POSITIVE SOLUTIONS TO GENERALIZED SECOND-ORDER THREE-POINT INTEGRAL BOUNDARY-VALUE PROBLEMS

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Abstract. In this article, by using Krasnoselskii’s fixed point theorem, we obtain single and multiple positive solutions to the nonlinear second-order three-point integral boundary value problem

\[ u''(t) + a(t)f(u(t)) = 0, \quad 0 < t < T, \]
\[ u(0) = \beta \int_0^\eta u(s) \, ds, \quad \alpha \int_0^\eta u(s) \, ds = u(T), \]

where \( 0 < \eta < T \), \( 0 < \alpha < \frac{2T}{\eta^2} \), \( 0 < \beta < \frac{2T - \alpha\eta^2}{\eta(2T - \eta)} \) are given constants. As an application, we give some examples that illustrate our results.

1. Introduction

We are interested in obtaining positive solutions of the second-order three-point integral boundary-value problem (BVP)

\[ u''(t) + a(t)f(u(t)) = 0, \quad t \in (0, T), \]  
\[ u(0) = \beta \int_0^\eta u(s) \, ds, \quad \alpha \int_0^\eta u(s) \, ds = u(T), \]

where \( 0 < \eta < T \), \( 0 < \alpha < \frac{2T}{\eta^2} \), \( 0 < \beta < \frac{2T - \alpha\eta^2}{\eta(2T - \eta)} \) and there exists a \( t_0 \in (0, T) \), such that \( a(t_0) > 0 \). Set

\[ f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}. \]

The study of the existence of solutions of multi-point boundary-value problems for linear second-order ordinary differential equations was initiated by Il’in and Moiseev [5]. Then Gupta [3] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, the existence of positive solutions for nonlinear second order three-point boundary-value problems has been studied by many authors by using a nonlinear alternative of the Leray-Schauder approach, coincidence degree theory, the fixed point theorem for cones and so on. We refer the reader to [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and the references therein. However, all of these papers are concerned with problems.

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with three-point boundary conditions consisting of restrictions on the slope of the solutions and the solutions themselves, for example:

\[ u(0) = 0, \quad \alpha u(\eta) = u(1); \]
\[ u(0) = \beta u(\eta), \quad \alpha u(\eta) = u(T); \]
\[ u'(0) = 0, \quad \alpha u(\eta) = u(1); \]
\[ u(0) - \beta u'(0) = 0, \quad \alpha u(\eta) = u(1); \]
\[ \alpha u(0) - \beta u'(0) = 0, \quad u'(\eta) + u'(1) = 0; \text{ etc.} \]

Recently, Tariboon [20] and the author proved the existence of positive solutions for the three-point boundary-value problem with integral condition

\[ u''(t) + a(t)f(u(t)) = 0, \quad t \in (0, 1), \]
\[ u(0) = 0, \quad \alpha \int_0^\eta u(s)\,ds = u(1), \quad \alpha \in (0, 1). \]

where \( 0 < \eta < 1 \) and \( 0 < \alpha < 2/\eta^2 \). We note that the three-point integral boundary conditions (1.2) and (1.4) are related to the area under the curve of solutions \( u(t) \) from \( t = 0 \) to \( t = \eta \).

The aim of this article is to establish some simple criteria for the existence of single positive solution for (1.1), (1.2) under \( f_0 = 0, f_\infty = \infty \) (\( f \) is superlinear) or \( f_0 = \infty, f_\infty = 0 \) (\( f \) is sublinear). Moreover, we establish the existence conditions of two positive solutions for (1.1), (1.2) under \( f_0 = f_\infty = \infty \) or \( f_0 = f_\infty = 0 \). Finally, we give some examples to illustrate our results. The key tool in our approach is the Krasnoselskii’s fixed point theorem in a cone.

**Theorem 1.1** (1). Let \( E \) be a Banach space, and let \( K \subset E \) be a cone. Assume \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1, \Omega_1 \subset \Omega_2 \), and let

\[ A : K \cap (\overline{\Omega_1} \setminus \Omega_2) \to K \]

be a completely continuous operator such that

(i) \( \|Au\| \leq \|u\|, \ u \in K \cap \partial \Omega_1 \), and \( \|Au\| \geq \|u\|, \ u \in K \cap \partial \Omega_2 \); or

(ii) \( \|Au\| \geq \|u\|, \ u \in K \cap \partial \Omega_1 \), and \( \|Au\| \geq \|u\|, \ u \in K \cap \partial \Omega_2 \).

Then \( A \) has a fixed point in \( K \cap (\overline{\Omega_2} \setminus \Omega_1) \).

2. Preliminaries

We now state and prove several lemmas before stating our main results.

