EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR SYSTEMS OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS ON AN INFINITE INTERVAL IN BANACH SPACES

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Abstract. The article shows the existence of positive solutions for systems of nonlinear singular differential equations with integral boundary conditions on an infinite interval in Banach spaces. Our main tool is the Mönch fixed point theorem combined with a monotone iterative technique. In addition, an explicit iterative approximation of the solution is provided.

1. Introduction

The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. There are many excellent results about the existence of positive solutions for integral boundary value problems in scalar case (see, for instance, [7, 9, 10, 13] and references therein). Very recently, by using Schauder fixed point theorem, Guo [2] obtained the existence of positive solutions for a class of nth-order nonlinear impulsive singular integro-differential equations in a Banach space.

Recently, Zhang et al. [14], using the cone theory and monotone iterative technique, investigated the existence of minimal nonnegative solution of the following nonlocal boundary value problems for second-order nonlinear impulsive differential equations on an infinite interval with an infinite number of impulsive times

\[-x''(t) = f(t, x(t), x'(t)), \quad t \in J, t \neq t_k,\]
\[\Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots,\]
\[\Delta x'|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots,\]
\[x(0) = \int_0^\infty g(t)x(t) \, dt, \quad x'(\infty) = 0,\]

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where \( J = [0, +\infty), f \in C(J \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+) \), \( \mathbb{R}^+ = [0, +\infty), 0 < t_1 < t_2 < \cdots < t_k < \ldots, t_k \to \infty, I_k \in C[\mathbb{R}^+, \mathbb{R}^+], \mathcal{J}_k \in C[\mathbb{R}^+, \mathbb{R}^+] \), \( g(t) \in C[\mathbb{R}^+, \mathbb{R}^+] \), with \( \int_0^\infty g(t) \, dt < 1 \).

To the best of our knowledge, only a few authors have studied integral boundary value problems in Banach spaces and results for systems of second-order differential equation are rarely seen. Motivated by Zhang and Guo’s work, in this paper, we consider the following singular integral boundary value problem on an infinite interval in a Banach space \( E \):

\[
\begin{align*}
  x''(t) + f(t, x(t), x'(t), y(t), y'(t)) &= 0, \\
  y''(t) + g(t, x(t), x'(t), y(t), y'(t)) &= 0, \quad t \in J_+ , \\
  x(0) &= \int_0^\infty q(t)x(t) \, dt, \quad x'(\infty) = x_\infty, \\
  y(0) &= \int_0^\infty h(t)y(t) \, dt, \quad y'(\infty) = y_\infty,
\end{align*}
\]

(1.1)

where \( J = [0, \infty), J_+ = (0, \infty) \), and the functions \( q(t), h(t) \) are in \( L[0, \infty) \) with \( \int_0^\infty |q(t)| \, dt < 1, \int_0^\infty |h(t)| \, dt < 1 \) and \( \int_0^\infty t g(t) \, dt < \infty, \int_0^\infty th(t) \, dt < \infty \). \( x'(\infty) = \lim_{t \to \infty} x'(t), \ y'(\infty) = \lim_{t \to \infty} y'(t) \). The nonlinear terms \( f(t, x_0, x_1, y_0, y_1) \) and \( g(t, x_0, x_1, y_0, y_1) \) permit singularities at \( t = 0 \) and \( x_i, y_i = \theta \) \((i = 0, 1)\), where \( \theta \) denotes the zero element of Banach space \( E \). By singularity, we mean that \( \|f(t, x_0, x_1, y_0, y_1)\| \to \infty \) as \( t \to 0^+ \) or \( x_i, y_i \to \theta \).

The main features of this article are as follows: Firstly, compared with [14], the systems of integral boundary value problem we discussed here is in Banach spaces and nonlinear term permits singularity not only at \( t = 0 \) but also at \( x_i, y_i = \theta \) \((i = 0, 1)\). Secondly, compared with [2], the problem we discussed here is systems of integral boundary value problem, since the problem we discuss is the integral boundary value problems, the construction of bounded convex closed set is different from that in [2]. Furthermore, the relative compact conditions we used are weaker. Finally, an iterative sequence for the solution under some normal type conditions is established which makes it convenient in applications.

2. Preliminaries

Let

\[
FC[J, E] = \{ x \in C[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{t + 1} < \infty \},
\]

\[
DC^1[J, E] = \{ x \in C^1[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{t + 1} < \infty \text{ and } \sup_{t \in J} \|x'(t)\| < \infty \}.
\]

Evidently, \( C^1[J, E] \subset FC[J, E] \) and \( DC^1[J, E] \subset FC[J, E] \). It is easy to see that \( FC[J, E] \) is a Banach space with norm

\[
\|x\|_F = \sup_{t \in J} \frac{\|x(t)\|}{t + 1},
\]

and \( DC^1[J, E] \) is also a Banach space with norm

\[
\|x\|_D = \max\{\|x\|_F, \|x'\|_1\},
\]

where

\[
\|x'(t)\|_1 = \sup_{t \in J} |x'(t)|.
\]
Let $X = DC^1[J, E] \times DC^1[J, E]$ with norm $\|(x, y)\|_X = \max\{\|x\|_D, \|y\|_D\}$, for $(x, y) \in X$. Then $(X, \|\cdot\|_X)$ is also a Banach space. The basic space using in this paper is $(X, \|\cdot\|_X)$.

Let $P$ be a normal cone in $E$ with normal constant $N$ which defines a partial ordering in $E$ by $x \leq y$. If $x \leq y$ and $x \neq y$, we write $x < y$. Let $P_+ = P \setminus \{\theta\}$. So, $x \in P_+$ if and only if $x > \theta$. For details on cone theory, see [4].

In what follows, we always assume that $x_0 \geq x^*_0, y_0 \geq y_0, x^*_0, y_0 \in P_+$. Let $
abla E = \{x \in P : x \geq x_0\}, \nabla F = \{y \in P : y \geq y_0\} (\lambda > 0)$. Obviously, $P_0, P_1 \subset P$ for any $\lambda > 0$. When $\lambda = 1$, we write $P_0 = P_0, P_1 = P_1$; i.e., $P_0 = \{x \in P : x \geq x_0\}, P_1 = \{y \in P : y \geq y_0\}$. Let $P(F) = \{x \in FC[J, E] : x(t) \geq \theta, \forall t \in J\}$, and $P(D) = \{x \in DC^1[J, E] : x(t) \geq \theta, x'(t) \geq \theta, \forall t \in J\}$. It is clear, $P(F)$ and $P(D)$ are cones in $FC[J, E]$ and $DC^1[J, E]$, respectively. A map $(x, y) \in DC^1[J, E] \cap C^2[J, E]$ is called a positive solution of (1.1) if $(x, y) \in P(D) \times P(D)$ and $(x(t), y(t))$ satisfies (1.1).

Let $\alpha, \alpha_F, \alpha_D, \alpha_X$ denote the Kuratowski measure of non-compactness in the sets $E, FC[J, E], DC^1[J, E]$ and $X$, respectively. For details on the definition and properties of the measure of non-compactness, the reader is referred to references [13, 14, 16]. Let

$$D_0 = \left(1 + \frac{\int_0^\infty q(t)\, dt}{1 - \int_0^\infty q(t)\, dt}\right), \quad D_1 = \left(1 + \frac{\int_0^\infty h(t)\, dt}{1 - \int_0^\infty h(t)\, dt}\right),$$

(2.1)

$$D^* = \max\{D_0, D_1\}.$$ 

Let us list some conditions for convenience.

