order of the diffusion part is lower than the first order hyperbolic part, so we do not expect any regularizing effect to take place. Indeed, for the fractional Burgers equation (1.8) with $\alpha < 1$ it is proved in [4, 40] that solutions can develop discontinuities in finite time. Consequently, one should employ a notion of entropy solutions for fractional conservation laws (1.9), i.e., weak solutions satisfying an additional entropy condition, to ensure the global-in-time well-posedness. This is well-known for conservation laws $\partial_t u + \operatorname{div} f(u) = 0$, cf. Kružkov [42], and the well-posedness theory of Kružkov was recently extended to fractional conservation laws in [1].

In recent years the theory of Kružkov [42] has been extended to quasilinear mixed hyperbolic-parabolic equations of the form

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}(a(u)\nabla u), \tag{1.10}$$

where f and a satisfy (1.3) and (1.4)-(1.5), respectively. Since the diffusion matrix $a(u)$ is not assumed to be strictly positive definite, (1.10) is strongly degenerate and will in general posses discontinuous solutions. In the isotropic case (with $a(\cdot)$ being a scalar function) the first general uniqueness result is due to Carrillo [22], who developed an original extension of Kružkov's method of doubling variables to prove his result, cf. [37, 38, 43, 44] for some additional applications of his techniques. The anisotropic case ($a(\cdot)$ being a matrix-valued function) was first treated by Chen and Perthame [25], who developed a kinetic formulation and established the uniqueness result using regularization by convolution. An alternative proof of the result of Chen and Perthame, adapting the device of doubling variables, was developed in [10], cf. also [24, 23, 45] some other papers dealing with the anisotropic case.

The main purpose of this paper is to extend the uniqueness and "continuous dependence on the nonlinearities" results of [10, 24, 23, 45] to fractional degenerate parabolic equations of the form (1.1). We introduce the notion of entropy solutions and state the main results in Section 2 . Sections 3 (existence), 4 (uniqueness), and 5 (continuous dependence on the nonlinearities and the Lévy measure) are devoted to the proofs of the main results.

2. Notion of solution and main results

For $i = 1, \dots, d$ and $k = 1, \dots, K$, define

$$\zeta^a_{ik}(u) := \int_0^u \sigma^a_{ik}(\xi)\, d\xi, \quad \zeta^{a,\psi}_{ik}(u) = \int_0^u \psi(\xi)\sigma^a_{ik}(\xi)\, d\xi, \quad u \in \mathbb{R},$$

for any $\psi \in C(\mathbb{R})$. Given any convex C^2 entropy function $\eta : \mathbb{R} \to \mathbb{R}$, we define the corresponding entropy fluxes $q = (q_i) : \mathbb{R} \to \mathbb{R}^d$ and $r = (r_{ij}) : \mathbb{R} \to \mathbb{R}^{d\times d}$ by

$$q'(u) = \eta'(u)f'(u), \quad r'(u) = \eta'(u)a(u).$$

We refer to (η, q, r) as an entropy-entropy flux triple. We now introduce the entropy formulation of (1.1)-(1.2).

Definition 2.1. An entropy solution of the initial value problem (1.1)-(1.2) is a measurable function $u : Q_T \to \mathbb{R}$ satisfying the following conditions:

(1) $u \in L^\infty(Q_T)$, $u \in L^\infty(0, T; L^1(\mathbb{R}^d))$,

$$\sum_{i=1}^{d} \partial_{x_i} \zeta^a_{ik}(u) \in L^2(Q_T), \quad k = 1, \dots, K, \tag{2.1}$$

and

$$\iint_{Q_T} \int_{\mathbb{R}^d\setminus\{0\}} (u(t, x + z) - u(t, x))^2 \, \pi(dz)\, dx\, dt < +\infty; \tag{2.2}$$

(2) For $k = 1, \dots, K$,

$$\sum_{i=1}^{d} \partial_{x_i} \zeta^{a,\psi}_{ik}(u) = \psi(u) \sum_{i=1}^{d} \partial_{x_i} \zeta^a_{ik}(u), \quad \text{a.e. in } Q_T \text{ and in } L^2(Q_T), \tag{2.3}$$

for any $\psi \in C(\mathbb{R})$;

(3) For any entropy-entropy flux triple (η, q, r),

$$\begin{aligned} &\iint_{Q_T} \Big(\eta(u)\partial_t\varphi + \sum_{i=1}^{d} q_i(u)\partial_{x_i}\varphi + \sum_{i,j=1}^{d} r_{ij}(u)\partial^2_{x_ix_j}\varphi\Big)\, dx\, dt \\ &+ \iint_{Q_T} \eta(u)\mathcal{L}[\varphi]\, dx\, dt + \int_{\mathbb{R}^d} \eta(u_0)\varphi(0, x)\, dx \ge n^u + m^u, \end{aligned} \tag{2.4}$$

for all non-negative $\varphi \in C^\infty_c([0, T) \times \mathbb{R}^d)$, where

$$\begin{aligned} n^u &= \iint_{Q_T} \eta''(u) \sum_{k=1}^{K} \Big(\sum_{i=1}^{d} \partial_{x_i}\zeta^a_{ik}(u)\Big)^2 \varphi(t, x)\, dx\, dt, \\ m^u &= \iint_{Q_T} \int_{\mathbb{R}^d\setminus\{0\}} \overline{\eta''}(u; z)\, (u(t, x + z) - u(t, x))^2\, \varphi(t, x)\pi(dz)\, dx\, dt, \end{aligned}$$

and

$$\overline{\eta''}(u; z) = \int_0^1 (1 - \tau)\eta''((1 - \tau)u(t, x) + \tau u(t, x + z))\, d\tau.$$

We remark that the chain rule (2.3) is automatically fulfilled when $a(\cdot)$ is a scalar or a diagonal matrix, cf. Chen and Perthame [25], and in this case we can drop (2) from the definition.

Our first result is the expected L^1 contraction property (and thus the uniqueness) of entropy solutions.

Theorem 2.2. *Suppose f and a satisfy* (1.3) *and* (1.4)-(1.5)*, respectively, and that the Lévy measure $\pi(dz)$ satisfies* (1.7)*. Then there exists an entropy solution* *of* (1.1)-(1.2)*. Let* u, v *be two entropy solutions of* (1.1) *with initial data* $u|_{t=0} = u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$, $v|_{t=0} = v_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$*. For a.e.* $t \in (0, T)$*, we have*

$$\int_{\mathbb{R}^d} (u(t,x) - v(t,x))^+ \, dx \le \int_{\mathbb{R}^d} (u_0 - v_0)^+ \, dx. \tag{2.5}$$

Consequently, if $u_0 \le v_0$ *a.e. in* $\mathbb{R}^d$ *then* $u \le v$ *a.e. in* Q_T*, so whenever* $u_0 = v_0$ *a.e. in* $\mathbb{R}^d$*, then* $u = v$ *a.e. in* Q_T*.*

This theorem generalizes to the "non-local diffusion" case the result of Chen and Perthame [25]. The proof follows that of Bendahmane and Karlsen [10].

Our second result, which is a refinement of the previous theorem, reveals how the entropy solution u depends on the Lévy measure $\pi(dz)$, and the nonlinear fluxes f, a (i.e., it is a "continuous dependence" estimate).

Theorem 2.3. *Suppose* f *and* a *satisfy* (1.3) *and* (1.4)-(1.5)*, respectively, and that the Lévy measure* $\pi(dz)$ *satisfies* (1.7)*. Let* $u \in L^\infty(0,T; BV(\mathbb{R}^d))$ *be the entropy solution of* (1.1) *with BV initial data* $u_0 \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^d)$ *and with a Lévy measure of the form* $\pi(dz) = m(z)\,dz$ *for some measurable function* $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$.

Replace the data set

$$(f, a, \pi, u_0), \quad a = \sigma^a(\sigma)^{\rm tr}, \quad \pi(dz) = m(z)\,dz$$

by another data set

$$(\tilde f, \tilde a, \tilde\pi(dz), v_0), \quad \tilde a = \sigma^{\tilde a}(\sigma^{\tilde a})^{\rm tr}, \quad \tilde\pi(dz) = \tilde m(z)\,dz,$$

where $\tilde f, \sigma^{\tilde a}, \tilde\pi, \tilde m$ *satisfy the same regularity conditions as* f, σ^a, π, m *and moreover* $v_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$*. Denote the corresponding entropy solution by* v*, and assume that* $v \in C([0,T]; L^1(\mathbb{R}^d))$*. Suppose* u *and* v *take values in a closed interval* $I \subset \mathbb{R}$*.*

For any $t \in (0,T)$*,*

$$\begin{aligned}
&\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R}^d)} \\
&\le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + C_1 t \|f - \tilde f\|_{W^{1,\infty}(I);\mathbb{R}^d)} \\
&\quad + C_2 \sqrt{t} \|\sigma^a - \sigma^{\tilde a}\|_{L^\infty(I;\mathbb{R}^{d\times K})} + C_3\sqrt{t}\Big(\int_{|z|<1} |z|^2 |m(z) - \tilde m(z)|\,dz\Big)^{1/2} \\
&\quad + C_4 t \int_{|z|\ge 1} |z||m(z) - \tilde m(z)|\,dz,
\end{aligned} \tag{2.6}$$

where the constants C_i*,* $i = 1, \dots, 4$*, depend on the* $L^\infty(0,T; BV(\mathbb{R}^d))$ *norm of* u*.*

This theorem generalizes results in [23, 24] to the "fractional case". In the hyperbolic case ($a, \tilde a \equiv 0$), a generalization of this theorem to nonlocal operators of the form $\mathcal{L}[A(u)]$, $A : \mathbb{R} \to \mathbb{R}$ Lipschitz and nondecreasing, can be found in [3].

3. Proof of Theorem 2.2 (existence)

Although a detailed version of the existence of entropy solutions to (1.1) is presented in [39], to motivate the entropy condition and to present a brief sketch, let us consider the following accompanying problem containing a uniformly parabolic operator depending on a small parameter $\rho > 0$:

$$\partial_t u_\rho + \operatorname{div} f(u_\rho) = \operatorname{div}(a(u_\rho)\nabla u_\rho) + \mathcal{L}[u_\rho(t,\cdot)] + \rho\Delta u_\rho. \tag{3.1}$$

It is standard to construct a smooth solution u_ρ to (3.1), for each fixed $\rho > 0$. Indeed, it can be done using the Galerkin method and the compactness argument, see Chapter 5 in [29] and [40].

