*Electronic Journal of Differential Equations*, Vol. 2011 (2011), No. 141, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# BIFURCATION FROM INFINITY AND MULTIPLE SOLUTIONS FOR FIRST-ORDER PERIODIC BOUNDARY-VALUE PROBLEMS

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Abstract. In this article, we study the existence and multiplicity of solutions for the first-order periodic boundary-value problem

$$u'(t) - a(t)u(t) = \lambda u(t) + g(u(t)) - h(t), \quad t \in (0, T),$$
$$u(0) = u(T).$$

## 1. Introduction

The first-order periodic differential equation

$$u'(t) = a(t)u(t) - f(u(t - \tau(t)))$$

has been proposed as models for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias, see [3, 8, 15]. Thus, the existence of periodic solutions of this periodic differential equation has been discussed by several authors; see for example [1, 2, 5, 6, 7, 9, 10, 11, 13, 14, 16] and the references therein.

In these articles, the condition $\int_0^T a(t)dt \neq 0$ is used for showing the existence of solutions. A natural question is what would happen if $\int_0^T a(t)dt = 0$. It is easy to check that if $\int_0^T a(t)dt = 0$, then the equation

$$-u'(t) + a(t)u(t) = 0, \quad u(0) = u(T)$$

has nontrivial solutions. Thus, the operator $Lu = -u'(t) + a(t)u(t)$ is not invertible.

In this article, using Leray-Schauder degree and bifurcation techniques and under the condition that $\int_0^T a(t)dt = 0$, we discuss the existence and multiplicity of solutions for the problem

$$u'(t) - a(t)u(t) = \lambda u(t) + g(u(t)) - h(t), \quad t \in (0, T), \tag{1.1}$$
$$u(0) = u(T), \tag{1.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous, $h \in L^1(0, T)$, and the parameter $\lambda$ is close to 0 which is the eigenvalue of

$$-u'(t) + a(t)u(t) = \lambda u(t), \quad u(0) = u(T).$$

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2000 *Mathematics Subject Classification.* 34B18.

*Key words and phrases.* First-order periodic problems; Landsman-Lazer type condition; Leray-Schauder degree; bifurcation; existence.



Submitted September 6, 2011. Published October 28, 2011.

In this article, we use the following assumptions:

(H1) $a(\cdot) \in C[0,T]$ and $\int_0^T a(t)dt = 0$;

(H2) $g : \mathbb{R} \to \mathbb{R}$ is continuous, and there exist $\alpha \in [0,1)$, $p, q \in (0,\infty)$, such that
$$|g(u)| \le p|u|^\alpha + q, \quad u \in \mathbb{R};$$

(H3) There exist constants $A, a, R, r$ such that $r < 0 < R$ and
$$\begin{gathered} g(u) \ge A, \quad \text{for all } u \ge R, \\ g(u) \le a, \quad \text{for all } u \le r; \end{gathered}$$

(H3') There exist constants $A, a, R, r$ such that $r < 0 < R$ and
$$\begin{gathered} g(u) \le A, \quad \text{for all } u \ge R, \\ g(u) \ge a, \quad \text{for all } u \le r. \end{gathered}$$

(H4)
$$\int_0^T \frac{g^{-\infty}}{\psi(s)} ds < \int_0^T \frac{h(s)}{\psi(s)} ds < \int_0^T \frac{g_{+\infty}}{\psi(s)} ds,$$
where
$$g^{-\infty} = \limsup_{s \to -\infty} g(s), \quad g_{+\infty} = \liminf_{s \to +\infty} g(s),$$
and $\psi(t) = e^{\int_0^t a(s)ds}$ is the solution of
$$-u' + a(t)u = 0, \quad u(0) = u(T).$$

(H4')
$$\int_0^T \frac{g^{+\infty}}{\psi(s)} ds < \int_0^T \frac{h(s)}{\psi(s)} ds < \int_0^T \frac{g_{-\infty}}{\psi(s)} ds,$$
where
$$g^{+\infty} = \limsup_{s \to +\infty} g(s), \quad g_{-\infty} = \liminf_{s \to -\infty} g(s).$$

Our main results are as follows.