(H1) $f, g \in C[J^+_+ \times P_0 \times P_0 \times P_0 \times P_1 \times P_1] \setminus P$ for any $\lambda > 0$ and there exist $a_i, b_i, c_i \in L(J^+_+)$ and $z_i \in C(J^+_+ \times J^+_+ \times J^+_+) (i = 0, 1)$ such that

$$\|f(t, x_0, x_1, y_0, y_1)\| \leq a_0(t) + b_0(t)z_0(\|x_0\|, \|x_1\|, \|y_0\|, \|y_1\|),$$

for all $t \in J^+_+, x_i \in P_0, y_i \in P_1$; and

$$\|g(t, x_0, x_1, y_0, y_1)\| \leq a_1(t) + b_1(t)z_1(\|x_0\|, \|x_1\|, \|y_0\|, \|y_1\|),$$

for all $t \in J^+_+, x_i \in P_0, y_i \in P_1$; and

$$\frac{\|f(t, x_0, x_1, y_0, y_1)\|}{c_0(t)(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|)} \rightarrow 0,$$

$$\frac{\|g(t, x_0, x_1, y_0, y_1)\|}{c_1(t)(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|)} \rightarrow 0,$$

as $x_i \in P_0, y_i \in P_1$ $(i = 0, 1), \|x_0\| + \|x_1\| + \|y_0\| + \|y_1\| \rightarrow \infty$, uniformly for $t \in J^+_+$; and for $i = 0, 1$:

$$\int_0^\infty a_i(t)\, dt = a^*_i < \infty, \quad \int_0^\infty b_i(t)\, dt = b^*_i < \infty, \quad \int_0^\infty c_i(t)(1 + t)\, dt = c^*_i < \infty.$$

(H2) For any $t \in J^+_+$ and countable bounded set $V_i \subset DC^1[J, P_0], W_i \subset DC^1[J, P_1]$ $(i = 0, 1)$, there exist $L_i(t), K_i(t) \in L(J, J) (i = 0, 1)$ such
that
\[ \alpha(f(t, V_0(t), V_1(t), W_0(t), W_1(t))) \leq \sum_{i=0}^{1} L_{\alpha}(t) \alpha(V_i(t)) + K_{\alpha}(t) \alpha(W_i(t)), \]
\[ \alpha(g(t, V_0(t), V_1(t), W_0(t), W_1(t))) \leq \sum_{i=0}^{1} L_{11}(t) \alpha(V_i(t)) + K_{11}(t) \alpha(W_i(t)), \]
with
\[ G^*_i = \int_0^{+\infty} \left[ (L_{i0}(s) + K_{i0}(s))(1 + s) + L_{i1}(s) + K_{i1}(s) \right] ds < \infty \quad (i = 0, 1), \]
\[ G^* = \max\{G^*_0, G^*_1\}. \]

(H3) \( t \in J_x, \lambda^*_0 x^*_0 \leq x_i \leq \bar{x}_i, \lambda^*_1 y^*_0 \leq y_i \leq \bar{y}_i \) \((i = 0, 1)\) imply
\[ f(t, x_0, x_1, y_0, y_1) \leq f(t, \bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1), \quad g(t, x_0, x_1, y_0, y_1) \leq g(t, \bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1). \]

In what follows, we write
\[ Q_1 = \{ x \in DC^1[J, P] : x^{(i)}(t) \geq \lambda^*_0 x^*_0, \forall t \in J, i = 0, 1 \}, \]
\[ Q_2 = \{ y \in DC^1[J, P] : y^{(i)}(t) \geq \lambda^*_1 y^*_0, \forall t \in J, i = 0, 1 \}, \]
and \( Q = Q_1 \times Q_2 \). Evidently, \( Q_1, Q_2 \) and \( Q \) are closed convex set in \( DC^1[J, E] \) and \( X \), respectively.

We shall reduce BVP (1.1) to a system of integral equations in \( E \). To this end, we first consider operator \( A \) defined by
\[ A(x, y)(t) = (A_1(x, y)(t), A_2(x, y)(t)), \tag{2.2} \]
where
\[ A_1(x, y)(t) = \frac{1}{1 - \int_0^{\infty} q(t) dt} \left\{ x_0 \int_0^{\infty} t g(t) dt \right. \]
\[ + \int_0^{\infty} \int_0^{\infty} q(t) G(t, s) f(s, x(s), x'(s), y(s), y'(s)) ds dt \left. \right\} \tag{2.3} \]
\[ + t x_0 + \int_0^{\infty} G(t, s) f(s, x(s), x'(s), y(s), y'(s)) ds, \]
and
\[ A_2(x, y)(t) = \frac{1}{1 - \int_0^{\infty} h(t) dt} \left\{ y_0 \int_0^{\infty} th(t) dt \right. \]
\[ + \int_0^{\infty} \int_0^{\infty} h(t) G(t, s) g(s, x(s), x'(s), y(s), y'(s)) ds dt \left. \right\} \tag{2.4} \]
\[ + t y_0 + \int_0^{\infty} G(t, s) g(s, x(s), x'(s), y(s), y'(s)) ds, \]
where
\[ G(t, s) = \begin{cases} t, & 0 \leq t \leq s < +\infty, \\ s, & 0 \leq s \leq t < +\infty. \end{cases} \]

**Lemma 2.1.** If (H1) is satisfied, then the operator \( A \) defined by (2.2) is a continuous operator from \( Q \) to \( Q \).
Proof. Let
\[
\varepsilon_0 = \min \left\{ \frac{1}{8c_0^2(1 + \int_{0}^{r} q(t) \, dt)} , \frac{1}{8c_0^2(1 + \int_{0}^{r} h(t) \, dt)} \right\},
\]
(2.5)
and
\[
r = \min \left\{ \frac{\lambda_0^*\|x_0^*\|}{N}, \frac{\lambda_1^*\|y_0^*\|}{N} \right\} > 0.
\]
(2.6)
By (H1), there exists a \( R > r \) such that
\[
\|f(t, x_0, x_1, y_0, y_1)\| \leq \varepsilon_0 c_0(t)(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|),
\]
for all \( t \in J_+ \), \( x_i \in P_{0\lambda_0^*}, y_i \in P_{1\lambda_1^*} \) \( (i = 0, 1) \), \( \|x_0\| + \|x_1\| + \|y_0\| + \|y_1\| > R \); and
\[
\|f(t, x_0, x_1, y_0, y_1)\| \leq a_0(t) + M_0 b_0(t),
\]
for all \( t \in J_+ \), \( x_i \in P_{0\lambda_0^*}, y_i \in P_{1\lambda_1^*} \) \( (i = 0, 1) \), \( \|x_0\| + \|x_1\| + \|y_0\| + \|y_1\| \leq R \), where
\[
M_0 = \max \{\varepsilon_0(u_0, u_1, v_0, v_1) : r \leq u_i, v_i \leq R \} \quad (i = 0, 1) \}.
\]
Hence
\[
\|f(t, x_0, x_1, y_0, y_1)\| \leq \varepsilon_0 c_0(t)(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|) + a_0(t) + M_0 b_0(t),
\]
(2.7)
for all \( t \in J_+ \), \( x_i \in P_{0\lambda_0^*}, y_i \in P_{1\lambda_1^*} \) \( (i = 0, 1) \). Let \((x, y) \in Q \). By (2.7) we have
\[
\|f(t, x(t), x'(t), y(t), y'(t))\| \\
\leq \varepsilon_0 c_0(t)(1 + t)\left\{ \frac{\|x(t)\|}{t + 1} + \frac{\|x'(t)\|}{t + 1} + \frac{\|y(t)\|}{t + 1} + \frac{\|y'(t)\|}{t + 1} \right\} + a_0(t) + M_0 b_0(t) \\
\leq \varepsilon_0 c_0(t)(1 + t)(\|x\|_F + \|x'\|_F + \|y\|_F + \|y'\|_F) + a_0(t) + M_0 b_0(t) \\
\leq 2\varepsilon_0 c_0(t)(1 + t)(\|x\|_D + \|y\|_D) + a_0(t) + M_0 b_0(t),
\]
(2.8)
\[
\leq 4\varepsilon_0 c_0(t)(1 + t)(\|x\|_X + a_0(t) + M_0 b_0(t),
\]
\[\forall t \in J_+,
\]
which together with (H1) implies the convergence of the infinite integral
\[
\int_{0}^{\infty} \|f(s, x(s), x'(s), y(s), y'(s))\| \, ds,
\]
(2.9)
which together with (2.3) and (H1) implies
\[
\|A_1(x, y)(t)\| \\
\leq \frac{1}{1 - \int_{0}^{\infty} q(t) \, dt} \left\{ \int_{0}^{\infty} \int_{0}^{\infty} q(t) G(t, s) \|f(s, x(s), x'(s), y(s), y'(s))\| \, ds \, dt \\
+ \|x_\infty\| \int_{0}^{\infty} t q(t) \, dt \right\} + t\|x_\infty\| + \int_{0}^{\infty} G(t, s) \|f(s, x(s), x'(s), y(s), y'(s))\| \, ds.
\]