As usual, the game is to pass to the limit as $\rho \to 0$ and identify the entropy condition satisfied by the limit function u. We will be brief in establishing the following estimates, since most of them are similar to the ones in [25] and we will assume $u_0 \in W^{2,1} \cap H^1 \cap L^\infty(\mathbb{R}^d)$, for general $u_0 \in L^1(\mathbb{R}^d)$ one can follow the approximation procedure presented in [25].

The following estimates can be established for sufficiently regular initial data:

$$\|u_\rho\|_{L^\infty(Q_T)} \le C; \quad |u_\rho(t,\cdot)|_{BV(\mathbb{R}^d)} \le C;$$
$$\|u_\rho(t_2,\cdot) - u_\rho(t_1,\cdot)\|_{L^1(\mathbb{R}^d)} \to 0, \quad \text{as } |t_2 - t_1| \to 0, \text{ uniformly in } \rho.$$

Hence there is a limit u such that, passing if necessary to a subsequence as $\rho \to 0$,

$$u_\rho \to u \quad \text{a.e. in } Q_T \text{ and in } L^p(Q_T) \text{ for any } p \in [1,\infty). \tag{3.2}$$

Next, we derive an energy estimate. To this end, fix a convex C^2 function η and define q, r by $q' = \eta' f'$, $r' = \eta' a$. Multiplying (3.1) by η' yields

$$\partial_t \eta(u_\rho) + \operatorname{div} q(u_\rho) = \sum_{i,j=1}^d \partial_{ij}^2 r_{ij}(u_\rho) + \mathcal{L}[\eta(u_\rho)] + \rho\Delta\eta(u_\rho) - \nu_\rho \tag{3.3}$$

where $\nu_\rho = \nu_\rho^1 + \nu_\rho^2 + \nu_\rho^3$ consists of three parts:

(i) the entropy dissipation term
$$\nu_\rho^1 := \rho\Delta\eta(u_\rho) - \rho\eta'(u_\rho)\Delta u_\rho = \rho\eta''(u_\rho)|\nabla u_\rho|^2;$$

(ii) the parabolic dissipation term
$$\nu_\rho^2 := \sum_{i,j=1}^d \partial_{ij}^2 r_{ij}(u_\rho) - \eta'(u_\rho)\operatorname{div}(a(u_\rho)\nabla u_\rho) = \eta''(u_\rho)\sum_{k=1}^K \Big(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u_\rho)\Big)^2;$$

(iii) the fractional parabolic dissipation term
$$\nu_\rho^3 = \int_{\mathbb{R}^d\setminus\{0\}} \overline{\eta''}(u_\rho;z)\,(u_\rho(t,x+z) - u_\rho(t,x))^2\,\pi(dz),$$
where $\overline{\eta''}(u_\rho;z) = \int_0^1 (1-\tau)\eta''((1-\tau)u_\rho(t,x) + \tau u_\rho(t,x+z))\,d\tau$.

In deriving (3.3), the "new" computation is the one showing that the commutator

$$\mathcal{L}[\eta(u_\rho)] - \eta'(u_\rho)\mathcal{L}[u_\rho]$$

equals ν_ρ^3, but this follows easily from Taylor's formula with integral reminder:

$$\eta(b) - \eta(a) = \eta'(a)(b-a) + \Big(\int_0^1 (1-\tau)\eta''((1-\tau)a + \tau b)\,d\tau\Big)(b-a)^2. \tag{3.4}$$

Specifying $\eta(z) = z^2/2$ in (3.3) gives

$$\int_0^T \int_{\mathbb{R}^d} \sum_{k=1}^K \Big(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u_\rho)\Big)^2\,dx\,dt \le C$$

and

$$\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u_\rho) \rightharpoonup \sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u) \quad \text{in } L^2(Q_T). \tag{3.5}$$

From this we easily see, as in [25], that (2.1) and (2.3) in Definition 2.1 hold.

Regarding the non-local operator $\mathcal{L}$, the same choice for η reveals that (2.2) in Definition 2.1 holds. Now set

$$\Pi(dz) := \left(|z|^2\mathbf{1}_{|z|<1} + |z|\,\mathbf{1}_{|z|\geq 1}\right)\pi(dz),$$

and note that $\Pi(dz)$ is a bounded Radon measure. Introducing the short-hand notation

$$D_\rho(t,x,z) = \frac{u_\rho(t,x+z) - u_\rho(t,x)}{|z|\mathbf{1}_{|z|<1} + \sqrt{|z|}\mathbf{1}_{|z|\geq 1}} \quad d\mu = \Pi(dz)\otimes dx\otimes dt,$$

Equation (2.2) translates into D_ρ being uniformly bounded in $L^2((0,T)\times\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\}); d\mu)$. Consequently, we may assume that there is a limit function D such that

$$D_\rho \rightharpoonup D \quad \text{in } L^2((0,T)\times\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\}); d\mu).$$

Let us identify D. To this end, fix a smooth function φ in $C_c^\infty(Q_T)$ and observe

$$\begin{aligned}
&\iint_{Q_T}\int_{\mathbb{R}^d\setminus\{0\}} \varphi(t,x)\frac{u_\rho(t,x+z) - u_\rho(t,x)}{|z|\mathbf{1}_{|z|<1} + \sqrt{|z|}\mathbf{1}_{|z|\geq 1}}\,\Pi(dz)\,dx\,dt \\
&= \iint_{Q_T}\int_{\mathbb{R}^d\setminus\{0\}} \frac{\varphi(t,x+z) - \varphi(t,x)}{|z|\mathbf{1}_{|z|<1} + \sqrt{|z|}\mathbf{1}_{|z|\geq 1}} u_\rho(t,x)\,\Pi(dz)\,dx\,dt.
\end{aligned}$$

Now, using that $u_\rho \xrightarrow{\rho\to 0} u$ a.e. in Q_T, we conclude that

$$D_\rho \rightharpoonup \frac{u(t,x+z) - u(t,x)}{|z|\mathbf{1}_{|z|<1} + \sqrt{|z|}\mathbf{1}_{|z|\geq 1}} \quad \text{in } L^2((0,T)\times\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\}); d\mu).$$

We are now in a position to pass to the distributional limit in (3.3) to recover the desired entropy condition satisfied by the limit $u = \lim_{\rho\to 0} u_\rho$. Note that to interpret (3.3) in the sense of distributions we use the formula

$$\int_{\mathbb{R}^d} \mathcal{L}[\Phi(x)]\phi(x)\,dx = \int_{\mathbb{R}^d} \Phi(x)\mathcal{L}[\phi(x)]\,dx, \tag{3.6}$$

which holds for all sufficiently regular (say, C^2) functions $\Phi, \phi : \mathbb{R}^d \to \mathbb{R}$. This relation is easily obtained by a change of variables $(t,x,z)\mapsto(t,x+z,-z)$ and an integration by parts in x.

We claim that the entropy condition satisfied by the limit $u = \lim_{\rho\to 0} u_\rho$ takes the following form: for any convex C^2 entropy function η and corresponding entropy fluxes q, r defined by $q' = \eta' f', r' = \eta' a$,

$$\partial_t\eta(u) + \operatorname{div} q(u) \leq \sum_{i,j}\partial_{x_i x_j} r_{ij}(u) + \mathcal{L}[\eta(u)] - n^{u,\eta} - m^{u,\eta} \tag{3.7}$$

in the sense of distributions, where

$$n^{u,\eta} = \eta''(u)\sum_{k=1}^K\Big(\sum_{i=1}^d \partial_{x_i}\zeta^a_{ik}(u)\Big)^2$$

is the parabolic dissipation measure with respect to u and

$$m^{u,\eta} = \int_{\mathbb{R}^d\setminus\{0\}} \overline{\eta''}(u;z)\left(u(t,x+z) - u(t,x)\right)^2\pi(dz)$$

is the fractional parabolic dissipation measure with respect to u.

In view of (3.2), to verify (3.7) we only need to argue that

$$\liminf_{\rho\to 0}\iint_{Q_T}\nu_\rho\,dx\,dt \geq \iint_{Q_T}\left(n^{u,\eta}+m^{u,\eta}\right)\,dx\,dt.$$

First, $\iint_{Q_T}\nu_\rho^1\,dx\,dt \geq 0$ for each $\rho>0$. Second, thanks to the weak convergence (3.5) and a standard weak lower semi-continuity result for quadratic functionals,

$$\begin{aligned}
&\liminf_{\rho\to 0}\int_0^T\int_{\mathbb{R}^d}\eta''(u_\rho)\sum_{k=1}^K\Big(\sum_{i=1}^d\partial_{x_i}\zeta_{ik}^a(u_\rho)\Big)^2\varphi\,dx\,dt\\
&\geq \int_0^T\int_{\mathbb{R}^d}\eta''(u)\sum_{k=1}^K\Big(\sum_{i=1}^d\partial_{x_i}\zeta_{ik}^a(u)\Big)^2\varphi\,dx\,dt,
\end{aligned}$$

for all test functions $\varphi\in C_c^\infty$. Similarly,

$$\begin{aligned}
&\liminf_{\rho\to 0}\iint_{Q_T}\int_{\mathbb{R}^d\setminus\{0\}}\overline{\eta''}(u_\rho;z)\left(u_\rho(t,x+z)-u_\rho(t,x)\right)^2\varphi\pi(dz)\,dx\,dt\\
&\geq \iint_{Q_T}\int_{\mathbb{R}^d\setminus\{0\}}\overline{\eta''}(u;z)\left(u(t,x+z)-u(t,x)\right)^2\varphi\pi(dz)\,dx\,dt,
\end{aligned}$$

for all test functions $\varphi\in C_c^\infty$. Combining, we deduce that (2.4) in Definition 2.1 holds. This completes the proof.