**Lemma 2.1.** Let \( \beta \neq \frac{2T-\alpha^2}{\eta(2T-\eta)} \). Then for \( \gamma \in C[0, T] \), the problem

\[ u'' + \gamma(t) = 0, \quad t \in (0, T), \]
\[ u(0) = \beta \int_0^\eta \gamma(s)\,ds, \quad \alpha \int_0^\eta u(s)\,ds = u(T), \]

has a unique solution

\[ u(t) = \frac{(\beta - \alpha)t - \beta T}{(2T - \alpha^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2 \gamma(s)\,ds \]
\[ + \frac{2(1 - \beta \eta)t + \beta \eta^2}{(2T - \alpha^2) - \beta \eta(2T - \eta)} \int_0^T (T - s) \gamma(s)\,ds - \int_0^t (t - s) \gamma(s)\,ds. \]
Hence, (2.1)-(2.2) has a unique solution.

For $t \in [0,T]$, integrating from 0 to $t$, we obtain

$$u'(t) = u'(0) - \int_0^t y(s)ds.$$  

From (2.1), we have

Proof.

By (2.2), from $u(T)$, we obtain

$$u(t) = u(0) + u'(0)t - \int_0^t \left( \int_0^x y(s)ds \right)dx;$$

i.e.,

$$u(t) = u(0) + u'(0)t - \int_0^t (t - s)y(s)ds := A + Bt - \int_0^t (t - s)y(s)ds. \quad (2.3)$$

Integrating (2.3) from 0 to $\eta$, where $\eta \in (0,T)$, we have

$$\int_0^\eta u(s)ds = \eta A + \frac{\eta^2}{2} B - \int_0^\eta \left( \int_0^x (x - s)y(s)ds \right)dx$$

$$= \eta A + \frac{\eta^2}{2} B - \frac{1}{2} \int_0^\eta (\eta - s)^2 y(s)ds.$$ 

Since $u(0) = A$,

$$u(T) = A + B\int_0^T (T - s)y(s)ds.$$ 

By (2.2), from $u(0) = \beta \int_0^\eta u(s)ds$ we have

$$(1 - \beta\eta)A - \frac{\beta\eta^2}{2} B = -\frac{\beta}{2} \int_0^\eta (\eta - s)^2 y(s)ds,$$

and from $u(T) = \alpha \int_0^\eta u(s)ds$ we have

$$(1 - \alpha\eta)A + \left( T - \frac{\alpha\eta^2}{2} \right) B = \int_0^T (T - s)y(s)ds - \frac{\alpha}{2} \int_0^\eta (\eta - s)^2 y(s)ds.$$ 

Therefore,

$$A = \frac{\beta\eta^2}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T (T - s)y(s)ds$$

$$- \frac{\beta T}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^\eta (\eta - s)^2 y(s)ds$$

$$B = \frac{2(1 - \beta\eta)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T (T - s)y(s)ds$$

$$+ \frac{(\beta - \alpha)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^\eta (\eta - s)^2 y(s)ds.$$ 

Hence, (2.1)-(2.2) has a unique solution

$$u(t) = \frac{(\beta - \alpha)t - \beta T}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^\eta (\eta - s)^2 y(s)ds$$

$$+ \frac{2(1 - \beta\eta)t + \beta\eta^2}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T (T - s)y(s)ds - \int_0^t (t - s)y(s)ds.$$ 

□
Lemma 2.2. Let $0 < \alpha < \frac{2T}{\eta'}$, $0 < \beta < \frac{2T-\alpha\eta^2}{\eta(2T-\eta)}$. If $y \in C(0, T)$ and $y(t) \geq 0$ on $(0, T)$, then the unique solution $u$ of (2.1)-(2.2) satisfies $u(t) \geq 0$ for $t \in [0, T]$.

Proof. It is known that the graph of $u$ is concave down on $[0, T]$ from $u''(t) = -y(t) \leq 0$, we obtain
\[
\int_0^\eta u(s)ds \geq \frac{1}{2} \eta(u(0) + u(\eta)),
\] (2.4)
where $\frac{1}{2}\eta(u(0) + u(\eta))$ is the area of the trapezoid under the curve $u(t)$ from $t = 0$ to $t = \eta$ for $\eta \in (0, T)$. Combining (2.4) with (2.2), we can get
\[
\begin{aligned}
u(0) &\geq \frac{\beta\eta}{2 - \beta\eta} u(\eta), \\
u(T) &\geq \frac{\alpha\eta}{2 - \beta\eta} u(\eta),
\end{aligned}
\] (2.5) (2.6)
such that
\[
2 - \beta\eta > 2 - \frac{2T - \alpha\eta^2}{2T - \eta} = \frac{2T - \eta + 2\eta^2}{2T - \eta} > 0.
\] (2.7)
From the graph of $u$ being concave down on $[0, T]$ again, we obtain
\[
\frac{u(\eta) - u(0)}{\eta} \geq \frac{u(T) - u(0)}{T}.
\] (2.8)
Using (2.5), (2.6) and (2.8), we obtain
\[
\frac{2 - 2\beta\eta}{\eta} u(\eta) \geq \frac{(\alpha - \beta)\eta}{T} u(\eta).
\]
If $u(0) < 0$, then $u(\eta) < 0$. It implies $\frac{2T - \alpha\eta^2}{\eta(2T - \eta)} \leq \beta$, a contradiction to $\beta < \frac{2T - \alpha\eta^2}{\eta(2T - \eta)}$.