**Theorem 1.1.** *Assume that* (H1)–(H4) *hold. Then there exists $\lambda_+, \lambda_-$ with $\lambda_+ > 0 > \lambda_-$ such that*

(i) (1.1), (1.2) *has at least one solution if $\lambda \in [0, \lambda_+]$;*
(ii) (1.1), (1.2) *has at least three solutions if $\lambda \in [\lambda_-, 0)$.*

**Theorem 1.2.** *Assume that* (H1), (H2), (H3'),(H4') *hold. Then there exists $\lambda_+, \lambda_-$ with $\lambda_+ > 0 > \lambda_-$ such that*

(i) (1.1), (1.2) *has at least one solution if $\lambda \in [\lambda_-, 0]$;*
(ii) (1.1), (1.2) *has at least three solutions if $\lambda \in (0, \lambda_+]$.*

The rest of the paper is arranged as follows. In section 2, we discuss the Lyapunov-Schmidt procedure for (1.1), (1.2). In section 3, the existence of solutions of (1.1), (1.2) is discussed under 'Landesman-Lazer' type conditions.

## 2. Lyapunov-Schmidt procedure

Let $X, Y$ be the Banach spaces $C[0,T]$, $L^1[0,T]$ with the norm $\|x\| = \max\{|x(t)| : t \in [0,T]\}$, $\|u\|_1 = \int_0^T |u(s)|ds$, respectively. Define linear operator $L : D(L) \subset X \to Y$ by

$$Lu = -u' + a(t)u, u \in D(L), \tag{2.1}$$

where $D(L) = \{u \in W^{1,1}(0,T) : u(0) = u(T)\}$. Let $N : X \to X$ be the nonlinear operator defined by

$$(Nu)(t) = g(u(t)), \quad t \in [0,T],\ u \in D(L). \tag{2.2}$$

It is easy to see that $N$ is continuous. Note that (1.1), (1.2) is equivalent to

$$Lu + \lambda u + Nu = h, u \in D(L). \tag{2.3}$$

**Lemma 2.1.** *Let $L$ be defined by* (2.1)*. Then*

$$\ker L = \{x \in X : x(t) = c\psi(t) : c \in \mathbb{R}\},$$

$$\operatorname{Im} L = \{y \in Y : \int_0^T \frac{y(s)}{\psi(s)} ds = 0\}.$$

*Proof.* It is easy to see that $\ker L = \{c\psi(t) : c \in \mathbb{R}\}$. The following will prove that $\operatorname{Im} L = \{y \in Y : \int_0^T \frac{y(s)}{\psi(s)} ds = 0\}$.

If $y \in \operatorname{Im} L$, then there exists $u \in D(L)$ such that $-u'(t) + a(t)u(t) = y(t)$. So

$$u(t) = u(0)\psi(t) - \int_0^t y(s)e^{\int_s^t a(\tau)d\tau} ds.$$

Combining with $u(0) = u(T)$, we have

$$\int_0^T \frac{y(s)}{\psi(s)} ds = 0.$$

On the other hand, if $y \in Y$ satisfies $\int_0^T \frac{y(s)}{\psi(s)} ds = 0$, then we set

$$u(t) := -\int_0^t y(s)e^{\int_s^t a(\tau)d\tau} ds.$$

It is not difficult to prove that $x \in D(L)$ and $Lu = y$. $\square$

Define operator $P : X \to \ker L$,

$$(Pu)(t) = u(0)\psi(t), \quad u \in X. \tag{2.4}$$

Let $Q : Y \to Y$ be such that

$$(Qy)(t) = \frac{1}{T}\psi(t) \int_0^T \frac{y(s)}{\psi(s)} ds. \tag{2.5}$$

Denote $X_1 = \{u \in X : u(0) = 0\}$.