Therefore,
\[
\frac{\|A_1(x, y)(t)\|}{1 + t} \leq \frac{1}{1 - \int_0^\infty q(t) \, dt} \left\{ \int_0^\infty \int_0^\infty q(t) \|f(s, x(s), x'(s), y(s), y'(s))\| \, ds \, dt + \|x_\infty\| \int_0^\infty t q(t) \, dt \right\} + \|x_\infty\| + \int_0^\infty \|f(s, x(s), x'(s), y(s), y'(s))\| \, ds
\]
\[
\leq \left(1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right) [4\varepsilon \|e_0^*\| (x, y) \|x + a_0^* + M_0b_0^*\] \tag{2.10}
\]
\[
+ \left(1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right) \|x_\infty\|
\]
\[
\leq \frac{1}{2} \| (x, y) \| x + \left(1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right) (a_0^* + M_0b_0^*)
\]
\[
+ \left(1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right) \|x_\infty\|.
\]
Differentiating (2.3), we obtain
\[
A_1'(x, y)(t) = \int_t^\infty f(s, x(s), x'(s), y(s), y'(s)) \, ds + x_\infty. \tag{2.11}
\]
Hence,
\[
\|A_1'(x, y)(t)\| \leq \int_0^{+\infty} \|f(s, x(s), x'(s), y(s), y'(s))\| \, ds + \|x_\infty\|
\]
\[
\leq 4\varepsilon \|e_0^*\| (x, y) \|x + a_0^* + M_0b_0^* + \|x_\infty\| \tag{2.12}
\]
\[
\leq \frac{1}{2} \| (x, y) \| x + a_0^* + M_0b_0^* + \|x_\infty\|, \quad \forall t \in J.
\]
It follows from (2.10) and (2.12) that
\[
\|A_1(x, y)\| \leq \frac{1}{2} \| (x, y) \| x + \left(1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right) (a_0^* + M_0b_0^*)
\]
\[
+ \left(1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right) \|x_\infty\|. \tag{2.13}
\]
So, \(A_1(x, y) \in DC^1[J, E]\). On the other hand, it can be easily seen that
\[
A_1(x, y)(t) \geq \left(\frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right) x_\infty \geq \lambda_0^* x_\infty \geq \lambda_0^* x_0^*, \quad \forall t \in J,
\]
\[
A_1'(x, y)(t) \geq x_\infty \geq \lambda_0^* x_\infty \geq \lambda_0^* x_0^*, \quad \forall t \in J.
\]
so, \(A_1(x, y) \in Q_1\). In the same way, we can easily obtain
\[
\|A_2(x, y)\| \leq \frac{1}{2} \| (x, y) \| x + \left(1 + \frac{\int_0^\infty h(t) \, dt}{1 - \int_0^\infty h(t) \, dt}\right) (a_0^* + M_0b_0^*)
\]
\[
+ \left(1 + \frac{\int_0^\infty h(t) \, dt}{1 - \int_0^\infty h(t) \, dt}\right) \|y_\infty\|. \tag{2.14}
\]
Under assumption Lemma 2.2. solution of

\[ A_2(x, y)(t) \geq \left( \frac{\int_0^\infty h(t) \, dt}{1 - \int_0^\infty h(t) \, dt} \right) y_\infty \geq \lambda_1^* y_\infty \geq \lambda_1^* y_0^*, \quad \forall \ t \in J, \]

\[ A_2'(x, y)(t) \geq y_\infty \geq \lambda_1^* y_\infty \geq \lambda_1^* y_0^*, \quad \forall t \in J. \]

where \( M_1 = \max\{z_i(u_0, u_1, v_0, v_1) : r \leq u_i, v_i \leq R \ (i = 0, 1)\}. \) Thus, we have proved that \( A \) maps \( Q \) to \( Q \) and we have

\[ \|A(x, y)\|_X \leq \frac{1}{2} \|(x, y)\|_X + \gamma, \]  \hspace{1cm} (2.15)

where

\[ \gamma = \max \left\{ \left(1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right)(a_0^* + M_0 b_0^*) + \left(1 + \frac{\int_0^\infty t q(t) \, dt}{1 - \int_0^\infty q(t) \, dt}\right)\|x_\infty\|, \right. \]

\[ \left. \left(1 + \frac{\int_0^\infty h(t) \, dt}{1 - \int_0^\infty h(t) \, dt}\right)(a_0^* + M_0 b_0^*) + \left(1 + \frac{\int_0^\infty h(t) \, dt}{1 - \int_0^\infty h(t) \, dt}\right)\|y_\infty\| \right\}. \]  \hspace{1cm} (2.16)

Finally, we show that \( A \) is continuous. Let \((x_m, y_m), (\overline{x}, \overline{y}) \in Q, \|(x_m, y_m) - (\overline{x}, \overline{y})\|_X \to 0 \ (m \to \infty). \) Then \( \{x_m, y_m\} \) is a bounded subset of \( Q. \) Thus, there exists \( r > 0 \) such that \( \sup_m \|(x_m, y_m)\|_X < r \) for \( m \geq 1 \) and \( \|(\overline{x}, \overline{y})\|_X \leq r + 1. \)

Similar to (2.10) and (2.12), it is easy to show that

\[ \|A_1(x_m, y_m) - A_1(\overline{x}, \overline{y})\|_X \]

\[ \leq \int_0^\infty \left\|f(s, x_m(s), x'_m(s), y_m(s), y'_m(s)) - f(s, \overline{x}(s), \overline{x}'(s), \overline{y}(s), \overline{y}'(s))\right\| \, ds \]

\[ + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt} \int_0^\infty \left\|f(s, x_m(s), x'_m(s), y_m(s), y'_m(s)) - f(s, \overline{x}(s), \overline{x}'(s), \overline{y}(s), \overline{y}'(s))\right\| \, ds. \]  \hspace{1cm} (2.17)

It is clear that

\[ f(t, x_m(t), x'_m(t), y_m(t), y'_m(t)) \to f(t, \overline{x}(t), \overline{x}'(t), \overline{y}(t), \overline{y}'(t)) \]  \hspace{1cm} (2.18)

as \( m \to \infty, \) for all \( t \in J_+. \) By (2.8), we obtain

\[ \|f(t, x_m(t), x'_m(t), y_m(t), y'_m(t)) - f(t, \overline{x}(t), \overline{x}'(t), \overline{y}(t), \overline{y}'(t))\| \]

\[ \leq 8\epsilon_0 c_0(1 + t) r + 2a_0(1 + t) + 2M_0 b_0(1 + t) \]

\[ = \sigma_0(t), \quad \sigma_0(t) \in L[J, J], \quad m = 1, 2, 3, \ldots, \forall t \in J_+. \]  \hspace{1cm} (2.19)

It follows from (2.18), (2.19), and the dominated convergence theorem that

\[ \lim_{m \to \infty} \int_0^\infty \|f(s, x_m(s), x'_m(s), y_m(s), y'_m(s)) - f(s, \overline{x}(s), \overline{x}'(s), \overline{y}(s), \overline{y}'(s))\| \, ds = 0. \]

It follows from the above inequality and (2.17) that \( \|A_1(x_m, y_m) - A_1(\overline{x}, \overline{y})\|_D \to 0 \) as \( m \to \infty. \) By the same method, we have \( \|A_2(x_m, y_m) - A_2(\overline{x}, \overline{y})\|_D \to 0 \) as \( m \to \infty. \) Therefore, the continuity of \( A \) is proved.