If $u(T) < 0$, then $u(\eta) < 0$, and the same contradiction emerges. Thus, it is true that $u(0) \geq 0$, $u(T) \geq 0$, together with the concavity of $u$, we have $u(t) \geq 0$ for $t \in [0, T]$. This proof is complete. □

Lemma 2.3. Let $\alpha\eta^2 \neq 2T$, $\beta > \max\left\{ \frac{2T - \alpha\eta^2}{\eta(2T - \eta)}, 0 \right\}$. If $y \in C(0, T)$ and $y(t) \geq 0$ for $t \in [0, T]$, then problem (2.1)-(2.2) has no positive solutions.

Proof. Suppose that (2.1)-(2.2) has a positive solution $u$ satisfying $u(t) \geq 0$, $t \in [0, T]$ and there is a $\tau_0 \in (0, T)$ such that $u(\tau_0) > 0$.

If $u(T) > 0$, then $\int_0^\eta u(s)ds > 0$. It implies
\[
u(0) = \beta \int_0^\eta u(s)ds > \frac{2T - \alpha\eta^2}{\eta(2T - \eta)} \int_0^\eta u(s)ds \geq \frac{\eta T(u(0) + u(\eta)) - \eta^2 u(T)}{\eta(2T - \eta)};\]
(2.9)
that is
\[
\frac{u(T) - u(0)}{T} \geq \frac{u(\eta) - u(0)}{\eta},
\] (2.10)
which is a contradiction to the concavity of $u$.

If $u(T) = 0$, then $\int_0^\eta u(s)ds = 0$. When $\tau_0 \in (0, \eta)$, we obtain $u(\tau_0) > u(T) = 0 > u(\eta)$, which contradicts the concavity of $u$. When $\tau_0 \in (\eta, T)$, we obtain $u(\eta) \leq 0 = u(0) < u(\tau_0)$, which contradicts the concavity of $u$ again. Therefore, no positive solutions exist. □

Let $E = C[0, T]$, then $E$ is a Banach space with respect to the norm
\[
\|u\| = \sup_{t \in [0, T]} |u(t)|.
\]
Lemma 2.4. Let $0 < \alpha < \frac{2T}{\eta^2}$, $0 < \beta < \frac{2T-\alpha\eta^2}{\eta(2\beta-\eta)}$. If $y \in C(0,T)$ and $y(t) \geq 0$ for $t \in [0,T]$, then the unique solution to problem \((2.1) - (2.2)\) satisfies
\[
\min_{t \in [0,T]} u(t) \geq \gamma \|u\|, \tag{2.11}
\]
where
\[
\gamma := \min \left\{ \frac{\alpha \eta (T - \eta)}{T(2 - \beta \eta) - \alpha \eta^2}, \frac{\beta \eta (T - \eta)}{(2 - \beta \eta) T - \gamma^2}, \frac{\beta \eta^2}{(2 - \beta \eta) T} \right\}. \tag{2.12}
\]

Proof. From the fact that $u''(t) = -y(t) \leq 0$, we know that the graph of $u(t)$ is concave down on $[0,T]$. If $u(t)$ is maximum at $t = \tau_1$, then $\|u\| = u(\tau_1)$. We divide the proof into two cases.

Case (i) If $u(0) \geq u(T)$ and $\min_{t \in [0,T]} u(t) = u(T)$, then either $0 \leq \tau_1 \leq \eta < T$, or $0 < \eta < \tau_1 < T$. If $0 \leq \tau_1 \leq \eta < T$, then
\[
u(\tau_1) \leq u(T) + \frac{u(T) - u(\eta)}{T - \eta} (\tau_1 - T)
\leq u(T) + \frac{u(T) - u(\eta)}{T - \eta} (0 - T)
= \frac{T u(\eta) - \eta u(T)}{T - \eta}
\leq \frac{T}{T - \eta} \left( \frac{2 - \beta \eta}{\alpha \eta} \right) u(T) \quad \text{(by (2.6))}
= \frac{T(2 - \beta \eta) - \alpha \eta^2}{\alpha \eta (T - \eta)} u(T).
\]
This implies
\[
\min_{t \in [0,T]} u(t) \geq \frac{\alpha \eta (T - \eta)}{T(2 - \beta \eta) - \alpha \eta^2} \|u\|.
\]
If $0 < \eta < \tau_1 < T$, from
\[
\frac{u(\eta)}{\eta} \geq \frac{u(\tau_1)}{\tau_1} \geq \frac{u(\tau_1)}{T},
\]
together with (2.6), we have
\[
u(T) \geq \frac{\alpha \eta^2}{(2 - \beta \eta) T} u(\tau_1).
\]
This implies
\[
\min_{t \in [0,T]} u(t) \geq \frac{\alpha \eta^2}{(2 - \beta \eta) T} \|u\|.
\]
Case (ii) If $u(0) \leq u(T)$ and $\min_{t \in [0,T]} u(t) = u(0)$, then either $0 < \tau_1 < \eta < T$, or $0 < \eta \leq \tau_1 \leq T$. If $0 < \tau_1 < \eta < T$, from
\[
\frac{u(\eta)}{T - \eta} \geq \frac{u(\tau_1)}{T - \tau_1} \geq \frac{u(\tau_1)}{T},
\]
jointly with (2.5), we have
\[
u(0) \geq \frac{\beta \eta (T - \eta)}{(2 - \beta \eta) T} u(\tau_1).
Hence
\[ \min_{t \in [0, T]} u(t) \geq \frac{\beta \eta (T - \eta)}{(2 - \beta \eta)T} \|u\|. \]
If \(0 < \eta \leq \tau_1 \leq T\), from
\[ \frac{u(\tau_1)}{T} \leq \frac{u(\tau_1)}{\tau_1} \leq \frac{u(\eta)}{\eta}, \]
together with (2.5), we have
\[ u(0) \geq \frac{\beta \eta^2}{(2 - \beta \eta)T} u(\tau_1). \]
This implies
\[ \min_{t \in [0, T]} u(t) \geq \frac{\beta \eta^2}{(2 - \beta \eta)T} \|u\|. \]
This completes the proof. \(\Box\)