**Lemma 2.2.** *Let operators $P$ and $Q$ be defined by* (2.4) *and* (2.5)*. Then*

$$X = X_1 \oplus \ker L, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

*We define linear operator $K : \operatorname{Im} L \to D(L) \cap X_1$*

$$(Ky)(t) = -\int_0^t y(s)e^{\int_s^t a(\tau)d\tau} ds, \quad y \in \operatorname{Im} L, \tag{2.6}$$

*satisfying $K = L_p^{-1}$, where $L_p = L|_{D(L)\cap X_1}$.*

*Proof.* Let $y_1(t) = y(t) - (Qy)(t), y \in Y$, then it is easy to verify that $y_1 \in \operatorname{Im} L$. Thus $Y = \operatorname{Im} L + \operatorname{Im} Q$. Also $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$. Hence $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. If $u \in D(L) \cap X_1$, then

$$(KL_p u)(t) = K\big(-u'(t) + a(t)u(t)\big) = u(t).$$

If $y \in \operatorname{Im} L$, then

$$(L_p K y)(t) = -\big(-\int_0^t y(s)e^{\int_s^t a(\tau)d\tau} ds\big)' - a(t)\int_0^t y(s)e^{\int_s^t a(\tau)d\tau} ds = y(t).$$

This indicates $K = L_p^{-1}$. □

Therefore, for every $u \in X$ , we have a unique decomposition $u(t) = \rho\psi(t) + v(t), t \in [0,T]$, where $\rho \in \mathbb{R}, v \in X_1$. Similarly, for every $h \in Y$, we have unique decomposition $h(t) = \tau\psi(t) + \bar{h}(t), t \in [0,T]$, where $\tau \in \mathbb{R}, \bar{h} \in \operatorname{Im} L$. The operator $Q, K$ be defined as (2.5), (2.6). Then $K(I-Q)N : X \to X$ is completely continuous, and (2.3) is equivalent to the system

$$v(t) + \lambda K v(t) + K(I-Q)N(\rho\psi(t) + v(t)) = K\bar{h}(t), \tag{2.7}$$

$$\lambda\rho\psi(t) + QN(\rho\psi(t) + v(t)) = \tau\psi(t). \tag{2.8}$$

**Lemma 2.3** ([4]). *Assume that* (H2), (H3) *hold. Then for each real number $s > 0$, there exists a decomposition $g(u) = q_s(u) + g_s(u)$ of $g$ by $q_s$ and $g_s$ satisfying the conditions:*

$$uq_s(u) \ge 0, u \in \mathbb{R}, \tag{2.9}$$

$$|q_s(u)| \le p|u| + q + s, u \ge 1, , \tag{2.10}$$

*there exists $\sigma_s$ depending on $a, A$ and $g$ such that*

$$|g_s(u)| \le \sigma_s, u \in \mathbb{R}. \tag{2.11}$$

**Lemma 2.4.** *Assume that* (H1)–(H4) *hold, and $\lambda$ satisfies*

$$0 \le \lambda \le \eta_1 := \frac{1}{2\|K\|_{\operatorname{Im} L \to X_1}}. \tag{2.12}$$

*Then there exists constant $R_0 > 0$ such that any solution $u$ of* (1.1) (1.2) *satisfies $\|u\| < R_0$.*

*Proof.* We divide the proof into several steps.

**Step 1.** By assumption (H2), there exists a constant $b$ such that

$$|g(u)| \le p|u| + b, u \in \mathbb{R},$$

where $p = \eta_1/4$. Using Lemma 2.3 with $s = 1$, (1.1), (1.2) is equivalent to

$$u'(t) - a(t)u(t) = \lambda u(t) + g_1(u(t)) + q_1(u(t)) - h(t), t \in [0,T], u \in D(L), \tag{2.13}$$

where $q_1$ and $g_1$ satisfying conditions (2.9) and (2.11). Moreover, by (2.10),

$$|q_1(u)| \le p|u| + b + 1. \tag{2.14}$$

Let $\bar{\delta} > 0$ and choose $B \in \mathbb{R}$ such that

$$(b+1)|\frac{1}{u}| \le \frac{1}{4}\bar{\delta} \tag{2.15}$$

for all $u \in \mathbb{R}$ with $|u| \ge B$. It follows from (2.14) and (2.15) that

$$0 \le q_1(u)u^{-1} \le p + \frac{1}{4}\bar{\delta} \tag{2.16}$$

for all $u \in \mathbb{R}$ with $|u| \ge B$.