**Lemma 2.2.** Under assumption (H1), \((x, y) \in Q \cap (C^2[J_i, E] \times C^2[J_i, E])\) is a solution of (1.1) if and only if \((x, y) \in Q\) is a fixed point of \( A.\)
Proof. Suppose that $(x, y) \in Q \cap (C^2[J_+, E] \times C^2[J_+, E])$ is a solution of (1.1). For $t \in J$, integrating (1.1) from 0 to $t$, we have

\[
-x'(t) + x'(0) = \int_0^t f(s, x(s), x'(s)) \, ds,
\]
\[
-y'(t) + y'(0) = \int_0^t g(s, x(s), x'(s)) \, ds.
\]

Taking the limit as $t \to \infty$, we obtain

\[
-x_\infty + x'(0) = \int_0^\infty f(s, x(s), x'(s)) \, ds,
\]
\[
-y_\infty + y'(0) = \int_0^\infty g(s, x(s), x'(s)) \, ds.
\]

Thus,

\[
x'(0) = x_\infty + \int_0^\infty f(s, x(s), x'(s)) \, ds,
\]
\[
y'(0) = y_\infty + \int_0^\infty g(s, x(s), x'(s)) \, ds.
\]

We obtain

\[
x'(t) = x_\infty + \int_0^t f(s, x(s), x'(s)) \, ds - \int_0^t f(s, x(s), x'(s)) \, ds,
\]
\[
y'(t) = y_\infty + \int_0^t g(s, x(s), x'(s)) \, ds - \int_0^t g(s, x(s), x'(s)) \, ds.
\]

Integrating (2.24) and (2.25) from 0 to $t$, we obtain

\[
x(t) = x(0) + tx_\infty + \int_0^t G(t, s)f(s, x(s), x'(s)) \, ds,
\]
\[
y(t) = y(0) + ty_\infty + \int_0^t G(t, s)g(s, x(s), x'(s)) \, ds,
\]

which together with the boundary-value condition implies that

\[
x(0) = \int_0^{\infty} q(t)x(t) \, dt = x(0) \int_0^{\infty} q(t) \, dt + x_\infty \int_0^{\infty} t q(t) \, dt + \int_0^{\infty} \int_0^{\infty} q(t)G(t, s)f(s, x(s), x'(s)) \, ds \, dt,
\]
\[
y(0) = \int_0^{\infty} h(t)x(t) \, dt = y(0) \int_0^{\infty} h(t) \, dt + y_\infty \int_0^{\infty} t h(t) \, dt + \int_0^{\infty} \int_0^{\infty} h(t)G(t, s)g(s, x(s), x'(s)) \, ds \, dt.
\]
Thus,
\[
x(0) = \frac{1}{1 - \int_0^\infty q(t) \, dt} \left\{ \int_0^\infty t q(t) \, dt \right\} (2.29)
\]

\[
y(0) = \frac{1}{1 - \int_0^\infty h(t) \, dt} \left\{ \int_0^\infty t h(t) \, dt \right\} (2.30)
\]

Substituting (2.29) and (2.30) in (2.26) and (2.27),
\[
x(t) = \frac{1}{1 - \int_0^\infty q(t) \, dt} \left\{ \int_0^\infty t q(t) \, dt \\
+ \int_0^\infty \int_0^\infty q(t) G(t, s)f(s, x(s), x'(s)) \, ds \, dt \right\} \\
+ tx_{\infty} + \int_0^\infty G(t, s)f(s, x(s), x'(s)) \, ds
\]
\[
y(t) = \frac{1}{1 - \int_0^\infty h(t) \, dt} \left\{ \int_0^\infty t h(t) \, dt \\
+ \int_0^\infty \int_0^\infty h(t) G(t, s)g(s, x(s), x'(s)) \, ds \, dt \right\} \\
+ ty_{\infty} + \int_0^\infty G(t, s)g(s, x(s), x'(s)) \, ds
\]

(2.31)

Obviously, the integral \(\int_0^t \int_s^\infty f(\tau, x(\tau), x'(\tau), y(\tau), y'(\tau)) \, d\tau \, ds\) and the integral \(\int_0^t \int_s^\infty g(\tau, x(\tau), x'(\tau), y(\tau), y'(\tau)) \, d\tau \, ds\) are convergent. Therefore, \((x, y)\) is a fixed point of operator \(A\). Conversely, if \((x, y)\) is fixed point of operator \(A\), then direct differentiation gives the proof. \qed

**Lemma 2.3.** Let (H1) be satisfied, and \(V\) be a bounded set of \(Q\). Then \(\frac{(A_i(V)(t))}{1+t}\) and \((A'_i(V)(t))\) are equicontinuous on any finite subinterval of \(J\); and for any \(\epsilon > 0\), there exists \(N = \max\{N_1, N_2\} > 0\) such that
\[
\|A_i(x, y)(t_1) - A_i(x, y)(t_2)\| < \epsilon, \quad \|A'_i(x, y)(t_1) - A'_i(x, y)(t_2)\| < \epsilon
\]
uniformly with respect to \((x, y) \in V\) as \(t_1, t_2 \geq N\).
Proof. We only give the proof for operator $A_1$. For $(x, y) \in V$, $t_2 > t_1$, we have
\[
\|A_1(x, y)(t_1) - A_1(x, y)(t_2)\| \leq \left| \frac{t_1}{1 + t_1} - \frac{t_2}{1 + t_2} \right| \left( \frac{\int_0^\infty tq(t) \, dt}{1 - \int_0^\infty q(t) \, dt} \right) \|x_{\infty}\|
+ \left| \frac{t_1}{1 + t_1} - \frac{t_2}{1 + t_2} \right| \left( 1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt} \right) \|x_{\infty}\|
\times \left\{ \left\| \int_{t_1}^{t_2} f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\| + \left\| \int_{t_1}^{t_2} s f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\| \right\}
\times \left\{ \left\| \int_{t_1}^{t_2} f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\| + \left\| \int_{t_1}^{t_2} s f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\| \right\}
(2.32)
\leq |t_1 - t_2| \left( 1 + \frac{\int_0^\infty tq(t) \, dt}{1 - \int_0^\infty q(t) \, dt} \right) \|x_{\infty}\| + \left( 1 + \frac{\int_0^\infty q(t) \, dt}{1 - \int_0^\infty q(t) \, dt} \right) \|x_{\infty}\|
\times \left\{ \left\| \int_{t_1}^{t_2} f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\| + \left\| \int_{t_1}^{t_2} s f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\| \right\}
\times \left\{ \left\| \int_{t_1}^{t_2} f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\| + \left\| \int_{t_1}^{t_2} s f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\| \right\}.
\]

Then, it is easy to see that by the above inequality and (H1), $\{\frac{A_1V(t)}{1+t}\}$ is equicontinuous on any finite subinterval of $J$.

Since $V \subset Q$ is bounded, there exists $r > 0$ such that for any $(x, y) \in V$, $\|(x, y)\|_x \leq r$. By (2.11), we obtain
\[
\|A_1(x, y)(t_1) - A_1(x, y)(t_2)\|
= \left\| \int_{t_1}^{t_2} f(s, x(s), x'(s), y(s), y'(s)) \, ds \right\|
\leq \int_{t_1}^{t_2} |4\epsilon r c_0(s)(1 + s) + a_0(s) + M_0 b_0(s)| \, ds.
(2.33)
\]
It follows from (2.33), (H1), and the absolute continuity of Lebesgue integral that $\{A_1V(t)\}$ is equicontinuous on any finite subinterval of $J$.