4. Proof of Theorem 2.2 (uniqueness)

We shall need C^2 approximations $\eta_\varepsilon^\pm(u)$ of the functions

$$\eta^\pm(u):=(u)^\pm=\max\left(\pm(u),0\right),\quad u\in\mathbb{R}.$$

We build these by picking nondecreasing C^1 approximations $\operatorname{sgn}_\varepsilon^\pm(u)$ of

$$\operatorname{sgn}^+(u):=\begin{cases}0, & \text{if } u\leq 0,\\ 1, & \text{if } u>0,\end{cases}\quad \operatorname{sgn}^-(u):=\begin{cases}-1, & \text{if } u\leq 0,\\ 0, & \text{if } u>0,\end{cases}$$

and defining

$$\eta_\varepsilon^\pm(u):=\int_0^u\operatorname{sgn}_\varepsilon^\pm(\xi)\,d\xi,\quad u\in\mathbb{R}.$$

For example, we can take

$$\operatorname{sgn}_\varepsilon^+(u)=\begin{cases}0, & \text{if } u<0,\\ \sin(\pi u/(2\varepsilon)), & \text{if } 0\leq u\leq\varepsilon,\\ 1, & \text{if } u>\varepsilon.\end{cases}$$

$$\operatorname{sgn}_\varepsilon^-(u)=\begin{cases}-1, & \text{if } u<-\varepsilon,\\ \sin(\pi u/(2\varepsilon)), & \text{if } -\varepsilon\leq u\leq 0,\\ 0, & \text{if } u>0.\end{cases}$$

The functions $\eta_\varepsilon^\pm$ are C^2 and convex. Moreover,

$$\eta_\varepsilon^\pm(u)\xrightarrow{\varepsilon\to 0}\eta^\pm(u),\quad u\in\mathbb{R}.$$

Observe that $(\eta_\varepsilon^\pm(\cdot - c))_{c\in\mathbb{R}}$ is a family of entropies. Given these entropies, we introduce the corresponding entropy fluxes

$$q_\varepsilon^\pm(u,c) = \int_c^u (\eta_\varepsilon^\pm)'(\xi - c) f'(\xi) d\xi, \quad u, c \in \mathbb{R},$$
$$r_\varepsilon^\pm(u,c) = \int_c^u (\eta_\varepsilon^\pm)'(\xi - c) a(\xi)\, d\xi, \quad u, c \in \mathbb{R}.$$

Clearly, as $\varepsilon \to 0$,

$$q_\varepsilon^\pm(u,c) \to q^\pm(u,c) := \operatorname{sgn}^\pm(u - c)(f(u) - f(c)), \quad u, c \in \mathbb{R},$$
$$r_\varepsilon^\pm(u,c) \to r^\pm(u,c) := \operatorname{sgn}^\pm(u - c)(A(u) - A(c)), \quad u, c \in \mathbb{R},$$

where the (matrix-valued) function $A(\cdot)$ is defined by $A(u) = \int_0^u a(\xi)\, d\xi$.

Observe that $(\eta_\varepsilon^\pm(\cdot - c), q_\varepsilon^\pm(\cdot, c), r_\varepsilon^\pm(\cdot, c))_{c\in\mathbb{R}}$ is a family of entropy-entropy flux triples, so choosing $\eta = \eta_\varepsilon^\pm$ in (2.4) yields

$$\begin{aligned}
&\iint_{Q_T} \Big(\eta_\varepsilon^\pm(u - c)\partial_t\varphi + \sum_{i=1}^d q_{\varepsilon,i}^\pm(u,c)\partial_{x_i}\varphi + \sum_{i,j=1}^d r_{\varepsilon,ij}^\pm(u,c)\partial_{x_ix_j}^2\varphi\Big)\, dx\, dt \\
&+ \iint_{Q_T} \eta_\varepsilon^\pm(u - c)\mathcal{L}[\varphi]\, dx\, dt + \int_{\mathbb{R}^d} \eta^\pm(u_0 - c)\varphi(0,x)\, dx \\
&\ge \iint_{Q_T} (\eta_\varepsilon^\pm)''(u - c) \sum_{k=1}^K \Big(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u)\Big)^2 \varphi\, dx\, dt \\
&\quad + \iint_{Q_T} \int_{\mathbb{R}^d\setminus\{0\}} \overline{(\eta_\varepsilon^\pm)''}(u - c; z)\, (u(t, x + z) - u(t,x))^2\, \varphi\pi(dz)\, dx\, dt.
\end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned}
\overline{(\eta_\varepsilon^\pm)''}(u - c; z) &= \int_0^1 (1-\tau)(\eta_\varepsilon^\pm)''\Big((1-\tau)u(t,x) + \tau u(t,x+z), c\Big)\, d\tau \\
&= \int_0^1 (1-\tau)(\operatorname{sgn}_\varepsilon^\pm)'\Big((1-\tau)(u(t,x) - c) + \tau(u(t,x+z) - c)\Big)\, d\tau.
\end{aligned}$$

To proceed, the following simple observations will be useful:

- $\operatorname{sgn}_\varepsilon^-(u - c) = -\operatorname{sgn}_\varepsilon^+(c - u)$ and $\eta_\varepsilon^-(u - c) = \eta_\varepsilon^+(c - u)$;
- $q_\varepsilon^-(u,c) = q_\varepsilon^+(c,u)$ and $r_\varepsilon^-(u,c) = r_\varepsilon^+(c,u)$;
- $(\eta_\varepsilon^-)''(u - c) = (\eta_\varepsilon^+)''(c - u)$.

Employing these observations, we can rewrite the "$-$" part of (4.1) as

$$\begin{aligned}
&\iint_{Q_T} \Big(\eta_\varepsilon^+(c - u)\partial_t\varphi + \sum_{i=1}^d q_{\varepsilon,i}^+(c,u)\partial_{x_i}\varphi + \sum_{i,j=1}^d r_{\varepsilon,ij}^+(c,u)\partial_{x_ix_j}^2\varphi\Big)\, dx\, dt \\
&+ \iint_{Q_T} \eta_\varepsilon^+(c - u)\mathcal{L}[\varphi]\, dx\, dt + \int_{\mathbb{R}^d} \eta_\varepsilon^+(c - u_0)\varphi(0,x)\, dx \\
&\ge \iint_{Q_T} (\eta_\varepsilon^+)''(c - u) \sum_{k=1}^K \Big(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u)\Big)^2 \varphi\, dx\, dt \\
&\quad + \iint_{Q_T} \int_{\mathbb{R}^d\setminus\{0\}} \overline{(\eta_\varepsilon^+)''}(c - u; z)\, (u(t, x + z) - u(t,x))^2\, \varphi\pi(dz)\, dx\, dt.
\end{aligned} \tag{4.2}$$

To establish the L^1 contraction property (2.5) we shall employ the doubling-of-variables device of Kružkov [42]. Let $u = u(t,x)$, $v = v(s,y)$ be two entropy solutions as stated in Theorem 2.2. Moreover, let $\varphi = \varphi(t,x,s,y)$ be a test function in the doubled variables (t,x,s,y). To simplify the presentation, we introduce the following notation (with ∇_{x+y} being short-hand for $\nabla_x + \nabla_y$)

$$\begin{aligned}
\mathcal{L}_x[\varphi] &:= \int_{\mathbb{R}^d\setminus\{0\}} \left[\varphi(t,x+z,s,y) - \varphi - z\cdot\nabla_x\varphi\mathbf{1}_{|z|<1}\right]\pi(dz),\\
\mathcal{L}_y[\varphi] &= \int_{\mathbb{R}^d\setminus\{0\}} \left[\varphi(t,x,s,y+z) - \varphi - z\cdot\nabla_y\varphi\mathbf{1}_{|z|<1}\right]\pi(dz),\\
\mathcal{L}_{x+y}[\varphi] &= \int_{\mathbb{R}^d\setminus\{0\}} \Big[\varphi(t,x+z,s,y+z) - \varphi - z\cdot\nabla_{x+y}\varphi\mathbf{1}_{|z|<1}\Big]\pi(dz),
\end{aligned}$$

In the "+" part of (4.1) written the entropy solution $u(t,x)$ we choose $c = v(s,y)$ and integrate the result over (s,y), obtaining

$$\begin{aligned}
&\iiiint \Big(\eta_\varepsilon^+(u-v)\partial_t\varphi + \sum_{i=1}^d q_{\varepsilon,i}^+(u,v)\partial_{x_i}\varphi + \sum_{i,j=1}^d r_{\varepsilon,ij}^+(u,c)\partial^2_{x_ix_j}\varphi\Big)\,dx\,dt\,dy\,ds\\
&+ \iiiint \eta_\varepsilon^+(u-v)\mathcal{L}_x[\varphi]\,dx\,dt\,dy\,ds + \iiint \eta_\varepsilon^+(u_0-v)\varphi(0,x,s,y)\,dx\,dy\,ds\\
&\ge \iint_{Q_T} (\eta_\varepsilon^+)''(u-v)\sum_{k=1}^K\Big(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u)\Big)^2\varphi\,dx\,dt\,dy\,ds\\
&\quad + \iiiint\int_{\mathbb{R}^d\setminus\{0\}} \overline{(\eta_\varepsilon^+)''}(u(t,\cdot)-v;z)\big(u(t,x+z)-u(t,x)\big)^2\varphi\pi(dz)\,dx\,dt\,dy\,ds.
\end{aligned}\tag{4.3}$$

Similarly, in (4.2) written for the entropy solution $v(s,y)$ we choose $c = u(t,x)$ and integrate over (t,x), thereby obtaining

$$\begin{aligned}
&\iiiint \Big(\eta_\varepsilon^+(u-v)\partial_s\varphi + \sum_{i=1}^d q_{\varepsilon,i}^+(u,v)\partial_{y_i}\varphi + \sum_{i,j=1}^d r_{\varepsilon,ij}^+(u,v)\partial^2_{y_iy_j}\varphi\Big)\,dx\,dt\,dy\,ds\\
&+ \iiiint \eta_\varepsilon^+(u-v)\mathcal{L}_y[\varphi]\,dx\,dt\,dy\,ds + \iiint \eta_\varepsilon^+(u-v_0)\varphi(t,x,0,y)\,dx\,dt\,dy\\
&\ge \iiiint (\eta_\varepsilon^+)''(u-v)\sum_{k=1}^K\Big(\sum_{i=1}^d \partial_{y_i}\zeta_{ik}^a(v)\Big)^2\varphi\,dx\,dt\,dy\,ds\\
&\quad + \iiiint\int_{\mathbb{R}^d\setminus\{0\}} \overline{(\eta_\varepsilon^+)''}(u-v(s,\cdot);z)\,\big(v(s,y+z)-v(s,y)\big)^2\,\varphi\pi(dz)\,dx\,dt\,dy\,ds.
\end{aligned}\tag{4.4}$$