In the rest of this article, we assume that \(0 < \alpha < 2T/\eta^2\), \(0 < \beta < \frac{2T - \alpha \eta^2}{\eta(2T - \eta)}\). It is easy to see that \((1.1)-(1.2)\) has a solution \(u = u(t)\) if and only if \(u\) is a solution of the operator equation
\[
u(t) = \frac{(\beta - \alpha)t - \beta T}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2 a(s) f(u(s)) ds \\
+ \frac{2(1 - \beta \eta)t + \beta \eta^2}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s)a(s) f(u(s)) ds \\
- \int_0^t (t - s)a(s) f(u(s)) ds \triangleq Au(t).
\]
Denote
\[ K = \{u \in E : u \geq 0, \min_{t \in [0, T]} u(t) \geq \gamma \|u\|\}, \tag{2.13} \]
where \(\gamma\) is defined in (2.12).

It is obvious that \(K\) is a cone in \(E\). Moreover from Lemma 2.2 and Lemma 2.4, \(A(K) \subset K\). It is also easy to check that \(A : K \to K\) is completely continuous. In the following, for the sake of convenience, set
\[
\Lambda_1 = \frac{2T + \beta(T + \eta^2)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s)a(s) ds, \\
\Lambda_2 = \frac{\gamma(2 - \beta \eta)(T - \eta)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T sa(s) ds.
\]

3. Main results

Now we are in the position to establish the main result.