**Step 2.** Let us define $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(u) = \begin{cases} u^{-1} q_1(u), & |u| \ge B; \\ B^{-1} q_1(B)(\frac{u}{B}) + (1 - \frac{u}{B})p, & 0 \le u < B; \\ B^{-1} q_1(-B)(\frac{u}{B}) + (1 + \frac{u}{B})p, & -B < u \le 0. \end{cases} \tag{2.17}$$

It is easy to see that $\gamma$ is continuous. Moreover, by (2.16), one has

$$0 \le \gamma(u) \le p + \frac{1}{4}\bar{\delta} \tag{2.18}$$

for all $u \in \mathbb{R}$. Defining $f : \mathbb{R} \to \mathbb{R}$ by

$$f(u) = g_1(u) + q_1(u) - \gamma(u)u, \tag{2.19}$$

it follows from (2.16) that for some $\sigma \in \mathbb{R}$,

$$|f(u)| \le \sigma \tag{2.20}$$

for all $u \in \mathbb{R}$, where $\sigma$ depends only on $p$ and $h$. Finally, (2.13) is equivalent to

$$u'(t) - a(t)u(t) = \lambda u(t) + f(u(t)) + \gamma(u(t))u(t) - h(t), t \in [0, T], u \in D(L).$$

**Step 3.** It is to see that $(L + \lambda I)|_{X_1 \cap D(L)} : X_1 \to \operatorname{Im} L$ is invertible. From (2.12),

$$\begin{aligned} \|(L + \lambda I)|^{-1}_{X_1 \cap D(L)}\|_{\operatorname{Im} L \to X_1} &= \|L^{-1}|_{X_1 \cap D(L)}(I + \lambda K)^{-1}\|_{\operatorname{Im} L \to X_1} \\ &= \|K\|_{\operatorname{Im} L \to X_1} \|(I + \lambda K)^{-1}\|_{\operatorname{Im} L \to X_1} \\ &\le 2\|K\|_{\operatorname{Im} L \to X_1}. \end{aligned}$$

Let $u = \rho\psi(t) + v$ be a solution of (2.13), where $\rho \in \mathbb{R}, v \in X_1$. Then from (2.7),

$$\begin{aligned} \|v\| &= \|(L + \lambda I)|^{-1}_{X_1 \cap D(L)}(I - Q)(\bar{h} - g(\rho\psi(t) + v(t)))\| \\ &\le \|(L + \lambda I)|^{-1}_{X_1 \cap D(L)}\|_{\operatorname{Im} L \to X_1} \|(I - Q)\|_{Y \to \operatorname{Im} L} [\|\bar{h}\|_1 + p(|\rho| \cdot \|\psi\| + \|v\|)^\alpha + q] \\ &\le 2\|K\|_{\operatorname{Im} L \to X_1} \|(I - Q)\|_{Y \to \operatorname{Im} L} [\|\bar{h}\|_1 + p(|\rho| \cdot \|\psi\| + \|v\|)^\alpha + q] \\ &= 2\|K\|_{\operatorname{Im} L \to X_1} \|(I - Q)\|_{Y \to \operatorname{Im} L} [\|\bar{h}\|_1 + p(|\rho| \cdot \|\psi\|)^\alpha (1 + \frac{\|v\|}{|\rho| \cdot \|\psi\|})^\alpha + q] \\ &\le 2\|K\|_{\operatorname{Im} L \to X_1} \|(I - Q)\|_{Y \to \operatorname{Im} L} [\|\bar{h}\|_1 + p(|\rho| \cdot \|\psi\|)^\alpha (1 + \frac{\alpha\|v\|}{|\rho| \cdot \|\psi\|}) + q] \\ &= 2\|K\|_{\operatorname{Im} L \to X_1} \|(I - Q)\|_{Y \to \operatorname{Im} L} [\|\bar{h}\|_1 + p(|\rho| \cdot \|\psi\|)^\alpha \\ &\quad \times \big(1 + \frac{\alpha}{(|\rho| \cdot \|\psi\|)^{1-\alpha}} \cdot \frac{\|v\|}{(|\rho| \cdot \|\psi\|)^\alpha}\big) + q]. \end{aligned}$$

