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REGULARITY AND SYMMETRY OF POSITIVE SOLUTIONS TO NONLINEAR INTEGRAL SYSTEMS

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Abstract. In this article, we consider the regularity and symmetry of positive solutions to the nonlinear integral system

$$u(x) = \int_{\mathbb{R}^n} K_\alpha(x-y)\frac{v(y)^q}{|y|^\beta}\,dy, \quad v(x) = \int_{\mathbb{R}^n} K_\alpha(x-y)\frac{u(y)^p}{|y|^\beta}\,dy$$

for $x \in \mathbb{R}^n$, where $K_\alpha(x)$ is the kernel of the operator $(-\Delta)^\alpha + id$ of order α, with $0 \le \beta < 2\alpha < n$, $1 < p$, $q < (n-\beta)/\beta$ and

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2\alpha+\beta}{n}.$$

We show that positive solution pairs $(u,v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ are locally Hölder continuous in $\mathbb{R}^N \setminus \{0\}$, radially symmetric about the origin, and strictly decreasing.

1. Introduction

In this article, we consider the regularity and symmetry of positive solutions to the nonlinear integral system

$$u(x) = \int_{\mathbb{R}^n} K_\alpha(x-y)\frac{v(y)^q}{|y|^\beta}\,dy, \quad v(x) = \int_{\mathbb{R}^n} K_\alpha(x-y)\frac{u(y)^p}{|y|^\beta}\,dy \tag{1.1}$$

for $x \in \mathbb{R}^n$, where $K_\alpha(x)$ is the kernel of the operator $(-\Delta)^\alpha + id$, $0 < \alpha < 1$, $0 \le \beta < 2\alpha < n$, $1 < p$, $q < \frac{n-\beta}{\beta}$ and

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2\alpha+\beta}{n}. \tag{1.2}$$

It can be shown that problem (1.1) is actually equivalent to the indefinite fractional elliptic systems

$$(-\Delta)^\alpha u + u = \frac{v^q}{|y|^\beta}, \quad (-\Delta)^\alpha v + v = \frac{u^p}{|y|^\beta}, \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

If $p = q$ and $\beta = 0$, problem (1.3) is of particular interest in fractional quantum mechanics in the study of particles on stochastic fields modelled by Lévy processes. A path integral over the Lévy flights paths and a fractional Schrödinger equation of fractional quantum mechanics are formulated by Laskin [11], see also [12]. It

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was shown in [10] that in the case $p = q$ and $\beta = 0$, problem (1.3) has at least a positive classical solution, which is radially symmetric and decays at infinity.

On the other hand, the problem

$$(-\Delta)^{\alpha/2}u = v^q, \quad (-\Delta)^{\alpha/2}v = u^p, \quad \text{in } \mathbb{R}^n \tag{1.4}$$

and its generalization have recently been extensively investigated in [1, 2, 4, 5, 6, 7] etc. Such a problem is equivalent to the integral system

$$u(x) = \int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^{n-\alpha}}\,dy, \quad v(x) = \int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^{n-\alpha}}\,dy, \quad \text{in } \mathbb{R}^n. \tag{1.5}$$

Solutions (u, v) of (1.5) are critical points of the functional associated with the well-known Hardy-Littlewood-Sobolev inequality, which is precisely stated as follows.

Proposition 1.1. *Let $0 < \lambda < n$ and let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$. Then*

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda}\,dx\,dy\Big| \le C_{q,\lambda,n}\|f\|_p\|g\|_q$$

for $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$.

Regularity and symmetry as well as classification of solutions of (1.5) and its generalization have been widely considered, see [2] and references therein. The Hardy-Littlewood-Sobolev inequality plays a key role in the study of these properties. Meanwhile, the moving plane method and the regularity lifting method for integral equations have been developed, see also [2] and references therein. Furthermore, the double weighted Hardy-Littlewood-Sobolev inequality, was introduced in [15], which is stated as follows.