**Theorem 3.1.** Problem \((1.1)-(1.2)\) has at least one positive solution under the assumptions:

(H1) \(f_0 = 0\) and \(f_\infty = \infty\) (superlinear); or
(H2) \(f_0 = \infty\) and \(f_\infty = 0\) (sublinear).
Proof. At first, let (H1) hold. Since \( f_0 = \lim_{u \to 0^+} (f(u)/u) = 0 \) for any \( \varepsilon \in (0, \Lambda^{-1}] \), there exists \( \rho_* \) such that
\[
\|u\| \leq \varepsilon u \quad \text{for} \quad u \in [0, \rho_*]. \tag{3.1}
\]
Let \( \Omega_{\rho_*} = \{ u \in E : \|u\| < \rho_* \} \) for any \( u \in K \cap \partial \Omega_{\rho_*} \). From (3.1), we obtain
\[
Au(t) \leq \frac{(\beta - \alpha)t - \beta T}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2 a(s) f(u(s)) ds
\]
\[
+ \frac{2(1 - \beta\eta)t + \beta \eta^2}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s)a(s)f(u(s)) ds
\]
\[
\leq \frac{\beta t}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2 a(s) f(u(s)) ds
\]
\[
+ \frac{2T + \beta \eta^2}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s)a(s)f(u(s)) ds
\]
\[
\leq \frac{\beta T}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2 a(s) f(u(s)) ds
\]
\[
+ \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^T (T^2 - sT)a(s)f(u(s)) ds
\]
\[
\leq \varepsilon \rho_* \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^T T(T - s)a(s) ds
\]
\[
= \varepsilon \Lambda_1 \rho_* \leq \rho_* = \|u\|,
\]
which yields
\[
\|Au\| \leq \|u\| \quad \text{for} \quad u \in K \cap \partial \Omega_{\rho_*}. \tag{3.2}
\]
Further, since \( f_\infty = \lim_{u \to \infty} (f(u)/u) = \infty \), for any \( M^* \in [\Lambda^{-1}_2, \infty) \), there exists \( \rho^* > \rho_* \) such that
\[
f(u) \geq M^* u \quad \text{for} \quad u \geq \gamma \rho^*. \tag{3.3}
\]
Set \( \Omega_{\rho^*} = \{ u \in E : \|u\| < \rho^* \} \) for any \( u \in K \cap \partial \Omega_{\rho^*} \). Since \( u \in K \), \( \min_{t \in [0,T]} u(t) \geq \gamma \|u\| = \gamma \rho^* \). Hence, for any \( u \in K \cap \Omega_{\rho^*} \), from (3.3) and (2.7), we obtain
\[
Au(\eta) = \frac{(\beta - \alpha)\eta - \beta T}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2 a(s) f(u(s)) ds
\]
\[
+ \frac{2\beta \eta - \beta T}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s)a(s)f(u(s)) ds
\]
\[
- \int_0^\eta (\eta - s)a(s)f(u(s)) ds
\]
\[
= \frac{2\beta \eta - \beta T}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s)a(s)f(u(s)) ds
\]
\[
+ \frac{1}{(2T - \alpha\eta^2) - \beta \eta(2T - \eta)} \times \int_0^\eta (\eta - s) \left[ -(2 - \beta\eta)T + (\beta(T - \eta) + \alpha\eta)a(s)f(u(s)) ds
\]
\[ \frac{(2 - \beta \eta)\eta}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s) a(s) f(u(s)) \, ds \]
\[ + \frac{(2 - \beta \eta)\eta}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)(2 - \beta \eta) a(s) f(u(s)) \, ds \]
\[ \geq \frac{(2 - \beta \eta)\eta}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s) a(s) f(u(s)) \, ds \]
\[ + \frac{(2 - \beta \eta)\eta}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)(2 - \beta \eta) a(s) f(u(s)) \, ds \]
\[ = \frac{(2 - \beta \eta)(T - \eta)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T sa(s) \, ds \]
\[ = M^* \Lambda_2 \rho^* \geq \rho^* = \|u\|, \]

which implies
\[ \|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial \Omega_{r^*}. \quad (3.4) \]

Therefore, from (3.2), (3.4) and Theorem 1.1, it follows that \( A \) has a fixed point in \( K \cap (\overline{\Omega}_r \setminus \Omega_{\rho^*}) \) such that \( \rho^* \leq \|u\| \leq r^* \).

Next, let (H2) hold. In view of \( f_0 = \lim_{u \to 0^+} f(u)/u = \infty \) for any \( M_* \in [\Lambda_{\alpha}^{-1}, \infty) \), there exists \( r_* > 0 \) such that
\[ f(u) \geq M_* u \quad \text{for } 0 \leq u \leq r_. \quad (3.5) \]

Set \( \Omega_{r_*} = \{ u \in E : \|u\| < r_* \} \) for \( u \in K \cap \partial \Omega_{r_*} \). Since \( u \in K \), it follows that \( \min_{t \in [0, T]} u(t) \geq \gamma \|u\| = \gamma r_* \). Thus from (3.5) for any \( u \in K \cap \partial \Omega_{r_*} \), we have
\[ Au(\eta) = \frac{(\beta - \alpha)\eta - \beta T}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2 a(s) f(u(s)) \, ds \]
\[ + \frac{(2 - \beta \eta)\eta}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s) a(s) f(u(s)) \, ds \]
\[ - \int_0^\eta (\eta - s) a(s) f(u(s)) \, ds \]
\[ \geq \gamma r_* M_* \frac{(2 - \beta \eta)(T - \eta)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T sa(s) \, ds \]
\[ = M_* \Lambda_2 r_* \geq r_* = \|u\|, \]

which yields
\[ \|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial \Omega_{r_*}. \quad (3.6) \]

Since \( f_\infty = \lim_{u \to \infty} f(u)/u = 0 \), for any \( \varepsilon_1 \in (0, \Lambda_{\alpha}^{-1}] \), there exists \( r_0 > r_* \) such that
\[ f(u) \leq \varepsilon_1 u \quad \text{for } u \in [r_0, \infty). \quad (3.7) \]

We have the next two cases:

Case (i): Suppose that \( f(u) \) is unbounded, then from \( f \in C([0, \infty), [0, \infty)) \), we know that there is \( r^* > r_0 \) such that
\[ f(u) \leq f(r^*) \quad \text{for } u \in [0, r^*]. \quad (3.8) \]
Since \( r^* > r_0 \), from (3.7) and (3.8), one has
\[
0 \leq f(u) \leq r^* \leq \varepsilon_1 r^* \quad \text{for} \quad u \in [0, r^*].
\]
(3.9)

For \( u \in K \), \( \|u\| = r^* \), from (3.9), we obtain
\[
Au(t) \leq \frac{2T + \beta(T + \eta^2)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T T(T-s)a(s)f(u(s))ds \leq \varepsilon_1 r^* \frac{2T + \beta(T + \eta^2)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T T(T-s)a(s)ds
\]
\[
= \varepsilon_1 \Lambda_1 \varepsilon_1 r^* \leq r^* = \|u\|.
\]