Therefore,

$$\frac{\|v\|}{(|\rho| \cdot \|\psi\|)^\alpha} \le \frac{c_0}{(|\rho| \cdot \|\psi\|)^\alpha} + c_1 + \frac{\alpha c_1}{(|\rho| \cdot \|\psi\|)^{1-\alpha}} \cdot \frac{\|v\|}{(|\rho| \cdot \|\psi\|)^\alpha},$$

where

$$\begin{gathered} c_0 = 2\|K\|_{\operatorname{Im} L \to X_1} \|(I - Q)\|_{Y \to \operatorname{Im} L} (\|\bar{h}\|_1 + q), \\ c_1 = 2p\|K\|_{\operatorname{Im} L \to X_1} \|(I - Q)\|_{Y \to \operatorname{Im} L}. \end{gathered}$$

If

$$|\rho| \ge \frac{(2\alpha c_1)^{\frac{1}{1-\alpha}}}{\|\psi\|} := \tilde{c},$$

then

$$\frac{\|v\|}{(|\rho|\cdot\|\psi\|)^{\alpha}} \le \frac{2c_0}{(\tilde{c}\|\psi\|)^{\alpha}} + 2c_1 := \bar{c}. \tag{2.21}$$

**Step 4.** If we now assume that the conclusion of the lemma is false, we obtain a sequence $\{\lambda_n\} : 0 \le \lambda_n \le \eta_1, \lambda_n \to 0$ and a sequence $\{u_n\} : u_n = \rho_n\psi(t) + v_n, \rho_n \in \mathbb{R}, v_n \in X_1$ with $\|u_n\| \to \infty$ such that

$$\lambda_n\rho_n\psi(t) + Qg(\rho_n\psi(t) + v_n(t)) = \tau\psi(t). \tag{2.22}$$

It follows immediately from (2.21) that

$$|\rho_n| \to \infty, \|v_n\|(|\rho_n|\cdot\|\psi\|)^{-1} \to 0, \quad n \to \infty. \tag{2.23}$$

So we infer that there exists sufficiently large $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$|v_n(t)|(|\rho_n|\psi(t))^{-1} \le 1, \quad t \in [0,T]. \tag{2.24}$$

Without loss of generality, let $\rho_n \to +\infty$ if $n \to +\infty$ (the other case be proved by similar method), then there exists sufficiently large $n_0 \in \mathbb{N}$. If $n \ge n_0$, $\lambda_n\rho_n \ge 0$; thus

$$\begin{gathered} \tau - \frac{1}{T}\int_0^T \frac{g(\rho_n\psi(s) + v_n(s))}{\psi(s)}ds \ge 0, \\ \tau \ge \frac{1}{T}\liminf_{n\to\infty}\int_0^T \frac{g(\rho_n\psi(s) + v_n(s))}{\psi(s)}ds. \end{gathered} \tag{2.25}$$