Adding (4.3) and (4.4) yields

$$I_{\rm time}(\varepsilon) + I_{\rm conv}(\varepsilon) + I_{\rm diff}(\varepsilon) + I_{\rm fdiff}(\varepsilon) + I_{\rm init}(\varepsilon) \ge I_{\rm diss}(\varepsilon) + I_{\rm fdiss}(\varepsilon), \tag{4.5}$$

where

$$I_{\mathrm{time}}(\varepsilon) = \iiiint \eta_\varepsilon^+(u-v)(\partial_t + \partial_s)\varphi \,dx\,dt\,dy\,ds$$

$$I_{\mathrm{conv}}(\varepsilon) = \iiiint \sum_{i=1}^d q_{\varepsilon,i}^+(u,v)(\partial_{x_i} + \partial_{y_i})\varphi \,dx\,dt\,dy\,ds$$

$$I_{\mathrm{diff}}(\varepsilon) = \iiiint \sum_{i,j=1}^d r_{\varepsilon,ij}^+(u,v)(\partial^2_{x_i x_j} + \partial^2_{y_i y_j})\varphi \,dx\,dt\,dy\,ds$$

$$I_{\mathrm{fdiff}}(\varepsilon) = \iiiint \eta_\varepsilon^+(u-v)\Big(\mathcal{L}_x[\varphi] + \mathcal{L}_y[\varphi]\Big) \,dx\,dt\,dy\,ds$$

$$\begin{aligned} I_{\mathrm{init}}(\varepsilon) &= \iiint \eta_\varepsilon^+(u_0-v)\varphi(0,x,s,y) \,dx\,dy\,ds \\ &\quad + \iiint \eta_\varepsilon^+(u-v_0)\varphi(t,x,0,y) \,dx\,dt\,dy \end{aligned}$$

$$\begin{aligned} I_{\mathrm{diss}}(\varepsilon) &= \iiiint (\eta_\varepsilon^+)''(u-v) \\ &\quad \times \sum_{k=1}^K \Big[\Big(\sum_i^d \partial_{x_i}\zeta_{ik}^a(u)\Big)^2 + \Big(\sum_{i=1}^d \partial_{y_i}\zeta_{ik}^a(v)\Big)^2\Big]\varphi \,dx\,dt\,dy\,ds \end{aligned}$$

$$\begin{aligned} I_{\mathrm{fdiss}}(\varepsilon) &= \iiiint\!\!\int_{\mathbb{R}^d\setminus\{0\}} \Big[\overline{(\eta_\varepsilon^+)''}(u(t,\cdot)-v;z)\,(u(t,x+z)-u(t,x))^2 \\ &\quad + \overline{(\eta_\varepsilon^+)''}(u,v(s,\cdot);z)\,(v(s,y+z)-v(s,y))^2\Big]\varphi\pi(dz)\,dx\,dt\,dy\,ds. \end{aligned}$$

In view of the inequality "$a^2+b^2 \ge 2ab$", we have $I_{\mathrm{diss}}(\varepsilon) \ge \widetilde{I}_{\mathrm{diss}}(\varepsilon)$, with

$$\widetilde{I}_{\mathrm{diss}}(\varepsilon) = 2\iiiint (\eta_\varepsilon^+)''(u-v)\sum_{k=1}^K \sum_{i,j=1}^d \partial_{x_i}\zeta_{ik}^a(u)\partial_{y_j}\zeta_{jk}^a(v)\varphi \,dx\,dt\,dy\,ds.$$

Arguing exactly as in [10], it follows that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\Big(I_{\mathrm{diff}}(\varepsilon) - \widetilde{I}_{\mathrm{diss}}(\varepsilon)\Big) \\ &\le \iiiint \sum_{i,j=1}^d r_{ij}^+(u,v)(\partial^2_{x_i x_j} + 2\partial^2_{x_i y_j} + \partial^2_{y_i y_j})\varphi \,dx\,dt\,dy\,ds. \end{aligned} \tag{4.6}$$

Fix a small number $\kappa > 0$, and let us split $\mathcal{L}$ into two parts

$$\begin{aligned} \mathcal{L}[\phi] &= \int_{|z|\le\kappa} \big[\phi(t,x+z) - \phi(t,x) - z\cdot\nabla\phi\mathbf{1}_{|z|<1}\big]\,\pi(dz) \\ &\quad + \int_{|z|>\kappa} \big[\phi(t,x+z) - \phi(t,x) - z\cdot\nabla\phi\mathbf{1}_{|z|<1}\big]\,\pi(dz) \\ &=: \mathcal{L}_\kappa[\phi] + \mathcal{L}^\kappa[\phi], \quad \forall \phi \in C^2, \end{aligned}$$

and similarly

$$\mathcal{L}_x = \mathcal{L}_{x,\kappa} + \mathcal{L}_x^\kappa, \quad \mathcal{L}_y = \mathcal{L}_{y,\kappa} + \mathcal{L}_y^\kappa, \quad \mathcal{L}_{x+y} = \mathcal{L}_{x+y,\kappa} + \mathcal{L}_{x+y}^\kappa.$$

The corresponding splitting of $I_{\mathrm{fdiff}}(\varepsilon)$ is written

$$I_{\mathrm{fdiff}}(\varepsilon) = I_{\mathrm{fdiff},\kappa}(\varepsilon) + I_{\mathrm{fdiff}}^\kappa(\varepsilon).$$

We also need to introduce the operator $\widetilde{\mathcal{L}}^\kappa$ defined by writing

$$\mathcal{L}^\kappa[\varphi] = \widetilde{\mathcal{L}}^\kappa[\varphi] - \Big(\int_{|z|>\kappa} z\mathbf{1}_{|z|<1}\,\pi(dz)\Big)\cdot\nabla_x\varphi,$$

with similar definitions for $\widetilde{\mathcal{L}}^\kappa_x$, $\widetilde{\mathcal{L}}^\kappa_y$, and $\widetilde{\mathcal{L}}^\kappa_{x+y}$. Observe that (3.6) continues to hold for all these operators. The function obtained by replacing $\mathcal{L}^\kappa$ with $\widetilde{\mathcal{L}}^\kappa$ in the definition of $I^\kappa_{\rm fdiff}(\varepsilon)$ will be named $\widetilde{I}^\kappa_{\rm fdiff}(\varepsilon)$.

Clearly, in view of (1.7),

$$|I_{\mathrm{fdiff},\kappa}(\varepsilon)| \le C\|D^2\varphi\|_{L^1(Q_T\times Q_T)}\int_{|z|\le\kappa}|z|^2\pi(dz) \overset{\kappa\to 0}{\longrightarrow} 0, \tag{4.7}$$

for some constant C independent of κ and ε.

Let us analyze $\widetilde{I}^\kappa_{\rm fdiff}(\varepsilon)$. By (3.6),

$$\widetilde{I}^\kappa_{\rm fdiff}(\varepsilon) = \iiiint \Big(\widetilde{\mathcal{L}}^\kappa_x\left[\eta^+_\varepsilon(u-v)\right] + \widetilde{\mathcal{L}}^\kappa_y\left[\eta^+_\varepsilon(u-v)\right]\Big)\varphi\,dt\,dx\,dy\,ds.$$

Specifying $a = u(t,x)-v(s,y)$ and $b = u(t,x+z)-v(s,y)$ in (3.4) yields

$$\begin{aligned}
&\eta^+_\varepsilon(u(t,x+z)-v(s,y)) - \eta^+_\varepsilon(u(t,x)-v(s,y))\\
&= (\eta^+_\varepsilon)'(u(t,x)-v(s,y))\,(u(t,x+z)-u(t,x))\\
&\quad + \overline{(\eta^+_\varepsilon)''}(u(t,\cdot)-v;z)\,(u(t,x+z)-u(t,x))^2 .
\end{aligned}\tag{4.8}$$

Similarly, taking $a = u(t,x)-v(s,y)$, $b = u(t,x)-v(s,y+z)$ in (3.4) yields

$$\begin{aligned}
&\eta^+_\varepsilon(u(t,x)-v(s,y+z)) - \eta^+_\varepsilon(u(t,x)-v(s,y))\\
&= -(\eta^+_\varepsilon)'(u(t,x)-v(s,y))\,(v(s,y+z)-v(s,y))\\
&\quad + \overline{(\eta^+_\varepsilon)''}(u-v(s,\cdot);z)\,(v(s,y+z)-v(s,y))^2 .
\end{aligned}\tag{4.9}$$

Adding the first term on the right-hand side of (4.8) to the first term on the right-hand side of (4.9) yields

$$\begin{aligned}
&(\eta^+_\varepsilon)'(u(t,x)-v(s,y))\,(u(t,x+z)-u(t,x))\\
&- (\eta^+_\varepsilon)'(u(t,x)-v(s,y))\,(v(s,y+z)-v(s,y))\\
&= (\eta^+_\varepsilon)'(u(t,x)-v(s,y))\Big[(u(t,x+z)-v(s,y+z)) - (u(t,x)-v(s,y))\Big]\\
&\le \eta^+_\varepsilon(u(t,x+z)-v(s,y+z)) - \eta^+_\varepsilon(u(t,x)-v(s,y)),
\end{aligned}$$

where we have used the convexity of η_ε to derive the last inequality.