We are in position to show that for any $\epsilon > 0$, there exists $N_1 > 0$ such that
\[
\|A_1(x, y)(t_1) - A_1(x, y)(t_2)\| < \epsilon, \quad \|A_1(x, y)(t_1) - A_1(x, y)(t_2)\| < \epsilon,
\]
uniformly with respect to \( x \in V \) as \( t_1, t_2 \geq N_1 \). Combining this with (2.32), we need only to show that for any \( \epsilon > 0 \), there exists sufficiently large \( N_1 > 0 \) such that
\[
\| \int_0^{t_1} \frac{s}{1 + t_1} f(s, x(s), x'(s), y(s), y'(s))ds \quad \text{and} \quad \int_0^{t_2} \frac{s}{1 + t_2} f(s, x(s), x'(s), y(s), y'(s))ds \| < \epsilon,
\]
for all \( x \in V \) as \( t_1, t_2 \geq N_1 \). The rest part of the proof is very similar to [8, Lemma 2.3], we omit the details.

The proof for operator \( A_2 \) can be given in a similar way. Then the proof is complete. \(\square\)

**Lemma 2.4.** Let (H1) be satisfied, \( V \) be a bounded set in \( DC^1[J, E] \times DC^1[J, E] \). Then
\[
\alpha_D(A_iV) = \max \{ \sup_{t \in J} \alpha \left( \frac{A_iV(t)}{1 + t} \right), \sup_{t \in J} \alpha((A_iV)'(t)) \} \quad (i = 0, 1).
\]

The proof of the above lemma is similar to that of [8, Lemma 2.4], we omit it.

**Lemma 2.5** (Mönch Fixed-Point Theorem [1, 3]). Let \( Q \) be a closed convex set of \( E \) and \( u \in Q \). Assume that the continuous operator \( F : Q \to Q \) has the following property: \( V \subset Q \) countable, \( V \subset \overline{\text{co}}(\{u\} \cup F(V)) \Rightarrow V \) is relatively compact. Then \( F \) has a fixed point in \( Q \).

**Lemma 2.6.** If (H3) is satisfied, then for \( x, y \in Q \), \( x^{(i)} \leq y^{(i)} \), \( t \in J \) (\( i = 0, 1 \)) imply \( (Ax)^{(i)} \leq (Ay)^{(i)} \), \( t \in J \) (\( i = 0, 1 \)).

It is easy to see that the above lemma follows from (2.3) (2.4) (2.11) and condition (H3).

**Lemma 2.7** ([5]). Let \( D \) and \( F \) be bounded sets in \( E \), then
\[
\bar{\alpha}(D \times F) = \max \{ \alpha(D), \alpha(F) \},
\]
where \( \bar{\alpha} \) and \( \alpha \) denote the Kuratowski measure of non-compactness in \( E \times E \) and \( E \), respectively.

**Lemma 2.8** ([5]). Let \( P \) be normal (fully regular) in \( E \), \( \bar{P} = P \times P \), then \( \bar{P} \) is normal (fully regular) in \( E \times E \).

### 3. Main results

**Theorem 3.1.** Assume (H1), (H2) and that \( 2D^* \cdot G^* < 1 \). Then \([1, 1]\) has a positive solution \((\overline{\tau}, \overline{\eta}) \in (DC^1[J, E] \cap C^2[J_+, E]) \times (DC^1[J, E] \cap C^2[J_+, E])\) satisfying \((\overline{\tau})^{(i)}(t) \geq \lambda_0^* x_0^*, (\overline{\eta})^{(i)}(t) \geq \lambda_1^* y_0^* \) for \( t \in J \) (\( i = 0, 1 \)).

**Proof.** By Lemma 2.1, the operator \( A \) defined by (2.2) is a continuous operator from \( Q \) to \( Q \). By Lemma 2.2 we need only to show that \( A \) has a fixed point \((\overline{\tau}, \overline{\eta})\) in \( Q \). Choose \( R > 2\gamma \) and let \( Q^* = \{(x, y) \in Q : \|(x, y)\|_X \leq R\} \). Obviously, \( Q^* \) is a bounded closed convex set in space \( DC^1[J, E] \times DC^1[J, E] \). It is easy to see that \( Q^* \) is not empty since \(((1 + t)x_\infty, (1 + t)y_\infty) \in Q^* \). It follows from (2.15) (2.16) that \((x, y) \in Q^* \) implies that \( A(x, y) \in Q^* \); i.e., \( A \) maps \( Q^* \) to \( Q^* \). Let
where \((A_1(t)) (t) = \{A_1(x, y) : m = 1, 2, \ldots \}, (A_1(t))' (t) = \{A_1'(x, y) : m = 1, 2, \ldots \}. \)

By Lemma 2.4, we have

\[
\alpha_D(A_1 V) = \max \left\{ \sup_{t \in J} \alpha((A_1 V)'(t)), \sup_{t \in J} \alpha \left( \frac{(A_1 V)(t)}{1 + t} \right) \right\},
\]

where \((A_1(t))(t) = \{A_1(x, y) : m = 1, 2, \ldots \}, (A_1(t))'(t) = \{A_1'(x, y) : m = 1, 2, \ldots \}. \)

Then, by Guo et al. [9, Theorem 1.2.3], (3.1), (3.2), (3.4), (H2), and Lemma 2.7, we obtain

\[
\alpha \left( \frac{(A_1 V)(t)}{1 + t} \right) \leq \frac{D_0}{1 + t} \left\{ 2 \int_0^T \alpha(f(t, x(t), x'(t), y(t), y'(t))) dt + 2 \varepsilon \right\}
\]

\[
\leq 2D_0 \int_0^\infty \alpha(f(t, x(t), x'(t), y(t), y'(t))) dt + 2 \varepsilon \tag{3.5}
\]

\[
\leq 2D_0 \alpha_X(V) \int_0^\infty (L_{00}(s) + K_{00}(s))(1 + s) + (L_{01}(s) + K_{01}(s)) dt + 2 \varepsilon.
\]

\[
\leq 2D_0 G_0^* \alpha_X(V) + 2 \varepsilon,
\]

and

\[
\alpha((A_1 V)'(t)) \leq 2 \int_0^\infty \alpha(f(t, x(t), x'(t), y(t), y'(t))) ds + 2 \varepsilon \leq 2G_0^* \alpha_X(V) + 2 \varepsilon.
\]

From this inequality, (3.3) and (3.5), it follows that

\[
\alpha_D(A_1 V) \leq 2D_0 \alpha_X(V) G_0^*.
\]

(3.6)

In the same way, we obtain

\[
\alpha_D(A_2 V) \leq 2D_1 \alpha_X(V) G_1^*.
\]

(3.7)

On the other hand, \(\alpha_X(V) \leq \alpha_X(\overline{\{u \cup (AV)\}}) = \alpha_X(AV). \) Then, (3.6), (3.7), (H2), and Lemma 2.7 imply \(\alpha_X(V) = 0; \) i.e., \(V \) is relatively compact in
Theorem 3.2. Let cone $P$ be normal and conditions (H1)–(H3) be satisfied. Then \( (1.1) \) has a positive solution \( (x, y) \in Q \cap (C^2[J_+] \times C^2[J_+]) \) which is minimal in the sense that \( u(i)(t) \geq x(i)(t), v(i)(t) \geq y(i)(t), t \in J \) \((i = 0, 1)\) for any positive solution \((u, v)\) \in \( Q \cap (C^2[J_+] \times C^2[J_+]) \) of \( (1.1) \). Moreover, \( \|x, y\|_X \leq 2 \gamma + \|(u_0, v_0)\|_X \), and there exists a monotone iterative sequence \( \{(u_m(t), v_m(t))\} \) such that \( u_m(t) \to x(t), v_m(t) \to y(t) \) as \( m \to \infty(i = 0, 1) \) uniformly on \( J \) and \( u_m''(t) \to x''(t), v_m''(t) \to y''(t) \) as \( m \to \infty \) for any \( t \in J_+ \), where

\[
\begin{align*}
\frac{1}{1 - \int_0^\infty q(t)\,dt} \int_0^x tq(t)\,dt + \int_0^\infty \int_0^\infty q(t)G(t, s)f(s)ds,