Case (ii) Suppose that \( f(u) \) is bounded, say \( f(u) \leq N \) for all \( u \in [0, \infty) \). Taking \( r^* \geq \max \{N/\varepsilon_1, r_*\} \), for \( u \in K \), \( \|u\| = r^* \), we have
\[
Au(t) \leq \frac{2T + \beta(T + \eta^2)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T T(T-s)a(s)f(u(s))ds \leq N \frac{2T + \beta(T + \eta^2)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T T(T-s)a(s)ds \leq \varepsilon_1 r^* \frac{2T + \beta(T + \eta^2)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T T(T-s)a(s)ds
\]
\[
= \varepsilon_1 \Lambda_1 \varepsilon_1 r^* \leq r^* = \|u\|.
\]

Hence, in either case, we always may set \( \Omega_{r^*} = \{u \in E : \|u\| < r^*\} \) such that
\[
\|Au\| \leq \|u\| \quad \text{for} \quad u \in K \cap \partial \Omega_{r^*}.
\]
(3.10)

Hence, from (3.6), (3.10) and Theorem 1.1, it follows that \( A \) has a fixed point in \( K \cap (\bar{\Omega}_{r^*} \setminus \Omega_{r_*}) \) such that \( r_* \leq \|u\| \leq r^* \). The proof is complete. \( \square \)

**Theorem 3.2.** Suppose that the following assumptions are satisfied:

(H3) \( f_0 = f_\infty = \infty \),

(H4) There exists a constant \( \rho_1 > 0 \), such that \( f(u) \leq \Lambda_1^{-1} \rho_1 \) for \( u \in [0, \rho_1] \).

Then 1.1, 1.2 has at least two positive solutions \( u_1 \) and \( u_2 \) such that
\[
0 < \|u_1\| < \rho_1 < \|u_2\|.
\]

**Proof.** At first, in view of \( f_0 = \lim_{u \to 0^+} (f(u)/u) = \infty \), for any \( M_* \in [\Lambda_2^{-1}, \infty) \), there exists \( \rho_* \in (0, \rho_1) \) such that
\[
f(u) \geq M_* u, \quad \text{for} \quad 0 \leq u \leq \rho_*.
\]
(3.11)

Set \( \Omega_{\rho_*} = \{u \in E : \|u\| < \rho_*\} \) for \( u \in K \cap \partial \Omega_{\rho_*} \). Since \( u \in K \), then \( \min_{t \in [0,T]} u(t) \geq \gamma \|u\| = \gamma \rho_* \). Thus from (3.11), for any \( u \in K \cap \partial \Omega_{\rho_*} \), we obtain
\[
Au(\eta) \leq \frac{(\beta - \alpha)\eta - \beta T}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2a(s)f(u(s))ds + \frac{(2 - \beta)\eta\eta}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T (T-s)a(s)f(u(s))ds
\]
\[
- \int_0^\eta (\eta - s)a(s)f(u(s))ds
\]
\[
\geq \gamma \rho_* M_* \frac{(2 - \beta)\eta(T - \eta)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T sa(s)ds
\]
Thus, from (3.12), (3.14) and (3.15), it follows from Theorem 1.1 that
\[ ||Au|| \geq ||u|| \text{ for } u \in K \cap \partial \Omega_{\rho_*}, \] (3.12)
Next, since \( f_\infty = \lim_{u \to -\infty} (f(u)/u) = \infty \), then for any \( M^* \in [\Lambda_2^{-1}, \infty) \), there exists \( \rho^* > \rho_1 \) such that
\[ f(u) \geq M^* u, \quad u \geq \gamma \rho^*. \] (3.13)
Set \( \Omega_{\rho^*} = \{ u \in E : ||u|| < \rho^* \} \) for \( u \in K \cap \partial \Omega_{\rho^*}. \) Since \( u \in K \), then \( \min_{t \in [0, T]} u(t) \geq \gamma ||u|| = \gamma \rho^* \). Thus from (3.13), for any \( u \in K \cap \partial \Omega_{\rho^*} \), we have
\[
Au(\eta) = \frac{(\beta - \alpha)\eta - \beta T}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^\eta (\eta - s)^2 a(s)f(u(s))ds \\
+ \frac{(2 - \beta \eta)\eta}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T (T - s)a(s)f(u(s))ds \\
- \int_0^\eta (\eta - s)a(s)f(u(s))ds \\
\geq \gamma \rho^* M^* \frac{(2 - \beta \eta)(T - \eta)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T sa(s)ds \\
= M^* \Lambda_2 \rho^* \geq \rho^* = ||u||,
\]
which implies
\[ ||Au|| \geq ||u|| \text{ for } u \in K \cap \partial \Omega_{\rho^*}. \] (3.14)
Finally, let \( \Omega_{\rho_1} = \{ u \in E : ||u|| < \rho_1 \} \) for any \( u \in K \cap \partial \Omega_{\rho^*}. \) Then from (H4) we obtain
\[
Au(t) \leq \frac{2T + \beta(T + \eta^2)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T T(T - s)a(s)f(u(s))ds \\
\leq \Lambda_1^{-1} \rho_1 \frac{2T + \beta(T + \eta^2)}{(2T - \alpha \eta^2) - \beta \eta(2T - \eta)} \int_0^T T(T - s)a(s)ds \\
\leq \rho_1 = ||u||,
\]
which yields
\[ ||Au|| \leq ||u|| \text{ for } u \in K \cap \partial \Omega_{\rho_*}. \] (3.15)
Thus, from (3.12), (3.14) and (3.15), it follows from Theorem 1.1 that \( A \) has a fixed point \( u_1 \) in \( K \cap (\Omega_{\rho_1} \setminus \Omega_{\rho^*}) \); and a fixed point \( u_2 \) in \( K \cap (\Omega_{\rho^*} \setminus \Omega_{\rho_1}) \). Both are positive solutions of (1.1), (1.2) and \( 0 < ||u_1|| < \rho_1 < ||u_2||. \) The proof is complete. \( \square \)