In view of these findings, we can rewrite $\widetilde{I}^\kappa_{\rm fdiff}(\varepsilon)$ as follows:

$$\begin{aligned}
\widetilde{I}^\kappa_{\rm fdiff}(\varepsilon) - I^\kappa_{\rm fdiss}(\varepsilon) &\le \iiiint \widetilde{\mathcal{L}}^\kappa_{x+y}\left[\eta^+_\varepsilon(u(t,\cdot)-v(s,\cdot))\right]\varphi\,dt\,dx\,dy\,ds\\
&\overset{(3.6)}{=} \iiiint \eta^+_\varepsilon(u-v)\widetilde{\mathcal{L}}^\kappa_{x+y}[\varphi]\,dt\,dx\,dy\,ds,
\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}
I^\kappa_{\rm fdiss}(\varepsilon) = \iiiint\int_{|z|>\kappa}&\Big[\overline{(\eta^+_\varepsilon)''}(u(t,\cdot)-v;z)\,(u(t,x+z)-u(t,x))^2\\
&+ \overline{(\eta^+_\varepsilon)''}(u-v(s,\cdot);z)\,(v(s,y+z)-v(s,y))^2\Big]\varphi\pi(dz)\,dx\,dt\,dy\,ds.
\end{aligned}$$

Consequently,

$$I^{\kappa}_{\rm fdiff}(\varepsilon) - I^{\kappa}_{\rm fdiss}(\varepsilon) \le \iiiint \eta^{+}_{\varepsilon}(u-v)\mathcal{L}^{\kappa}_{x+y}[\varphi]\,dt\,dx\,dy\,ds,$$

The next step is to first send $\kappa \to 0$ and then $\varepsilon \to 0$. Related to this, observe that

$$\lim_{\kappa\to 0} I^{\kappa}_{\rm fdiff}(\varepsilon) = I_{\rm fdiff}(\varepsilon), \quad \lim_{\kappa\to 0} I^{\kappa}_{\rm fdiss}(\varepsilon) = I_{\rm fdiss}(\varepsilon)$$

for each fixed $\varepsilon > 0$, by the dominated convergence theorem. Moreover, we clearly have $\lim_{\kappa\to 0}\mathcal{L}^{\kappa}_{x+y}[\varphi] = \mathcal{L}_{x+y}[\varphi]$. In view of this and (4.7), we conclude that

$$I_{\rm fdiff}(\varepsilon) - I_{\rm fdiss}(\varepsilon) \le \iiiint \eta^{+}_{\varepsilon}(u-v)\mathcal{L}_{x+y}[\varphi]\,dt\,dx\,dy\,ds. \tag{4.11}$$

By (4.6) and (4.11), it follows from (4.5) and sending $\varepsilon \to 0$ that

$$\begin{aligned} &\iiiint \Big((u-v)^{+}(\partial_t+\partial_s)\varphi + \sum_{i=1}^{d} q^{+}_i(u,v)(\partial_{x_i}+\partial_{y_i})\varphi \\ &+ \sum_{i,j=1}^{d} r^{+}_{ij}(u,v)(\partial^2_{x_ix_j} + 2\partial^2_{x_iy_j} + \partial^2_{y_iy_j})\varphi + (u-v)^{+}\mathcal{L}_{x+y}[\varphi]\Big)\,dx\,dt\,dy\,ds \\ &+ \iiint (u_0-v)^{+}\varphi(0,x,s,y)\,dx\,dy\,ds + \iiint (u-v_0)^{+}\varphi(t,x,0,y)\,dx\,dt\,dy \\ &\ge 0. \end{aligned} \tag{4.12}$$

Let us specify the test function $\varphi = \varphi(t,x,s,y)$. To this end, fix a nonnegative test function $\phi = \phi(t,x) \in C^{\infty}_c([0,\infty)\times\mathbb{R}^d)$, and pick two sequences $\{\theta_\nu\}_{\nu>0} \subset C^{\infty}_c(0,\nu)$, $\{\delta_\mu\}_{\mu>0} \subset C^{\infty}_c(B(0,\mu))$ of approximate delta functions, where $B(0,\mu)$ denotes the open ball centered at the origin with radius μ. Then take

$$\varphi(t,x,s,y) = \theta_\nu(s-t)\delta_\mu(y-x)\phi(t,x). \tag{4.13}$$

Simple calculations reveal that

$$\begin{gathered} (\partial_t+\partial_s)\varphi = \theta_\nu(s-t)\delta_\mu(y-x)\partial_t\phi(t,x), \\ (\partial_{x_i}+\partial_{y_i})\varphi = \theta_\nu(s-t)\delta_\mu(y-x)\partial_{x_i}\phi(t,x), \\ (\partial^2_{x_ix_j} + 2\partial^2_{x_iy_j} + \partial^2_{y_iy_j})\varphi = \theta_\nu(s-t)\delta_\mu(y-x)\partial^2_{x_ix_j}\phi(t,x), \\ \varphi(t,x+z,s,y+z) - \varphi(t,x,s,y) = \theta_\nu(s-t)\delta_\mu(y-x)\left(\phi(t,x+z)-\phi(t,x)\right). \end{gathered}$$

Note that $\theta_\nu = 0$ on $(-\infty,0]$ and so $\varphi(t,x,0,y) \equiv 0$. By the choice of the test function φ and the observations above, we deduce from (4.12) that

$$\begin{aligned} &\iiiint (u-v)^{+}\theta_\nu(s-t)\delta_\mu(y-x)\partial_t\phi(t,x)\,dx\,dt\,dy\,ds \\ &+ \iiiint \sum_{i=1}^{d} q^{+}_i(u,v)\theta_\nu(s-t)\delta_\mu(y-x)\partial_{x_i}\phi(t,x)\,dx\,dt\,dy\,ds \\ &+ \iiiint \sum_{i,j=1}^{d} r^{+}_{ij}(u,v)\theta_\nu(s-t)\delta_\mu(y-x)\partial^2_{x_ix_j}\phi(t,x)\,dx\,dt\,dy\,ds \\ &+ \iiiint (u-v)^{+}\theta_\nu(s-t)\delta_\mu(y-x)\mathcal{L}[\phi]\,dx\,dt\,dy\,ds + I_{u_0,v}(\nu,\mu) \ge 0, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} I_{u_0,v}(\nu,\mu) &:= \iiint (u_0-v)^+\theta_\nu(s)\delta_\mu(y-x)\phi(0,x)\,dx\,dy\,ds \\ &= -\iiint (u_0-v)^+\partial_s\Big(\tilde\phi_\nu(s)\delta_\mu(y-x)\phi(0,x)\Big)\,dx\,dy\,ds, \end{aligned}$$

with

$$\tilde\phi_\nu(s) := \int_s^T \theta_\nu(\tau)\,d\tau = \int_{\min(s,\nu)}^{\nu}\theta_\nu(\tau)\,d\tau \overset{\nu\to 0}{\longrightarrow} 1.$$

Specifying $\varphi = \phi(t,x)\tilde\phi_\nu(s)\delta_\mu(y-x)$ in the entropy inequality for v and noting that $\theta_\nu(s)$ vanishes for $s>\nu$, we obtain

$$\begin{aligned} &\iint (u_0-v)^+\partial_s\varphi(s,x,y)\,dy\,ds \\ &\le \iint (u_0-v_0)^+\theta_\nu(s)\delta_\mu(y-x)\phi(0,x)\,dy\,ds + o(\nu) \\ &\overset{\nu\to 0}{\longrightarrow} \iint (u_0-v_0)^+\delta_\mu(y-x)\phi(0,x)\,dy\,ds, \end{aligned} \tag{4.15}$$

where the "$o(\nu)$" term follows from an integrability argument.

Hence, sending $\nu,\mu\to 0$, we deduce

$$\begin{aligned} \limsup_{\mu\to 0}\limsup_{\nu\to 0} I_{u_0,v}(\nu,\mu) &\le \limsup_{\mu\to 0}\iint (u_0-v_0)^+\delta_\mu(y-x)\phi(0,x)\,dx\,dy \\ &= \int (u_0-v_0)^+\phi(0,x)\,dx, \end{aligned} \tag{4.16}$$

with $u_0=u_0(x)$ and $v_0=v_0(x)$.

Keeping in mind (4.16) when sending $\mu,\nu\to 0$ in (4.14), we conclude that

$$\begin{aligned} &\iint_{Q_T}\Big((u-v)^+\partial_t\phi + \sum_{i=1}^d q_i^+(u,v)\partial_{x_i}\phi \\ &+ \sum_{i,j=1}^d r_{ij}^+(u,v)\partial^2_{x_ix_j}\phi + (u-v)^+\mathcal{L}[\phi]\Big)\,dx\,dt + \int_{\mathbb{R}^d}(u_0-v_0)^+\phi(0,x)\,dx \\ &\ge 0, \end{aligned} \tag{4.17}$$

where all the involved functions depend on (t,x). It now only takes a standard argument to conclude from (4.17) that Theorem 2.2 holds. Indeed, one chooses a sequence of functions $0\le\phi\le 1$ from $C_c^\infty([0,T)\times\mathbb{R}^d)$ that converges to $\mathbf{1}_{[0,t)\times\mathbb{R}^d}$ for a Lebesgue point t of $\int_{\mathbb{R}^d}(u-v)^+\,dx$ and then use the integrability of u,v to conclude the proof.

This concludes the proof of Theorem 2.2.

5. Proof of Theorem 2.3 (continuous dependence)

We again employ the doubling of variables device as in the previous section, but with a slightly different choice of the entropy function. For each $\varepsilon>0$, define

$$\operatorname{sgn}_\varepsilon(\xi) = \begin{cases} -1, & \text{if } \xi < -\varepsilon \\ \sin(\pi\xi/(2\varepsilon)), & \text{if } |\xi|\le\varepsilon \\ 1, & \text{if } \xi>\varepsilon, \end{cases}$$

which is a C^1 approximation of $\operatorname{sgn}(\cdot)$. This choice gives rise to a C^2 approximation $\eta_\varepsilon(z) = \int_0^z \operatorname{sgn}_\varepsilon(\xi)\,d\xi$ of the entropy flux $|z|$. As before, we introduce the corresponding entropy flux functions $\eta^\varepsilon(u,c)$, $q_i^\varepsilon(u,c)$, and $r_{ij}^\varepsilon(u,c)$. We now employ the doubling variables technique using the test function

$$\varphi(t,x,s,y) = \theta_\nu(s-t)\delta_\mu(y-x)\Theta_\alpha(t),$$

where θ_ν, δ_μ are symmetric approximate delta functions with support in $(-\nu,\nu)$ and $B(0,\mu)$, respectively. Fix a time τ from $(0,T)$. For any $\alpha > 0$ with $0 < \alpha < \min(\tau_0, T-\tau)$, we define

$$\Theta_\alpha(t) = H_\alpha(t) - H_\alpha(t-\tau), \quad H_\alpha(t) = \int_{-\infty}^t \theta_\alpha(\sigma)\,d\sigma.$$

so that $\Theta_\alpha'(t) = \theta_\alpha(t) - \theta_\alpha(t-\tau)$.