\lambda_1^*y_0^*,\lambda_1^*y_0^*\,ds\,dt + tx_\infty + \int_0^\infty G(t, s)f(s, x_0^*, x_0^*\,ds, y_0^*, y_0^*)\,ds, \\
\lambda_1^*y_0^*,\lambda_1^*y_0^*\,ds\,dt + ty_\infty + \int_0^\infty G(t, s)g(s)\,ds, y_0^*, y_0^*)\,ds, \\
\lambda_1^{i+1}y_0^*\,ds\,dt + tx_\infty + \int_0^\infty G(t, s)f(s, u_{m-1}(s), u_{m-1}'(s), v_{m-1}(s), v_{m-1}'(s))\,ds,
\end{align*}
\]

(3.8)

(3.9)

and

\[
\begin{align*}
\frac{1}{1 - \int_0^\infty q(t)\,dt} \int_0^x tq(t)\,dt + \int_0^\infty \int_0^\infty q(t)G(t, s)f(s, u_{m-1}(s), u_{m-1}'(s), v_{m-1}(s), v_{m-1}'(s))\,ds,

\lambda_1^*y_0^*,\lambda_1^*y_0^*\,ds\,dt + ty_\infty + \int_0^\infty G(t, s)g(s, \lambda_1^*y_0^*, \lambda_1^*y_0^*\,ds, y_0^*, y_0^*)\,ds, \\
\lambda_1^{i+1}y_0^*\,ds\,dt + tx_\infty + \int_0^\infty G(t, s)f(s, u_{m-1}(s), u_{m-1}'(s), v_{m-1}(s), v_{m-1}'(s))\,ds,
\end{align*}
\]

(3.10)

(3.11)

Proof. From (3.8), (3.9) one sees that \((u_0, v_0) \in C[J, E] \times C[J, E] \) and

\[
u_0'(t) = \int_t^\infty f(s, x_0^*, x_0^*\,ds, y_0^*, y_0^*)\,ds + x_\infty.
\]

(3.12)
By (3.8) and (3.12), we have \( u_0^{(i)}(i = 0, 1) \) and
\[
\frac{\|u_0(t)\|}{1 + t} \leq \frac{1}{1 - \int_0^\infty q(t) \, dt} \left\{ \int_0^\infty \int_0^\infty q(t) G(t, s) \|f(s, \lambda_0^* x_0^*, \lambda_0^* y_0^*, \lambda_1^* y_0^*)\| \, ds \, dt \\
+ \|x_\infty\| \int_0^\infty t q(t) \, dt \right\} + t \|x_\infty\| + \int_0^\infty G(t, s) \|f(s, \lambda_0^* x_0^*, \lambda_0^* y_0^*, \lambda_1^* y_0^*)\| \, ds,
\]
\[
\leq \left(1 + \frac{\int_0^\infty g(t) \, dt}{1 - \int_0^\infty g(t) \, dt} \right) \int_0^\infty a_0(s) + b_0(s) z_0(\|\lambda_0^* x_0^*\|, \|\lambda_0^* y_0^*\|, \|\lambda_1^* y_0^*\|) \, ds + \|x_\infty\|,
\]
which implies \( \|u_0\|_D < \infty \). Similarly, we have \( \|v_0\|_D < \infty \). Thus, \((u_0, v_0) \in DC^1(J,E) \times DC^1(J,E)\). It follows from (2.3) and (3.10) that
\[
(u_m, v_m)(t) = A(u_{m-1}, v_{m-1})(t), \quad \forall t \in J, \quad m = 1, 2, 3, \ldots \tag{3.13}
\]
By Lemma 2.4, we obtain \((u_m, v_m) \in Q\) and
\[
\|(u_m, v_m)\|_Q = \|A(u_{m-1}, v_{m-1})\|_Q \leq \frac{1}{2} \|(u_{m-1}, v_{m-1})\|_Q + \gamma. \tag{3.14}
\]
By (H3) and (3.13), we have
\[
u_1(t) = A_1(u_0(t), v_0(t)) \geq A_1(\lambda_0^* x_0^*, \lambda_1^* y_0^*) = u_0(t), \quad \forall t \in J, \tag{3.15}
\]
and
\[
u_1(t) = A_2(u_0(t), v_0(t)) \geq A_2(\lambda_0^* x_0^*, \lambda_1^* y_0^*) = v_0(t), \quad \forall t \in J. \tag{3.16}
\]
By induction, we obtain
\[
(\lambda_0^* x_0^*, \lambda_1^* y_0^*) \leq (u_0(t), v_0(t)) \leq (u_1(t), v_1(t)) \leq \cdots \leq (u_m(t), v_m(t)) \leq \cdots, \tag{3.17}
\]
for all \( t \in J \). By induction and Lemma 2.6 and (3.13), we have
\[
(\lambda_0^* x_0^*, \lambda_1^* y_0^*) \leq (v_0^{(i)}(t), v_0^{(i)}(t)) \leq (u_1^{(i)}(t), v_1^{(i)}(t)) \leq \cdots \leq (u_m^{(i)}(t), v_m^{(i)}(t)) \leq \cdots, \tag{3.18}
\]
for all \( t \in J \). It follows from (3.14), by induction, that
\[
\|(u_m, v_m)\|_Q \leq \gamma + \frac{1}{2} \gamma + \cdots + \left(\frac{1}{2}\right)^{m-1} \gamma + \frac{1}{2}^m \|(u_0, v_0)\|_Q \leq \gamma + \frac{1}{2}^m \|(u_0, v_0)\|_Q \leq 2 \gamma + \|(u_0, v_0)\|_Q \quad (m = 1, 2, 3, \ldots). \tag{3.19}
\]
Let \( K = \{(x, y) \in Q : \|(x, y)\|_Q \leq 2 \gamma + \|(u_0, v_0)\|_Q \} \). Then \( K \) is a bounded closed convex set in space \( DC^1(J,E) \times DC^1(J,E) \) and operator \( A \) maps \( K \) into \( K \). Clearly, \( K \) is not empty since \((u_0, v_0) \in K\). Let \( W = \{(u_m, v_m) : m = 0, 1, 2, \ldots \}, AW = \{A(u_m, v_m) : m = 0, 1, 2, \ldots \} \). Obviously, \( W \subset K \) and \( W = \{(u_0, v_0) \cup A(W) \). As
in the proof of Theorem [3.1], we obtain \( \alpha_X(AW) = 0 \); i.e., \( W \) is relatively compact in \( DC^1[J, E] \times DC^1[J, E] \). So, there exists a \((\bar{\varphi}, \bar{\psi}) \in DC^1[J, E] \times DC^1[J, E] \) and a subsequence \( \{(u_{m_j}, v_{m_j}) : j = 1, 2, 3, \ldots \} \subset W \) such that \( \{(u_{m_j}^{(i)}(t), v_{m_j}^{(i)}(t)) : j = 1, 2, 3, \ldots \} \) converges to \((\bar{\varphi}^{(i)}(t), \bar{\psi}^{(i)}(t)) \) uniformly on \( J \) \((i = 0, 1) \). Since that \( P \) is normal and \( \{(u_{m_j}^{(i)}(t), v_{m_j}^{(i)}(t)) : j = 1, 2, 3, \ldots \} \) is nondecreasing, by Lemma 2.8 it is easy to see that the entire sequence \((u_{m_j}^{(i)}(t), v_{m_j}^{(i)}(t)) : j = 1, 2, 3, \ldots \) converges to \((\bar{\varphi}^{(i)}(t), \bar{\psi}^{(i)}(t)) \) uniformly on \( J \) \((i = 0, 1) \). Since \((u_m, v_m) \in K \) and \( K \) is a closed convex set in space \( DC^1[J, E] \times DC^1[J, E] \), we have \((\bar{\varphi}, \bar{\psi}) \in K \). It is clear, \[
f(s, u_m(s), u_m'(s), v_m(s), v_m'(s)) \to f(s, \bar{\varphi}(s), \bar{\varphi}'(s), \bar{\psi}(s), \bar{\psi}'(s)), \tag{3.20}
\]
as \( m \to \infty \), for all \( s \in J_+ \). By (H1) and (3.19), we have \[
\|f(s, u_m(s), u_m'(s), v_m(s), v_m'(s)) - f(s, \bar{\varphi}(s), \bar{\varphi}'(s), \bar{\psi}(s), \bar{\psi}'(s))\| \leq \varepsilon \|u_m(s), v_m(s)\| + 2\varepsilon_0(s) + 2M_0b_0(s) \tag{3.21}
\]
Noticing (3.20) and (3.21) and taking limit as \( m \to \infty \) in (3.10), we obtain \[
\bar{\varphi}(t) = \frac{1}{1 - \int_0^\infty q(t) dt} \left\{ \frac{1}{x} \int_0^\infty tq(t) dt + \int_0^\infty \int_0^\infty q(t)G(t, s)f(s, \bar{\varphi}(s), \bar{\varphi}'(s), \bar{\psi}(s), \bar{\psi}'(s)) ds dt \right\} \tag{3.22}
\]
In the same way, taking limit as \( m \to \infty \) in (3.11), we obtain \[
\bar{\psi}(t) = \frac{1}{1 - \int_0^\infty h(t) dt} \left\{ \frac{1}{y} \int_0^\infty th(t) dt + \int_0^\infty \int_0^\infty h(t)G(t, s)g(s, \bar{\varphi}(s), \bar{\psi}(s), \bar{\psi}'(s)) ds dt \right\} \tag{3.23}
\]
which together with (3.22) and Lemma 2.2 implies that \((\bar{\varphi}, \bar{\psi}) \in K \cap C^2[J_+, E] \times C^2[J_+, E] \) and \((\bar{\varphi}(t), \bar{\psi}(t)) \) is a positive solution of (1.1). Differentiating (3.10) twice, we obtain \[
u''(t) = -f(t, u_{m-1}(t), u'_{m-1}(t), v_{m-1}(t), v'_{m-1}(t)), \quad \forall t \in J', \ m = 1, 2, 3, \ldots
\]
Hence, by (3.20), we obtain \[
\lim_{m \to \infty} u''(t) = -f(t, \bar{\varphi}(t), \bar{\varphi}'(t), \bar{\psi}(t), \bar{\psi}'(t)) = \bar{\varphi}''(t), \quad \forall t \in J'.
\]
Similarly, we have \[
\lim_{m \to \infty} v''(t) = -g(t, \bar{\varphi}(t), \bar{\varphi}'(t), \bar{\psi}(t), \bar{\psi}'(t)) = \bar{\psi}''(t), \quad \forall t \in J'_+.
\]
Let \((m(t), n(t)) \) be any positive solution of (1.1). By Lemma 2.2 we have \((m, n) \in Q \) and \((m(t), n(t)) = A(m, n)(t), \) for \( t \in J \). It is clear that \( m(t) \geq \lambda_1 x_0 > \theta, \)
\( n(t) \geq \lambda_1 y_0 > \theta \) for any \( t \in J \) \((i = 0, 1) \). So, by Lemma 2.6 we have \( m(t) \geq
value problem of form (1.1) in normal constant 1. Now we consider (4.1) which can be regarded as a boundary-