**Theorem 3.3.** Suppose that the following assumptions are satisfied:
(H5) \( f_0 = f_\infty = 0 \),
(H6) There exists a constant \( \rho_2 > 0 \), such that
\[ f(u) \geq \Lambda_2^{-1} \rho_2 \text{ for } u \in [\gamma \rho_2, \rho_2]. \]
Then (1.1), (1.2) has at least two positive solutions \( u_1 \) and \( u_2 \) such that
\[ 0 < ||u_1|| < \rho_2 < ||u_2||. \]

**Proof.** Firstly, since \( f_0 = \lim_{u \to 0^+} (f(u)/u) = 0 \), for any \( \varepsilon \in (0, \Lambda_1^{-1}] \), there exists \( \rho_* \in (0, \rho_2) \) such that
\[ f(u) \leq \varepsilon u, \quad u \in [0, \rho_2]. \] (3.16)
Let $\Omega_{\rho_\ast} = \{ u \in E : \| u \| < \rho_\ast \}$ for any $u \in K \cap \partial \Omega_{\rho_\ast}$. Then from (3.16), we obtain

$$Au(t) \leq \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T T(T - s) a(s) f(u(s)) ds$$

$$\leq \varepsilon \rho_\ast \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T T(T - s) a(s) ds$$

$$\leq \varepsilon \Lambda_1 \rho_\ast \leq \rho_\ast = \| u \|,$$

which implies

$$\| Au \| \leq \| u \| \quad \text{for } u \in K \cap \partial \Omega_{\rho_\ast}. \quad (3.17)$$

Secondly, in view of $f_\infty = \lim_{u \to \infty} (f(u)/u) = 0$, for any $\varepsilon_1 \in (0, \Lambda_1^{-1}]$ there exists $\rho_0 > \rho_2$, such that

$$f(u) \leq \varepsilon_1 u, \quad \text{for } u \in [\rho_0, \infty). \quad (3.18)$$

We consider the next two cases.

Case (i): Suppose that $f(u)$ is unbounded. Then from $f \in C([0, \infty), [0, \infty))$, there exists $\rho^* > \rho_0$ such that

$$f(u) \leq f(\rho^*), \quad \text{for } u \in [0, \rho^*]. \quad (3.19)$$

Since $\rho^* > \rho_0$, from (3.18) and (3.18) one has

$$f(u) \leq f(\rho^*) \leq \varepsilon_1 \rho^*, \quad \text{for } u \in [0, \rho^*]. \quad (3.20)$$

For $u \in K$, and $\| u \| = \rho^*$, from (3.20), we obtain

$$Au(t) \leq \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T T(T - s) a(s) f(u(s)) ds$$

$$\leq \varepsilon_1 \rho^* \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T T(T - s) a(s) ds$$

$$\leq \varepsilon_1 \Lambda_1 \rho^* \leq \rho^* = \| u \|.$$

Case (ii): Suppose that $f(u)$ is bounded, say $f(u) \leq L$ for all $u \in [0, \infty)$. Taking $\rho^* \geq \max\{L/\varepsilon_1, \rho_0\}$, for $u \in K$ with $\| u \| = \rho^*$, we have

$$Au(t) \leq \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T T(T - s) a(s) f(u(s)) ds$$

$$\leq L \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T T(T - s) a(s) ds$$

$$\leq \varepsilon_1 \rho^* \frac{2T + \beta(T + \eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T T(T - s) a(s) ds$$

$$\leq \varepsilon_1 \Lambda_1 \rho^* \leq \rho^* = \| u \|.$$

Hence, in either case, we always may set $\Omega_{\rho^*} = \{ u \in E : \| u \| < \rho^* \}$ such that

$$\| Au \| \leq \| u \| \quad \text{for } u \in K \cap \partial \Omega_{\rho^*}. \quad (3.21)$$