Proceeding as in the previous section (cf. also [23]) and sending $\varepsilon \to 0$, we find

$$-\iiiint |u-v|\theta_\nu(s-t)\delta_\mu(y-x)\Theta_\alpha'(t)\,dx\,dt\,dy\,ds \le I_{\text{conv}} - I_{\text{diff}} + I_{\text{fdiff}},$$

where

$$\begin{gathered}
I_{\text{conv}} := \iiiint [G(u,v) - F(u,v)] \cdot \nabla_x \delta_\mu(y-x)\theta_\nu(s-t)\Theta_\alpha(t)\,dx\,dt\,dy\,ds, \\
F(u,v) := \operatorname{sgn}(u-v)\left(f(u) - f(v)\right), \quad G(u,v) := \operatorname{sgn}(u-v)\left(g(u) - g(v)\right),
\end{gathered}$$

$$I_{\text{diff}} := \iiiint \sum_{i,j=1}^d \Theta_\alpha(t)\theta_\nu(s-t)\partial^2_{x_i x_j}\delta_\mu(y-x) \int_v^u \operatorname{sgn}(\xi - v)\varepsilon_{ij}^{a-b}(\xi)\,d\xi\,dx\,dt\,dy\,ds,$$

$$\varepsilon_{ij}^{a-b}(\xi) := \sum_{k=1}^K \left(\sigma_{ik}^a(\xi)\sigma_{jk}^a(\xi) - 2\sigma_{ik}^a(\xi)\sigma_{jk}^b(\xi) + \sigma_{ik}^b(\xi)\sigma_{jk}^b(\xi)\right).$$

and $I_{\text{fdiff}} = I_{\text{fdiff}_1} + I_{\text{fdiff}_2}$ with

$$\begin{aligned}
I_{\text{fdiff}_1} := \iiiint \int_{|z|<1} & |u-v|\theta_\nu(s-t)\Theta_\alpha(t) \\
& \times \left[\delta_\mu(y-x-z) - \delta_\mu(y-x) - \nabla\delta_\mu(y-x)\cdot z\right](m(z) - \tilde{m}(z))\,dz\,dx\,dt\,dy\,ds
\end{aligned}$$

and

$$\begin{aligned}
I_{\text{fdiff}_2} := \iiiint \int_{|z|\ge 1} & |u-v|\theta_\nu(s-t)\Theta_\alpha(t)\left[\delta_\mu(y-x-z) - \delta_\mu(y-x)\right] \\
& \times (m(z) - \tilde{m}(z))\,dz\,dx\,dt\,dy\,ds.
\end{aligned}$$

By the triangle inequality

$$\begin{aligned}
& - \iiiint |u(t,x) - v(s,y)|\theta_\nu(s-t)\delta_\mu(y-x)\Theta_\alpha'(t)\,dx\,dt\,dy\,ds \\
& \ge - \iiiint |u(t,y) - v(t,y)|\theta_\nu(s-t)\delta_\mu(y-x)|\Theta_\alpha'(t)|\,dx\,dt\,dy\,ds \\
& \quad - \iiiint |v(t,y) - v(s,y)|\theta_\nu(s-t)\delta_\mu(y-x)|\Theta_\alpha'(t)|\,dx\,dt\,dy\,ds \\
& \quad - \iiiint |u(t,x) - u(t,y)|\theta_\nu(s-t)\delta_\mu(y-x)|\Theta_\alpha'(t)|\,dx\,dt\,dy\,ds \\
& =: L + R_t + R_x.
\end{aligned}$$

Keeping in mind that $v \in C(L^1)$ and $u \in L^\infty(BV)$, it is standard to show that

$$\lim_{\nu\to 0} R_t = 0, \quad \limsup_{\alpha\to 0} |R_x| \le C\mu$$

and moreover, since also $u(t) \to u_0, v(t) \to v_0$ as $t \to 0$,

$$\lim_{\alpha\to 0} L = \|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^1(\mathbb{R}^d)} - \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}.$$

Following [23], using $u \in L^\infty(BV)$ we conclude that

$$\lim_{\alpha\to 0}\lim_{\nu\to 0} |I_{\mathrm{conv}}| \le C\tau\|f - g\|_{\mathrm{Lip}(I)},$$

and, exploiting also that $\int |\partial_{x_i}\delta_\mu| \le C/\mu$,

$$\lim_{\alpha\to 0}\lim_{\nu\to 0} |I_{\mathrm{diff}}| \le \frac{C}{\mu}\tau\|(\sigma^a - \sigma^b)(\sigma^a - \sigma^b)^{\mathrm{tr}}\|_{L^\infty(I;\mathbb{R}^{d\times d})}.$$

It remains to estimate $|I_{\mathrm{fdiff}}|$. First, we nconsider I_{fdiff_1}. Using the Taylor and Fubini theorems we obtain

$$\begin{aligned} |I_{\mathrm{fdiff}_1}| = &\iiint \int_{|z|<1}\int_0^1 (1-\tau)\theta_\mu(s-t)\Theta_\alpha(t)(\tilde{m}(z) - m(z)) \\ &\times \Big(\int_{\mathbb{R}^d} |u(t,x) - v(s,y)| D^2\delta_\mu(y - x - \tau z)\, z\cdot z\, dx\Big)\, d\tau\, dz\, dy\, ds\, dt. \end{aligned}$$

Thanks to $|u(t,\cdot) - v(s,y)| \in BV(\mathbb{R}^d)$, an integration by parts yields

$$\begin{aligned} &I_{\mathrm{fdiff}_1} \\ &= \iiint \int_{|z|<1}\int_0^1 (1-\tau)\theta_\mu(s-t)\Theta_\alpha(t)(\tilde{m}(z) - m(z)) \\ &\quad \times \Big(\int_{\mathbb{R}^d} \nabla\delta_\mu(y - x - \tau z)\cdot z\, D_x\left(|u(t,x) - v(s,y)|\right)\cdot z\, dx\Big)\, d\tau\, dz\, dy\, ds\, dt, \end{aligned} \tag{5.1}$$

where the inner integral is taken with respect to the bounded Borel measure $D\left(|u(t,\cdot) - v(s,y)|\right)\cdot z$. Since $|D(u(t,\cdot) - v(s,y))| \le |D(u(t,\cdot))|$, the term inside the parentheses in (5.1), is upper bounded by

$$|z|^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla\delta_\mu(y - x - \tau z)|\, |dD(u(t,\cdot))(x)|\, dy \le |z|^2 |u(t,\cdot)|_{BV(\mathbb{R}^d)}\|\nabla\delta_\mu\|_{L^1(\mathbb{R}^d)},$$

where we have used that $|Du(t,\cdot)|$ is finite and the Fubini's theorem to first integrate with respect to y. Hence,

$$\lim_{\alpha\to 0}\lim_{\nu\to 0} |I_{\mathrm{fdiff}_1}| \le \frac{C}{\mu}\tau \int_{|z|<1} |z|^2 |m(z) - \tilde{m}(z)|\, dz,$$

where $C > 0$ is a finite constant.

Similarly, relying again on the $L^\infty(BV)$ regularity of u, it is not difficult to deduce via an integration by parts the estimate

$$\lim_{\nu\to 0}\lim_{\alpha\to 0} |I_{\mathrm{fdiff}_2}| \le C\tau \int_{|z|\ge 1} |z||m(z) - \tilde{m}(z)|\, dz.$$

Finally, we collect the bounds we have obtained so far and then optimize over μ to obtain the desired continuous dependence estimate (2.6).

6. More general equations

To motivate what follows, we recall that (formally) the Lévy part of the entropy condition (2.4) comes from multiplying the nonlocal operator $\mathcal{L}[u]$ by $\eta'(u)$ and computing the commutator $\mathcal{L}[\eta(u)] - \eta'(u)\mathcal{L}[u]$. As an alternative, we can replace the term

$$\iint_{Q_T} \eta(u)\mathcal{L}[\varphi]\,dx\,dt - m^u$$

occurring in (2.4) by

$$\begin{aligned} &\iint_{Q_T} \eta(u)\mathcal{L}_\kappa[\varphi]\,dx\,dt + \iint_{Q_T} \eta'(u)\widetilde{\mathcal{L}}^\kappa[u]\varphi\,dx\,dt \\ &+ u(t,x)\Big(\int_{|z|>\kappa} z\mathbf{1}_{|z|<1}\pi(dz)\Big)\cdot\nabla\varphi, \quad \kappa\in(0,1), \end{aligned} \tag{6.1}$$

cf. the proofs of Theorems 2.2 and 2.3 for the relevant notation.

This formulation of the nonlocal term is directly related to the formulation used in [1] for fractional conservation laws. The proof of Theorem 2.2 works equally well with this formulation of the Lévy part of the entropy condition. In fact, (the Lévy part of) the proof relies on two main properties, which both are available with (6.1): First, as $\kappa \to 0$, cf. (4.7),

$$|\iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(x)-u(y)|\Big(\mathcal{L}_{x,\kappa}[\phi] + \mathcal{L}_{y,\kappa}[\phi]\Big)\,dx\,dy| = o(\kappa), \tag{6.2}$$

and second the monotonicity property, cf. (4.10),

$$\begin{aligned} &\iint_{\mathbb{R}^d\times\mathbb{R}^d} \operatorname{sgn}(u(x)-v(y))\Big(\widetilde{\mathcal{L}}^\kappa[u](x) - \widetilde{\mathcal{L}}^\kappa[v](y)\Big)\phi(x,y)\,dx\,dy \\ &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(x)-v(y))|\widetilde{\mathcal{L}}^\kappa_{x+y}[\phi](x,y)\phi(x,y)\,dx\,dy, \end{aligned} \tag{6.3}$$

for $u,v\in L^\infty(\mathbb{R}^d)$ and $0\le\phi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$.

Now let $\beta:\mathbb{R}\to\mathbb{R}$ be a nondecreasing Lipschitz continuous function, and consider the equation

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}(a(u)\nabla u) + \mathcal{L}[\beta(u)], \tag{6.4}$$

where

$$\mathcal{L}[\beta(u)] = \int_{\mathbb{R}^d\setminus\{0\}} \big[\beta(u(t,x+z)) - \beta(u(t,x)) - z\cdot\nabla\beta(u)\,\mathbf{1}_{|z|<1}\big]\,\pi(dz). \tag{6.5}$$

Recently the authors of [3][1] analyzed the equation

$$\partial_t u + \operatorname{div} f(u) = \mathcal{L}[\beta(u)],$$

which is a special case of (6.4) (set $a\equiv 0$). Actually, the work [3] allowed for slightly more general Levy measures than we do in our framework, but we will not be concerned with this refinement here. The work [3] provides a series of results regarding stability and continuous dependence estimates.

Inspired by [3], our aim here is to outline a uniqueness (stability) proof for the more general equation (6.4). Combining this proof with arguments from [3], it is moreover possible to generalize the continuous dependence estimates in [3] to (6.4) (we leave the details to the interested reader).

---

[1]We are grateful to an anonymous referee for drawing our attention to this paper.