To apply Fatou's lemma to (2.25), we need a function $\hat{K} \in L^1[0,T]$ such that for $s \in [0,T]$, $\frac{g(u_n(s))}{\psi(s)} \ge \hat{K}(s)$. Indeed, from the relation (2.24), one has that there exists nonnegative function $k_1 \in L^1[0,T]$ such that for $n \ge n_0$,

$$|v_n(t)|(\rho_n\psi(t))^{-1} \le k_1(t), \quad t \in [0,T],$$

and for every $s \in [0,T]$,

$$\begin{aligned} \gamma(u_n(s))u_n(s) + f(u_n(s)) &= \gamma(u_n(s))(\rho_n\psi(s) + v_n(s)) + f(u_n(s)) \\ &\ge \gamma(u_n(s))\frac{\rho_n\psi(s) + v_n(s)}{|\rho_n|\psi(s)} + f(u_n(s)) \\ &\ge \gamma(u_n(s))(1 - k_1(s)) - |f(u_n(s))| \\ &\ge -(p + \frac{1}{4}\bar{\delta})(1 - k_1(s)) - \sigma := \hat{K}(s). \end{aligned}$$

It follows from $\psi(s) > 0$ that

$$\frac{1}{\psi(s)}g(\rho_n\psi(s) + v_n(s)) \ge \frac{1}{\psi(s)}\hat{K}(s), s \in [0,T].$$

Thus, applying Fatou's lemma to (2.25), we have

$$\begin{aligned} \tau &\ge \frac{1}{T}\liminf_{n\to\infty}\int_0^T \frac{g(\rho_n\psi(s) + v_n(s))}{\psi(s)}ds \\ &\ge \frac{1}{T}\int_0^T \liminf_{n\to\infty}\frac{g(\rho_n\psi(s) + v_n(s))}{\psi(s)}ds \\ &\ge \frac{1}{T}\int_0^T \frac{g_{+\infty}}{\psi(s)}ds. \end{aligned}$$

This contradicts with (H4). □

**Lemma 2.5.** *Assume that* (H1), (H2), (H3'), (H4') *hold, and* $\lambda$ *satisfies*

$$0 \leq \lambda \leq \eta_1 := \frac{1}{2\|K\|_{\operatorname{Im} L \to X_1}}.$$

*Then there exists constant* $R_0 > 0$ *such that any solution* $u$ *of* (1.1) (1.2) *satisfy* $\|u\| < R_0$.

## 3. The Proof of the Main Result

**Lemma 3.1.** *Assume that* (H1)–(H4) *hold. Then there exists* $R_1 : R_1 \geq R_0$ *such that for* $0 \leq \lambda \leq \delta$, *and* $R \geq R_1$ *one has*

$$\deg(L + \lambda I + N - h, B(R), 0) = \deg(L + \delta I, B(R), 0) = \pm 1,$$

*where* $B(R) = \{u \in C[0,T] : \|u\| < R\}$, *and the* deg *denotes Leray-Schauder degree when* $\lambda \neq 0$ *and coincidence degree when* $\lambda = 0$. *Then* (1.1),(1.2) *has a solution in* $\bar{B}(R)$ *for* $0 \leq \lambda \leq \delta$.

*Proof.* From Lemma 2.4 and the definition of $L$, if $\lambda \in [0, \delta]$,

$$\deg(L + \delta I, B(R), 0)$$

is defined and depends on $\lambda$. Let $(\mu, u) \in [0,1] \times X$ be a solution of (2.3). Then

$$Lu + \delta u + \mu(Nu - h) = 0.$$

So

$$\|u\| = \mu\|(L + \delta)^{-1}(h - Nu)\| \leq \|(L + \delta)^{-1}\|_{Y \to X}(\|h\|_1 + p\|u\|^{\alpha} + q).$$

Therefore there exists $R_0' > 0$ such that $\|u\| < R_0'$. Choosing $R_1 = \max\{R_0', R_0\}$, then for arbitrary $R > R_1$,

$$\deg(L + \lambda I + N - h, B(R), 0) = \deg(L + \delta I, B(R), 0) = \pm 1.$$

□

**Lemma 3.2.** *Assume that* (H1), (H2), (H3'),(H4') *hold. Then there exists* $R_1$ : $R_1 \geq R_0$ *such that for* $0 \leq \lambda \leq \delta$, *and* $R \geq R_1$ *one has*

$$\deg(L + \lambda I + N - h, B(R), 0) = \deg(L + \delta I, B(R), 0) = \pm 1,$$

*where* $B(R) = \{u \in C[0,T] : \|u\| < R\}$.