Finally, set $\Omega_{\rho_2} = \{ u \in E : \| u \| < \rho_2 \}$ for $u \in K \cap \partial \Omega_{\rho_2}$. Since $u \in K$, $\min_{t \in [0,T]} u(t) \geq \gamma \| u \| = \gamma \rho_2$. Hence, for any $u \in K \cap \partial \Omega_{\rho_2}$, and (H6), we have

$$Au(\eta) = \frac{(\beta - \alpha)\eta - \beta T}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^\eta (\eta - s)^2 a(s) f(u(s)) ds$$

$$+ \frac{(2 - \beta\eta)\eta}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T (T - s) a(s) f(u(s)) ds.$$
Thus, since \( \rho < \rho^* \) and from (3.17), (3.21) and (3.22), it follows from Theorem 1.1 that \( A \) has a fixed point \( u_1 \) in \( K \cap (\Omega_{\rho^*} \setminus \Omega_\rho) \), and a fixed point \( u_2 \) in \( K \cap (\Omega_{\rho^*} \setminus \Omega_\rho) \). Both are positive solutions of (1.1), (1.2) and \( 0 < \|u_1\| < \rho_2 < \|u_2\| \).

The proof is complete. \( \square \)

4. Some examples

In this section, to illustrate our results, we consider some examples.

Example 4.1. Consider the boundary-value problem

\[ u''(t) + t^2 u^p = 0, \quad 0 < t < e^2, \quad (4.1) \]

\[ u(0) = \frac{2}{9} \int_0^e u(s)ds, \quad u(e^2) = \frac{2}{3} \int_0^e u(s)ds. \quad (4.2) \]

Set \( \alpha = \frac{2}{3}, \quad \beta = \frac{2}{9}, \quad \eta = e, \quad T = e^2, \quad a(t) = t^2, \quad f(u) = u^p \). We can show that

\[ 0 < \alpha = \frac{2}{3} < 2 = \frac{2T}{\eta^2}, \quad 0 < \beta = \frac{2}{9} < \frac{4}{3(2e - 1)} = \frac{2T - \alpha \eta^2}{\eta(2T - \eta)}. \]

Case I: \( p \in (1, \infty) \). In this case, \( f_0 = 0, \quad f_\infty = \infty \) and (H1) holds. Then (4.1), (4.2) has at least one positive solution.

Case II: \( p \in (0, 1) \) In this case, \( f_0 = \infty, \quad f_\infty = 0 \) and (H2) holds. Then (4.1), (4.2) has at least one positive solution.

Example 4.2. Consider the boundary-value problem

\[ u''(t) + \frac{1}{85}(4 - t)^{1/2}(u^{1/2} + u^2) = 0, \quad 0 < t < 4, \quad (4.3) \]

\[ u(0) = \frac{1}{10} \int_0^1 u(s)ds, \quad u(4) = 2 \int_0^1 u(s)ds. \quad (4.4) \]

Set \( \alpha = 2, \quad \beta = \frac{1}{10}, \quad \eta = 1, \quad T = 4, \quad a(t) = \frac{1}{85}(4 - t)^{1/2}, \quad f(u) = u^{1/2} + u^2 \). We can show that \( 0 < \alpha = 2 < 8 = 2T/\eta^2 \), \( 0 < \beta = \frac{1}{10} < 6/7 = (2T - \alpha \eta^2)/(\eta(2T - \eta)) \). Since \( f_0 = f_\infty = \infty \), then (H3) holds. Again \( \Lambda_1^{-1} = ((2T - \alpha \eta^2) - \beta \eta(2T - \eta))/(2T + \beta(\eta^2)) \). Hence, by Theorem 3.2 BVP (4.3), (4.4) has at least two positive solutions \( u_1 \) and \( u_2 \) such that \( 0 < \|u_1\| < 4 < \|u_2\| \).

Example 4.3. Consider the boundary-value problem

\[ u''(t) + e^{32}u^2e^{-u} = 0, \quad 0 < t < \frac{4}{5}, \quad (4.5) \]
Set $\alpha = 20, \beta = 2, \eta = 1/5, T = 4/5, a(t) \equiv e^{3t}$. We can show that $0 < \alpha = 20 < 40 < \eta = 20/7 = (2T - \eta)(2T - \eta)$, $\gamma = \min\{\eta(T - \eta)/(2(2 - \beta\eta))\} = (2T - \eta)/2T = 1/16$. Since $f_0 = f_{\infty} = 0$, then (H5) holds. Again $\Lambda_2^{-1} = (2T - \eta)/(2 - \beta\eta)T - \eta)$, so $f(u) = 1024e^{-32} > 400e^{-32} = \Lambda_2^{-1}$. which implies (H6) holds. Hence, by Theorem 3.3, BVP (4.5), (4.6) has at least two positive solutions $u_1$ and $u_2$ such that $0 < \|u_1\| < 32 < \|u_2\|.$

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