Let $\eta \in C^2(\mathbb{R})$ be convex and $u \in C^2(\mathbb{R})$. Then

$$\eta'(u(x))\mathcal{L}[\beta(u)] = \eta'(u(x))\mathcal{L}_\kappa[\beta(u)] + \eta'(u(x))\mathcal{L}^\kappa[\beta(u)], \quad \kappa \in (0,1).$$

Define $q_\beta, S_\beta : \mathbb{R} \to \mathbb{R}$ by

$$q_\beta' = \eta'\beta', \quad S_\beta' = \eta' \circ \beta^{-1},$$

where β^{-1} denotes, say, the left-continuous inverse of the nondecreasing function β, see also [6, Lemma 2.2]. One can check that $S_\beta(\beta(u)) = q_\beta(u)$.

By the convexity of S_β,

$$\begin{aligned}
&\eta'(u(t,x))\Big(\beta(u(t,x+z)) - \beta(u(t,x)) - z \cdot \nabla\beta(u(t,x))\mathbf{1}_{|z|<1}\Big)\\
&= S_\beta'(\beta(u(t,x))\Big(\beta(u(t,x+z)) - \beta(u(t,x)) - z \cdot \nabla\beta(u(t,x))\mathbf{1}_{|z|<1}\Big)\\
&\le S_\beta(\beta(u(t,x+z))) - S_\beta(\beta(u(t,x))) - z \cdot \nabla S_\beta(\beta(u(t,x)))\mathbf{1}_{|z|<1}\\
&= q_\beta(u(t,x+z)) - q_\beta(u(t,x)) - z \cdot \nabla q_\beta(u(t,x))\mathbf{1}_{|z|<1}.
\end{aligned}$$

Therefore,

$$\eta'(u(t,x))\mathcal{L}_\kappa[\beta(u)] \le \mathcal{L}_\kappa[q_\beta(u)],$$

and so for any non-negative $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \eta'(u(x))\mathcal{L}_\kappa[\beta(u)]\phi(x)\,dx \le \int_{\mathbb{R}^d} q_\beta(u(x))\mathcal{L}_\kappa[\phi]\,dx.$$

Summarizing, for any $\kappa \in (0,1)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \eta'(u(x))\mathcal{L}[\beta(u)]\phi(x)\,dx\\
&\le \int_{\mathbb{R}^d} q_\beta(u(x))\mathcal{L}_\kappa[\phi]\,dx + \int_{\mathbb{R}^d} \eta'(u(x))\mathcal{L}^\kappa[\beta(u)]\phi(x)\,dx\\
&\le \int_{\mathbb{R}^d} q_\beta(u(x))\mathcal{L}_\kappa[\phi]\,dx + \int_{\mathbb{R}^d} \eta'(u(x))\widetilde{\mathcal{L}}^\kappa[\beta(u)]\phi(x)\,dx + \int_{\mathbb{R}^d} q_{\beta,\kappa}(u) \cdot \nabla\phi\,dx,
\end{aligned}$$

where $q_{\beta,\kappa} : \mathbb{R} \to \mathbb{R}$ is defined by

$$q_{\beta,\kappa}'(u) = q_\beta'(u) \cdot \Big(\int_{|z|>\kappa} z\mathbf{1}_{|z|<1}\pi(dz)\Big) \overset{z\mapsto -z}{=} q_\beta'(u) \cdot \Big(-\int_{|z|>\kappa} z\mathbf{1}_{|z|<1}\pi(dz)\Big).$$

The above formal calculations motivate the following definition.

Definition 6.1. An entropy solution of the initial value problem (6.4)-(1.2) is a measurable function $u : Q_T \to \mathbb{R}$ satisfying the following conditions:

(1) $u \in L^\infty(Q_T)$, $u \in L^\infty(0,T;L^1(\mathbb{R}^d))$,

$$\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u) \in L^2(Q_T), \quad k = 1,\dots,K;$$

(2) For $k = 1,\dots,K$,

$$\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{a,\psi}(u) = \psi(u)\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^a(u), \quad \text{a.e. in } Q_T \text{ and in } L^2(Q_T),$$

for any $\psi \in C(\mathbb{R})$;

(3) For any entropy-entropy flux triple (η, q, r),

$$
\begin{aligned}
&\iint_{Q_T} \Big(\eta(u)\partial_t\varphi + \Big(q(u) + q_{\beta,\kappa}(u)\Big)\cdot\nabla\varphi + \sum_{i,j=1}^{d} r_{ij}(u)\partial^2_{x_i x_j}\varphi\Big)\,dx\,dt \\
&+ \iint_{Q_T} q_\beta(u)\mathcal{L}_\kappa[\varphi]\,dx\,dt + \iint_{Q_T} \eta'(u)\widetilde{\mathcal{L}}^\kappa[\beta(u)]\varphi\,dx\,dt \\
&+ \int_{\mathbb{R}^d} \eta(u_0)\varphi(0,x)\,dx \ge \iint_{Q_T} \eta''(u)\sum_{k=1}^{K}\Big(\sum_{i=1}^{d}\partial_{x_i}\zeta^a_{ik}(u)\Big)^2\varphi(t,x)\,dx\,dt,
\end{aligned}
$$

for all $0 \le \varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$ and any $\kappa \in (0,1)$.

Equipped with this definition, we can repeat many of the steps in the proof of Theorem 2.2. Indeed, in the present context, inequality (4.5) reads

$$
I_{\rm time}(\varepsilon) + I_{\rm conv}(\varepsilon) + I_{\rm diff}(\varepsilon) + I_{\rm fdiff}(\varepsilon) + I_{\rm init}(\varepsilon) \ge I_{\rm diss}(\varepsilon), \tag{6.6}
$$

where $I_{\rm time}, I_{\rm diff}, I_{\rm diss}$ are exactly as before, whereas

$$
\begin{aligned}
I_{\rm conv}(\varepsilon) &= \iiiint \Big(q_\varepsilon(u,v) + q_{\beta,\kappa}(u,v)\Big)\cdot\nabla_{x+y}\varphi\,dx\,dt\,dy\,ds, \\
I_{\rm fdiff}(\varepsilon) &= I(\kappa) + \iiiint \eta'_\varepsilon(u(t,x) - v(s,y)) \\
&\qquad \times \Big(\widetilde{\mathcal{L}}^\kappa[\beta(u)](t,x) - \widetilde{\mathcal{L}}^\kappa[\beta(v)](s,y)\Big)\varphi\,dx\,dt\,dy\,ds, \\
I(\kappa) &= \iiiint q_{\beta,\varepsilon}(u-v)\Big(\mathcal{L}_{x,\kappa}[\varphi] + \mathcal{L}_{y,\kappa}[\varphi]\Big)\,dx\,dt\,dy\,ds,
\end{aligned}
$$

with $q_{\beta,\varepsilon}$ defined by $q'_{\beta,\varepsilon} = \eta'_\varepsilon\beta'$; η_ε defined in the proof of Theorem 2.3; and the test function $\varphi \in C_c^\infty(Q_T\times Q_T)$ defined in (4.13).

Observe that

$$
|I(\kappa)| = o(\kappa) \quad \text{(independently of } \varepsilon), \tag{6.7}
$$

and

$$
\begin{aligned}
&\eta'_\varepsilon(u(t,x) - v(s,y))\Big(\widetilde{\mathcal{L}}^\kappa[\beta(u)](t,x) - \widetilde{\mathcal{L}}^\kappa[\beta(v)](s,y)\Big) \\
&= \int_{|z|>\kappa} \Big(\eta'_\varepsilon(u(t,x) - v(s,y))\,(\beta(u(t,x+z)) - \beta(v(s,y+z))) \\
&\quad - \eta'_\varepsilon(u(t,x) - v(s,y))\,(\beta(u(t,x)) - \beta(v(s,y)))\Big)\pi(dz) \\
&\le \int_{|z|>\kappa} \Big(|\beta(u(t,x+z)) - \beta(v(s,y+z))| \\
&\quad - \eta'_\varepsilon(u(t,x) - v(s,y))\,(\beta(u(t,x)) - \beta(v(s,y)))\Big)\pi(dz) \\
&\overset{\varepsilon\to 0}{\to} \int_{|z|>\kappa} \Big(|\beta(u(t,x+z)) - \beta(v(s,y+z))| \\
&\quad - |\beta(u(t,x)) - \beta(v(s,y))|\Big)\pi(dz) \quad \text{in } L^1(Q_T\times Q_T) \\
&= \widetilde{\mathcal{L}}^\kappa_{x+y}[|\beta(u) - \beta(v)|],
\end{aligned}
$$

where we have used $|\eta_\varepsilon'(\cdot)| \le 1$ and $\eta_\varepsilon(\cdot) \to \operatorname{sgn}(\cdot)$ as $\varepsilon \to 0$. Consequently,

$$\limsup_{\varepsilon\to 0} I_{\mathrm{fdiff}}(\varepsilon) \le \iiiint |\beta(u)-\beta(v)| \widetilde{\mathcal{L}}^{\kappa}_{x+y}[\varphi]\, dx\, dt\, dy\, ds + o(\kappa). \tag{6.8}$$

In view of (6.7) and (6.8), we can proceed as in the proof of Theorem 2.2 and eventually send $\varepsilon \to 0$ (keeping κ fixed) in (6.6), resulting in the inequality

$$\begin{aligned}
&\iiiint \Big(|u-v|(\partial_t+\partial_s)\varphi + q(u,v)\cdot\nabla_{x+y}\varphi \\
&+ |\beta(u)-\beta(v)|\Big(-\int_{|z|>\kappa} z\mathbf{1}_{|z|<1}\pi(dz)\Big)\cdot \nabla_{x+y}\varphi \\
&+ \sum_{i,j=1}^d r_{ij}(u,v)(\partial^2_{x_ix_j} + 2\partial^2_{x_iy_j} + \partial^2_{y_iy_j})\varphi \\
&+ |\beta(u)-\beta(v)|\widetilde{\mathcal{L}}^{\kappa}_{x+y}[\varphi]\Big)\, dx\, dt\, dy\, ds \\
&+ \iiint |u_0-v|\varphi(0,x,s,y)\, dx\, dy\, ds \\
&+ \iiint |u-v_0|\varphi(t,x,0,y)\, dx\, dt\, dy \ge o(\kappa),
\end{aligned} \tag{6.9}$$

where the integrands on the second and fourth lines added together becomes

$$|\beta(u)-\beta(v)|\mathcal{L}^{\kappa}_{x+y}[\varphi].$$

Sending $\kappa \to 0$ in (6.9) we arrive at, compare with (4.12),

$$\begin{aligned}
&\iiiint \Big(|u-v|(\partial_t+\partial_s)\varphi + \sum_{i=1}^d q_i(u,v)(\partial_{x_i}+\partial_{y_i})\varphi \\
&+ \sum_{i,j=1}^d r_{ij}(u,v)(\partial^2_{x_ix_j} + 2\partial^2_{x_iy_j} + \partial^2_{y_iy_j})\varphi + |\beta(u)-\beta(v)|\mathcal{L}_{x+y}[\varphi]\Big)\, dx\, dt\, dy\, ds \\
&+ \iiint |u_0-v|\varphi(0,x,s,y)\, dx\, dy\, ds + \iiint |u-v_0|\varphi(t,x,0,y)\, dx\, dt\, dy \ge 0.
\end{aligned}$$