**Lemma 3.3.** *Assume that* (H1)–(H4) *hold. Then there exists* $\mu \geq 0$ *such that for* $-\mu \leq \lambda \leq 0$ *one has*

$$\deg(L + \lambda I + N - h, B(R), 0) = \deg(L + \delta I, B(R), 0) = \pm 1,$$

*where* $R$ *be defined in Lemma 3.1. Then* (1.1),(1.2) *has a solution in* $B(R)$ *for* $-\mu \leq \lambda \leq \delta$.

*Proof.* Let

$$\tau_0 = \inf_{u \in \partial B(R) \cap X} \|Lu + Nu - h\|.$$

It is easy to verify that $\tau_0 > 0$. Choosing sufficiently small $\mu > 0$ such that $\mu R < \tau_0$, then if $\lambda \in [-\mu, \mu]$,

$$\deg(L + \lambda I + N - h, B(R), 0) = \deg(L + N - h, B(R), 0).$$

Combined with Lemma 3.1, the result can be proved. That is to see that if $\lambda \in [-\mu, \delta]$, (2.3) has at least one solution in $\bar{B}(R)$. □

**Lemma 3.4.** *Assume that* (H1), (H2), (H3')(H4') *hold. Then there exists* $\mu \geq 0$ *such that for* $-\mu \leq \lambda \leq 0$*, one has*

$$\deg(L + \lambda I + N - h, B(R), 0) = \deg(L + \delta I, B(R), 0) = \pm 1,$$

*where* $R$ *be defined in Lemma 3.1. Then* (1.1), (1.2) *has a solution in* $B(R)$ *for* $-\mu \leq \lambda \leq \delta$.

**Remark 3.5.** Since $g$ is L-completely continuous and satisfies (H2) and since $\lambda = 0$ is a simple eigenvalue of L, it follows from bifurcation results of [4] that there exist two connected sets $\mathcal{C}_+, \mathcal{C}_- \subset \mathbb{R} \times X$ of solutions of (1.1), (1.2) such that for all sufficiently small $\epsilon > 0$,

$$\mathcal{C}_+ \cap U_\epsilon \neq \emptyset, \quad \mathcal{C}_- \cap U_\epsilon \neq \emptyset,$$

where $U_\epsilon := \{(\lambda, u) \in \mathbb{R} \times X, |\lambda| < \epsilon, \|u\| > 1/\epsilon\}$.

*Proof of Theorem 1.1.* Set $\lambda^+ = \delta$, then it follows from Lemma 3.1 and Lemma 3.3 that (1.1), (1.2) has at least one solution in $B(R)$ for $\lambda \in [-\mu, \lambda_+]$. On the other hand, Remark 3.5 shows that there exists two connected sets $\mathcal{C}+$ and $\mathcal{C}-$ of solutions of (1.1), (1.2) bifurcating from infinity at $\lambda = 0$. Hence by Lemma 2.4, the connected sets $\mathcal{C}+$ and $\mathcal{C}-$ of Remark 3.5 must satisfy

$$\mathcal{C}_+, \mathcal{C}_- \subset \{(\lambda, u) : \|u\| \geq 1/\epsilon, -\mu < \lambda < 0\}.$$

and hence, if $1/\epsilon \geq R$; i.e., $\epsilon \leq 1/k$. Choosing $\lambda_- = \max\{-\mu, -1/k\}$, we obtain two solutions $u_1, u_2 : u_1 \in \mathcal{C}_+, u_2 \in \mathcal{C}_-$, and $\|u_i\| \geq R$ $(i = 1, 2)$. □

Theorem 1.2 can be proved by a similar method.

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