t for $t \in J$ ($i = 0, 1$). From Lemma 2.6 it follows that

$$(A_1(m,n)(t), A_2(m,n)(t)) \geq (A_1(u_{m-1}, v_{m-1}))(t), A_2(u_{m-1}, v_{m-1}))(t)$$

for $t \in J$ ($i = 0, 1$); i.e., $(m^{(i)}(t), n^{(i)}(t)) \geq (u^{(i)}(t), v^{(i)}(t))$ for $t \in J$ ($i = 0, 1$).

Hence, by induction, we obtain

$$m^{(i)}(t) \geq \pi_i(t), n^{(i)}(t) \geq \pi_i(t) \quad \forall t \in J (i = 0, 1; m = 0, 1, 2, \ldots).$$  (3.24)

Now, taking limits in (3.24), we obtain $m^{(i)}(t) \geq \pi_i(t), n^{(i)}(t) \geq \pi_i(t)$ for $t \in J (i = 0, 1)$. The proof is complete. \qed

**Theorem 3.3.** Let cone $P$ be fully regular and conditions (H1) and (H3) be satisfied. Then the conclusion of Theorem 3.2 holds.

**Proof.** The proof is almost the same as that of Theorem 3.2. The only difference is that, instead of using condition (H2), the conclusion $\alpha_X(W) = 0$ is implied directly by (3.18) and (3.19), the full regularity of $P$ and Lemma 2.8. \qed

4. An example

Consider the infinite system of scalar singular second-order three-point boundary value problems:

$$-x''_n(t) = \frac{1}{9n^3 \sqrt{2\pi} t^9} \left(5 + y_n(t) + x'_{2n}(t) + y'_{2n}(t)\right)$$

$$+ \frac{2}{3n^2 x_n(t)} + \frac{4}{7n^5 x'_{2n}(t)} \right)^{1/2} + \frac{1}{9n^7 t(1 + 3t)^3} \ln \left[(1 + 3t)x_n(t)\right],

-y''_n(t) = \frac{5}{16n^4 y'_{2n}(t)} \right)^{1/3} + \frac{1}{7n^7 t(3 + 3t)} \ln \left[(3 + 4t)y'_{2n}(t)\right],

x_n(0) = \int_0^\infty e^{-t^2} x_n(t) dt, \quad x'_n(\infty) = \frac{1}{n},

y_n(0) = \int_0^\infty \frac{1}{2} e^{-\frac{t^2}{2}} y_n(t) dt, \quad y'_n(\infty) = \frac{1}{2n}, \quad (n = 1, 2, \ldots).$$  (4.1)

**Proposition 4.1.** System (4.1) has a minimal positive solution $(x_n(t), y_n(t))$ satisfying $x_n(t), x'_n(t) \geq 1/n, y_n(t), y'_n(t) \geq 1/(2n)$ for $0 \leq t < +\infty$ ($n = 1, 2, 3, \ldots$).