We are now in a position to conclude the L^1 stability and uniqueness of entropy solutions as in the proof of Theorem 2.2. An existence proof can be given along the lines outlined in Section 3. We summarize with

Theorem 6.2. *Suppose f and a satisfy* (1.3) *and* (1.4)–(1.5)*, respectively, and that the Lévy measure $\pi(dz)$ satisfies* (1.7)*. Moreover, suppose $\beta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function. Then there exists an entropy solution of* (6.4)-(1.2)*. Let u, v be two entropy solutions of* (6.4) *with initial data $u|_{t=0} = u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$, $v|_{t=0} = v_0 \in (L^1\cap L^\infty)(\mathbb{R}^d)$. For a.e. $t \in (0,T)$, we have*

$$\int_{\mathbb{R}^d} |u(t,x)-v(t,x)|\, dx \le \int_{\mathbb{R}^d} |u_0-v_0|\, dx.$$

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References

[1] N. Alibaud. Entropy formulation for fractal conservation laws. *J. Evol. Equ.*, 7(1):145–175, 2007.

[2] N. Alibaud. Existence, uniqueness and regularity for nonlinear parabolic equations with nonlocal terms equations with nonlocal terms. *NoDEA Nonlinear Differential Equations Appl.*, 14(3-4):259–289, 2007.

[3] N. Alibaud, S. Cifani, and E. Jacobsen. Continuous dependence estimates for nonlinear fractional convection-diffusion equations. preprint available in arXiv:1105.2288, 2011.

[4] N. Alibaud, J. Droniou, and J. Vovelle. Occurrence and non-appearance of shocks in fractal Burgers equations. *J. Hyperbolic Differ. Equ.*, 4(3):479–499, 2007.

[5] O. Alvarez and A. Tourin. Viscosity solutions of nonlinear integro-differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(3):293–317, 1996.

[6] B. Andreianov, M. Bendahmane, and K. H. Karlsen. Discrete duality finite volume schemes for doubly nonlinear degenerate hyperbolic-parabolic equations. *J. Hyperbolic Differ. Equ.*, 7(1):1—67, 2010.

[7] M. Arisawa. A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(5):695–711, 2006.

[8] G. Barles, R. Buckdahn, and E. Pardoux. Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.*, 60(1-2):57–83, 1997.

[9] G. Barles and C. Imbert. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(3):567–585, 2008.

[10] M. Bendahmane and K. H. Karlsen. Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations. *SIAM J. Math. Anal.*, 36(2):405–422 (electronic), 2004.

[11] F. E. Benth, K. H. Karlsen, and K. Reikvam. Optimal portfolio selection with consumption and nonlinear integrodifferential equations with gradient constraint: a viscosity solution approach. *Finance and Stochastic*, 5:275–303, 2001.

[12] F. E. Benth, K. H. Karlsen, and K. Reikvam. Portfolio optimization in a Lévy market with intertemporal substitution and transaction costs. *Stoch. Stoch. Rep.*, 74(3-4):517–569, 2002.

[13] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[14] P. Biler, T. Funaki, and W. A. Woyczynski. Fractal Burgers equations. *J. Differential Equations*, 148(1):9–46, 1998.

[15] P. Biler, G. Karch, and W. A. Woyczyński. Asymptotics for conservation laws involving Lévy diffusion generators. *Studia Math.*, 148(2):171–192, 2001.

[16] M. Bossy and B. Jourdain. Rate of convergence of a particle method for the solution of a 1D viscous scalar conservation law in a bounded interval. *Ann. Probab.*, 30(4):1797–1832, 2002.

[17] L. Brandolese and G. Karch. Far field asymptotics of solutions to convection equation with anomalous diffusion. *J. Evol. Equ.*, 8(2):307–326, 2008.

[18] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[19] L. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[20] L. A. Caffarelli, S. Salsa, and L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.*, 171(2):425–461, 2008.

[21] L. Caffarelli and L. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math.*, 171:1903–1930, 2010.

[22] J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Ration. Mech. Anal.*, 147(4):269–361, 1999.

[23] G.-Q. Chen and K. H. Karlsen. Quasilinear anisotropic degenerate parabolic equations with time-space dependent diffusion coefficients. *Commun. Pure Appl. Anal.*, 4(2):241–266, 2005.

[24] G.-Q. Chen and K. H. Karlsen. L^1-framework for continuous dependence and error estimates for quasilinear anisotropic degenerate parabolic equations. *Trans. Amer. Math. Soc.*, 358(3):937–963 (electronic), 2006.

[25] G.-Q. Chen and B. Perthame. Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(4):645–668, 2003.

[26] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[27] J. Droniou, T. Gallouet, and J. Vovelle. Global solution and smoothing effect for a non-local regularization of a hyperbolic equation. *J. Evol. Equ.*, 3(3):499–521, 2003. Dedicated to Philippe Bénilan.

[28] J. Droniou and C. Imbert. Fractal first-order partial differential equations. *Arch. Ration. Mech. Anal.*, 182(2):299–331, 2006.

[29] L. C. Evans. Partial Differential Equations. *Providence, RI: American Mathematical Society*, 2002.

[30] M. G. Garroni and J. L. Menaldi. *Second order elliptic integro-differential problems*, volume 430 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[31] N. Jacob. *Pseudo differential operators and Markov processes. Vol. I.* Imperial College Press, London, 2001. Fourier analysis and semigroups.

[32] N. Jacob. *Pseudo differential operators & Markov processes. Vol. II.* Imperial College Press, London, 2002. Generators and their potential theory.

[33] N. Jacob. *Pseudo differential operators and Markov processes. Vol. III.* Imperial College Press, London, 2005. Markov processes and applications.

[34] E. R. Jakobsen and K. H. Karlsen. Continuous dependence estimates for viscosity solutions of integro-PDEs. *J. Differential Equations*, 212(2):278–318, 2005.

[35] E. R. Jakobsen and K. H. Karlsen. A "maximum principle for semicontinuous functions" applicable to integro-partial differential equations. *NoDEA Nonlinear Differential Equations Appl.*, 13(2):137–165, 2006.

[36] G. Karch, C. Miao, and X. Xu. On convergence of solutions of fractal Burgers equation toward rarefaction waves. *SIAM J. Math. Anal.*, 39(5):1536–1549, 2008.

[37] K. H. Karlsen and M. Ohlberger. A note on the uniqueness of entropy solutions of nonlinear degenerate parabolic equations. *J. Math. Anal. Appl.*, 275(1):439–458, 2002.

[38] K. H. Karlsen and N. H. Risebro. On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. *Discrete Contin. Dyn. Syst.*, 9(5):1081–1104, 2003.

[39] K.H. Karlsen and S. Ulusoy. Existence and numerics for entropy solutions to fractional quasi-linear anisotropic degenerate parabolic equations. in progress.

[40] A. Kiselev, F. Nazarov, and R. Shterenberg. Blow up and regularity for fractal Burgers equation. *Dyn. Partial Differ. Equ.*, 5(3):211–240, 2008.

[41] A. Kiselev, F. Nazarov, and R. Volberg. Global well-posedness for critical 2D dissipative quasi-geostrophic equation. *Invent. Math.*, 167(3):445–453, 2007.

[42] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[43] C. Mascia, A. Porretta, and A. Terracina. Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations. *Arch. Ration. Mech. Anal.*, 163(2):87–124, 2002.

[44] A. Michel and J. Vovelle. Entropy formulation for parabolic degenerate equations with general Dirichlet boundary conditions and application to the convergence of FV methods. *SIAM J. Numer. Anal.*, 41(6):2262–2293 (electronic), 2003.

[45] B. Perthame and P. E. Souganidis. Dissipative and entropy solutions to non-isotropic degenerate parabolic balance laws. *Arch. Ration. Mech. Anal.*, 170(4):359–370, 2003.

[46] H. Pham. Optimal stopping of controlled jump diffusion processes: a viscosity solution approach. *J. Math. Systems Estim. Control*, 8(1):27 pp. (electronic), 1998.

[47] K.-i. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[48] A. Sayah. Équations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. I. Unicité des solutions de viscosité. *Comm. Partial Differential Equations*, 16(6-7):1057–1074, 1991.

[49] A. Sayah. Équations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. II. Existence de solutions de viscosité. *Comm. Partial Differential Equations*, 16(6-7):1075–1093, 1991.

[50] L. Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana Univ. Math. J.*, 55(3):1155–1174, 2006.
[51] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.
[52] H. M. Soner. Optimal control of jump-Markov processes and viscosity solutions. In *Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986)*, volume 10 of *IMA Vol. Math. Appl.*, pages 501–511. Springer, New York, 1988.
[53] J. Wu. Inviscid limits and regularity estimates for the solutions of the 2-D dissipative Quasi-geostrophic equations. *Indiana Univ. Math. J.*, 46(2):1113–1124, 1997.
[54] J. Wu. Dissipative quasi-geostrophic equations with L^p data. *Electronic J. Diff. Eqns*, 56:1–13, 2001.

Kenneth Hvistendahl Karlsen
Centre of Mathematics for Applications, University of Oslo
P.O. Box 1053, Blindern, N-0316 Oslo, Norway
E-mail address: `kennethk@math.uio.no` `http://folk.uio.no/kennethk`

Süleyman Ulusoy
Department of Mathematics, Faculty of Education, Zirve University
Sahinbey, Gaziantep, 27270, Turkey
E-mail address: `suleyman.ulusoy@zirve.edu.tr` `http://person.zirve.edu.tr/ulusoy/`