Proposition 1.2. *Let $0 < \lambda < n$, $1 < p < \infty, \tau < \frac{n}{p'}, \beta < \frac{n}{q}, \tau + \beta \ge 0$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{p} + \frac{1}{q'} + \frac{\lambda+\tau+\beta}{n} = 2$. If $p \le q < \infty$ and $f \in L^p(\mathbb{R}^n), g \in L^{q'}(\mathbb{R}^n)$, Then, there exists a constant C independent of f and g such that the following inequality holds*

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\tau|x-y|^\lambda|y|^\beta}dx\,dy\Big| \le C\|f\|_p\|g\|_{q'}. \tag{1.6}$$

Critical points of the functional associated with inequality (1.6) will yield solutions of (1.4) if $\tau = \beta = 0$. Essentially, problem (1.4) is related to the Riesz potentials $\mathcal{I}_\alpha(f) = (-\Delta)^{-\alpha}$, $0 < \alpha < \frac{n}{2}$, which is defined by

$$\mathcal{I}_\alpha(f)(x) = \frac{1}{C(\alpha)}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2\alpha}}\,dy$$

for some $C(\alpha) > 0$. It is known that

$$\|\mathcal{I}_\alpha f\|_q \le C\|f\|_p, \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{n}.$$

In [8], the authors studied the regularity and radial symmetry of solutions of

$$u(x) = \int_{\mathbb{R}^n} G_\alpha(x-y)\frac{v(y)^q}{|y|^\beta}\,dy, \quad v(x) = \int_{\mathbb{R}^n} G_\alpha(x-y)\frac{u(y)^p}{|y|^\beta}\,dy, \tag{1.7}$$

where G_α is the Bessel kernel; that is,

$$G_\alpha(x) = \frac{(\sqrt{2\pi})^{-n}}{\Gamma(\frac{n}{2})}\int_0^\infty e^{-s}e^{-|x|^2/(4s)}s^{(\alpha-n)/2}\frac{ds}{s},$$

which is associated with the operator $(-\Delta+id)^{\frac{\alpha}{2}}$. While problem (1.1) is connected with the kernel K_α of the operator $(-\Delta)^\alpha + id$, such an operator enjoys different features, for instant, it is not clear whether the ground state solution of

$$(-\Delta)^\alpha u + u = u^p, \quad \text{in } \mathbb{R}^n$$

is exponentially decaying at infinity, see [10] for further properties of the operator $(-\Delta)^\alpha + id$ and results for one equation case. We will consider in this paper the regularity and radial symmetry of positive solutions of (1.1), which involves in Hardy type weights. To this purpose, we first establish the following Hardy-Littlewood-Sobolev inequality for the potential K_α with double weights.

Theorem 1.3. *Let $0 < \alpha < 1$, $1 < p, q < \frac{n}{2\alpha}$, $\tau, \beta \geq 0$. In addition $n(1 - \frac{1}{p} - \frac{1}{q} + \frac{2\alpha}{n}) > \beta + \tau > n(1 - \frac{1}{p} - \frac{1}{q})$. Then, there exists a constant C, independent of $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, such that the following inequality holds*

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)K_\alpha(x-y)h(y)}{|x|^\tau |y|^\beta} dx\, dy\Big| \leq C\|f\|_p \|h\|_q. \tag{1.8}$$

Furthermore, let

$$Th(x) = \int_{\mathbb{R}^n} \frac{K_\alpha(x-y)h(y)}{|x|^\tau |y|^\beta} dy,$$

then

$$\|Th\|_{p'} = \sup_{\|f\|_p = 1} |\langle Th, f\rangle| \leq C\|h\|_q,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $1 + \frac{1}{p'} \geq \frac{1}{q} + \frac{n-2\alpha+\beta+\tau}{n}$ and $h \in L^q(\mathbb{R}^n)$.

Next, we use Theorem 1.3 to investigate properties of positive solutions of (1.1). In the following, we always assume $1 < p, \ q < \frac{n-\beta}{\beta}$ and that (1.2) holds. We have the following result.

Theorem 1.4. *If $(u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ is a solution pair of (1.1), then $(u, v) \in L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$.*

Results in Theorem 1.4 hold also for sign-changing solutions of (1.1). In the proof of Theorem 1.4, we first lift the integrability of a suitable cut-off function of the solution by the regularity lifting method to some L^{q_0}, and then we show that the solution is actually in L^∞. From Theorem 1.4, one may expect the solution to be smooth. Our next result asserts that the solution is locally Hölder continuous. Precisely, let $\gamma = 1 - \frac{\beta}{n}$, under the conditions in Theorem 1.4, we have the following result.