**Proof.** Let $E = c_0 = \{x = (x_1, \ldots, x_n, \ldots) : x_n \to 0\}$ with the norm $\|x\| = \sup_n |x_n|$. Obviously, $(E, \| \cdot \|)$ is a real Banach space. Choose $P = \{x = (x_n) \in c_0 : x_n \geq 0, n = 1, 2, 3, \ldots\}$. It is easy to verify that $P$ is a normal cone in $E$ with normal constant 1. Now we consider (4.1) which can be regarded as a boundary-value problem of form (1.1) in $E$ with $x_\infty = (1, 1/2, 1/3, \ldots), y_\infty = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots)$. In this situation, $x = (x_1, \ldots, x_n, \ldots), u = (u_1, \ldots, u_n, \ldots), y = (y_1, \ldots, y_n, \ldots), \ldots$
v = (v₁, . . . , vₙ, . . .), f = (f₁, . . . , fₙ, . . .), in which
\[
f_n(t, x, u, y, v) = \frac{1}{9n³\sqrt{v}e^{2t}(2 + 5t)^9} \left(5 + y_n + u_{2n} + v_{3n} + \frac{2}{3n²x_n} + \frac{4}{7n³u_{2n}}\right)^{1/2}
\]
\[+ \frac{1}{9\sqrt{t}(1 + 3t)^2} \ln[(1 + 3t)x_n],
\]
and
\[
g_n(t, x, u, y, v) = \frac{1}{7n^4\sqrt{v}e^{2t}(4 + 5t)^8} \left(6 + x_{3n} + u_{4n} + \frac{1}{8n³y_{2n}} + \frac{5}{16n^4v_{4n}}\right)^{1/3}
\]
\[+ \frac{1}{7\sqrt{t}(3 + 4t)^3} \ln[(3 + 4t)v_{4n}].
\]
Let \(x_0 = x_{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \ldots), y_0 = y_{\infty} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots).\) Then \(P_{0\lambda} = \{x = (x₁, x₂, \ldots, xₙ, . . .) : xₙ ≥ \frac{λ}{n}, \ n = 1, 2, 3, \ldots\}, P_{1λ} = \{y = (y₁, y₂, \ldots, yₙ, . . .) : yₙ ≥ \frac{λ}{n}, \ n = 1, 2, 3, . . .\}, \) for \(λ > 0.\) By a simple computation, we have \(D₀ = 8.7912, D₁ = 2.6787, \int₀^∞ e^{-t²} dt \approx 0.8863 < 1, \int₀^∞ xe^{-t²} dt = 0.5, \int₀^∞ \frac{1}{2}e^{-t²} dt \approx 0.6267 < 1, \int₀^∞ \frac{1}{2}e^{-t²} dt = 0.5, λ₀ = λ₁ = 1. It is clear, \(f, g ∈ C[J⁺ × P₀λ × P₁λ × P_{0\lambda} × P_{1\lambda}, P]\) for any \(λ > 0.\) Note that \(\sqrt{v}e^{2t} > \sqrt{t}\) for \(t > 0\) for \(t > 0,\) by [4.2] and [4.3], we obtain
\[
\|f(t, x, u, y, v)\| ≤ \frac{1}{9\sqrt{t}(1 + 3t)^2} \left\{\frac{167}{21} + \|y\| + \|u\| + \|v\|\right\}^{1/2}
\]
\[+ \ln \left[(1 + 3t)\|x\|]\right],
\]
and
\[
\|g(t, x, u, y, v)\| ≤ \frac{1}{7\sqrt{t}(3 + 4t)^2} \left\{\left(9 + \|x\| + \|u\|\right)^{1/3} + \ln \left[(3 + 4t)\|v\|\right]\right\},
\]
which imply (H1) is satisfied for \(a₀(t) = 0, b₀(t) = c₀(t) = \frac{1}{9\sqrt{t}(3 + 3t²)²}, a₁(t) = 0, b₁(t) = c₁(t) = \frac{1}{7\sqrt{t}(3 + 4t)^2}\)
and
\[
z₀(u₀, u₁, u₂, u₃) = \left(\frac{167}{21} + u₁ + u₂ + u₃\right)^{1/2} + \ln[(1 + 3t)u₀],
\]
\[
z₁(u₀, u₁, u₂, u₃) = (9 + u₀ + u₁)^{1/3} + \ln[(3 + 4t)u₃).
\]
Let \(f^1 = \{f₁^1, f₂^1, \ldots, fₙ^1, \ldots\}, f^2 = \{f₁^2, f₂^2, \ldots, fₙ^2, \ldots\}, g^1 = \{g₁^1, g₂^1, \ldots, gₙ^1, \ldots\},
\]
g² = \{g₁², g₂², \ldots, gₙ², \ldots\}, where
\[
f_n^1(t, x, u, y, v) = \frac{1}{9n³\sqrt{v}e^{2t}(2 + 5t)^9} \left(5 + y_n + u_{2n} + v_{3n} + \frac{2}{3n²x_n} + \frac{4}{7n³u_{2n}}\right)^{1/2},
\]
\[
f_n^2(t, x, u, y, v) = \frac{1}{9\sqrt{t}(1 + 3t)^2} \ln[(1 + 3t)x_n],
\]
\[
g_n^1(t, x, u, y, v) = \frac{1}{7n^4\sqrt{v}e^{2t}(4 + 5t)^8} \left(6 + x_{3n} + u_{4n} + \frac{1}{8n³y_{2n}} + \frac{5}{16n^4v_{4n}}\right)^{1/3},
\]
\[ g_n^2(t, x, u, y, v) = \frac{1}{7 \sqrt[3]{(3 + 4t)^3}} \ln[(3 + 4t)v_n]. \] (4.9)

Let \( t \in J_+ \), and \( R > 0 \), and \( \{z^{(m)}\} \) be any sequence in \( f^1(t, P^r_{0R}, P^r_{0R}, P^r_{1R}, P^r_{1R}) \), where \( z^{(m)} = (z_1^{(m)}, \ldots, z_n^{(m)}, \ldots) \). By \( (4.6) \), we have
\[ 0 \leq z_n^{(m)} \leq \frac{1}{9n^3 \sqrt{e^2(t + 5t)^9}} \left( \frac{167}{21} + 3R \right)^{1/2} (n, m = 1, 2, 3, \ldots). \] (4.10)

So, \( \{z_n^{(m)}\} \) is bounded, and by the diagonal method we can choose a subsequence \( \{m_i\} \subset \{m\} \) such that
\[ \{z_n^{(m_i)}\} \to \pi_n \quad \text{as } i \to \infty \quad (n = 1, 2, 3, \ldots), \] (4.11)
which by \( (4.10) \) implies
\[ 0 \leq \pi_n \leq \frac{1}{9n^3 \sqrt{e^2(t + 5t)^9}} \left( \frac{167}{21} + 3R \right)^{1/2} (n = 1, 2, 3, \ldots). \] (4.12)

Hence \( \pi = (\pi_1, \ldots, \pi_n, \ldots) \in c_0 \). It is easy to see from \( (4.10)-(4.12) \) that
\[ \|z^{(m_i)} - \pi\| = \sup_n |z_n^{(m_i)} - \pi_n| \to 0 \quad \text{as } i \to \infty. \]

Thus, we have proved that \( f^1(t, P^r_{0R}, P^r_{0R}, P^r_{1R}, P^r_{1R}) \) is relatively compact in \( c_0 \).

For any \( t \in J_+, R > 0, x, y, \pi, \bar{y} \in D \subset P^r_{0R} \), by \( (4.7) \) we have
\[
|f_n^2(t, x, u, v) - f_n^2(t, \pi, \bar{y}, \pi, \bar{y})| = \frac{1}{9 \sqrt[3]{7}(1 + 3t)^2} \left[ \ln[(1 + 3t)x_n] - \ln[(1 + 3t)\pi_n] \right] \\
\leq \frac{1}{9 \sqrt[3]{7}(1 + 3t)(1 + 3t)x_n},
\] (4.13)

where \( \xi_n \) is between \( x_n \) and \( \pi_n \). By \( (4.13) \), we obtain
\[
\|f^2(t, x, y, u, v) - f^2(t, \pi, \bar{y}, \pi, \bar{y})\| \leq \frac{1}{9 \sqrt[3]{7}(1 + 3t)} \|x - \pi\|, \quad x, y, \pi, \bar{y} \in D. \] (4.14)

In the same way, we can prove that \( g^1(t, P^r_{0R}, P^r_{0R}, P^r_{1R}, P^r_{1R}) \) is relatively compact in \( c_0 \). Also we can obtain
\[
\|g^2(t, x, y, u, v) - g^2(t, \pi, \bar{y}, \pi, \bar{y})\| \leq \frac{1}{9 \sqrt[3]{7}(3 + 4t)^2} \|v - \pi\|, \quad x, y, \pi, \bar{y} \in D.
\]

From this inequality and \( (4.14) \), it is easy to see that (H2) holds for \( L_{00}(t) = 1/\left(9 \sqrt[3]{7}(1 + 3t)^2\right) \), \( L_{10}(t) = 1/(7 \sqrt[3]{7}(3 + 4t)^2) \). By a simple computation, we have \( G^*_0 \approx 0.044, G^*_1 \approx 0.013, 2G^* \cdot D^* \approx 0.7736 < 1 \). Our conclusion follows from Theorem 3.1.

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