Theorem 1.5. *$u, v \in C^{0,\gamma}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$.*

Furthermore, we show that the solution is radially symmetric by the moving plane method.

Theorem 1.6. *Both u and v are radially symmetric and strictly decreasing about the origin.*

In Section 2, we establish the weighted Hardy-Littlewood-Sobolev inequality related to the kernel K_α. Then, using the inequality, we prove Theorem 1.4 in section 3. In section 4, we prove Theorem 1.5 by Theorem 1.3 and the regularity lifting method. Theorem 1.6 is shown in section 5 by the moving plane method.

2. Hardy-Littlewood-Sobolev inequality for the Bessel potential

In this section, we establish a weighted Hardy-Littlewood-Sobolev inequality for the potential K_α. Let $\alpha \in (0,1)$, the kernel K_α associated with the operator $(-\Delta)^\alpha + id$ is defined as

$$K_\alpha(x) = \mathcal{F}^{-1}\big(\frac{1}{1+|\xi|^\alpha}\big),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transformation. It is known from [10] that the kernel K_α is radially symmetric, non-negative and non-increasing in $r = |x|$. Furthermore, for appropriate constants C_1 and C_2, the kernel K_α satisfies

$$K_\alpha(x) \le \begin{cases} C_1|x|^{-n+2\alpha} & \text{when } |x| \le 1, \\ C_2|x|^{-n-2\alpha} & \text{when } |x| \ge 1. \end{cases} \tag{2.1}$$

Proof of Theorem 1.3. By (2.1), we have

$$\begin{aligned} &\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)K_\alpha(x-y)h(y)}{|x|^\tau|y|^\beta}\,dy\,dx\Big| \\ &= \Big|\int_{\mathbb{R}^n}\int_{\{y:|x-y|\ge 1\}} \frac{f(x)K_\alpha(x-y)h(y)}{|x|^\tau|y|^\beta}\,dy\,dx \\ &+ \int_{\mathbb{R}^n}\int_{\{y:|x-y|\le 1\}} \frac{f(x)K_\alpha(x-y)h(y)}{|x|^\tau|y|^\beta}\,dy\,dx\Big| \\ &\le C\int_{\mathbb{R}^n}\int_{\{y:|x-y|\ge 1\}} \frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n+2\alpha}|y|^\beta}\,dy\,dx \\ &\quad + C\int_{\mathbb{R}^n}\int_{\{y:|x-y|\le 1\}} \frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n-2\alpha}|y|^\beta}\,dy\,dx := C(I+J). \end{aligned} \tag{2.2}$$

Firstly, we estimate I. We write

$$\begin{aligned} I = &\int_{\{x:|x|\le\frac12\}}\int_{\{y:|x-y|\ge 1\}} \frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n+2\alpha}|y|^\beta}\,dy\,dx \\ &+ \int_{\{x:|x|\ge\frac12\}}\int_{\{y:|x-y|\ge 1\}} \frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n+2\alpha}|y|^\beta}\,dy\,dx := I_1 + I_2. \end{aligned} \tag{2.3}$$

Notice that if $|x| \le \frac12, |y-x| \ge 1$, then $|y| \ge \frac12$. While the function $|x|^{-n-2\alpha}$ is in $L^r(R^n\backslash B_1(0))$ for all $\frac{n}{n+2\alpha} < r < \infty$, we have by Young's inequality that

$$\|(|\cdot|^{-n-2\alpha}\chi_{\{|\cdot|\ge 1\}}) * g\|_s \le C\|g\|_q\Big(\int_{\{x:|x|\ge 1\}} \frac{1}{|x|^{(n+2\alpha)r}}dx\Big)^{1/r},$$

where $1+\frac1s = \frac1r + \frac1q$. Choosing in Young's inequality that $r = n/(n-2\alpha)$, then $s = nq/(n-2\alpha q)$, we obtain

$$\begin{aligned} I_1 &\le \int_{\{x:|x|\le\frac12\}} \frac{|f(x)|}{|x|^{\tau+\beta}}\,dx\Big(\int_{\{y:|y-x|\ge 1\}} \frac{|h(y)|}{|x-y|^{n+2\alpha}}dy\Big)\,dx \\ &\le \|f\|_p \||\cdot|^{-n-2\alpha}\chi_{\{|\cdot|\ge 1\}} * |h|\|_{\frac{nq}{n-2\alpha q}}\Big(\int_{\{x:|x|\le\frac12\}} |x|^{-\frac{\tau+\beta}{1-\frac1p-\frac1q+\frac{2\alpha}{n}}}\,dx\Big)^{1-\frac1p-\frac1q+\frac{2\alpha}{n}} \\ &\le C\|f\|_p\|h\|_q\Big(\int_{\{x:|x|\ge 1\}} \frac{1}{|x|^{(n+2\alpha)r}}\,dx\Big)^{1/r} \end{aligned}$$

$$\times\Big(\int_{\{x:|x|\leq 1\}}|x|^{-\frac{\tau+\beta}{1-\frac{1}{p}-\frac{1}{q}+\frac{2\alpha}{n}}}\,dx\Big)^{1-\frac{1}{p}-\frac{1}{q}+\frac{2\alpha}{n}}.$$

Since $n-\frac{\tau+\beta}{1-\frac{1}{p}-\frac{1}{q}+\frac{2\alpha}{n}}>0$ if and only if $\beta+\tau<n(1-\frac{1}{p}-\frac{1}{q}+\frac{2\alpha}{n})$, it yields

$$I_1\leq C\|f\|_p\|h\|_q.$$

We decompose I_2 as follows.

$$\begin{aligned}I_2&=\int_{\{x:|x|\geq\frac12\}}\int_{\{y:|x-y|\geq 1,|y|\geq\frac12\}}\frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n+2\alpha}|y|^\beta}\,dy\,dx\\&\quad+\int_{\{x:|x|\geq\frac12\}}\int_{\{y:|x-y|\geq 1,|y|\leq\frac12\}}\frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n+2\alpha}|y|^\beta}\,dy\,dx\\&:=I_2^1+I_2^2.\end{aligned}\tag{2.4}$$

Furthermore,

$$\begin{aligned}I_2^1&=\int_{\{x:|x|\geq\frac12\}}\int_{\{y:|x-y|\geq 1,|y|\geq\frac12,|y|\geq|x|\}}\frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n+2\alpha}|y|^\beta}\,dy\,dx\\&\quad+\int_{\{x:|x|\geq\frac12\}}\int_{\{y:|x-y|\geq 1,|y|\geq\frac12,|y|\leq|x|\}}\frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n+2\alpha}|y|^\beta}\,dy\,dx\\&:=I_2^{11}+I_2^{12}.\end{aligned}\tag{2.5}$$

Now we estimate I_2^{11} and I_2^{12}, respectively. We deduce by Young's inequality that

$$\begin{aligned}I_2^{11}&\leq\int_{\{x:|x|\geq\frac12\}}\frac{|f(x)|}{|x|^{\tau+\beta}}\Big(\int_{\{y:|y|\geq\frac12,|x-y|\geq 1\}}\frac{|h(y)|}{|x-y|^{n+2\alpha}}\,dy\Big)\,dx\\&\leq\|f\|_p\|(|\cdot|^{-n-2\alpha}\chi_{\{|\cdot|\geq 1\}})*|h|\|_q\Big(\int_{\{x:|x|\geq 1\}}|x|^{-(\tau+\beta)\frac{pq}{pq-p-q}}\,dx\Big)^{1-\frac1p-\frac1q}\\&\leq C\|f\|_p\|h\|_q\Big(\int_{\{x:|x|\geq 1\}}|x|^{-(\tau+\beta)\frac{pq}{pq-p-q}}\,dx\Big)^{1-\frac1p-\frac1q}.\end{aligned}$$

Since $\tau+\beta>n(1-\frac1p-\frac1q)$, we have $n-(\tau+\beta)\frac{pq}{pq-p-q}<0$. Hence,

$$I_2^{11}\leq C\|f\|_p\|h\|_q.$$

By the Fubini theorem, we find

$$\begin{aligned}I_2^{12}&\leq\int_{\{x:|x|\geq\frac12\}}\int_{\{y:|y|\geq\frac12,|x-y|\geq 1\}}\frac{|f(x)||h(y)|}{|x-y|^{n+2\alpha}|y|^{\tau+\beta}}|\,dy\,dx\\&\leq\int_{\{y:|y|\geq\frac12\}}\frac{|h(y)|}{|y|^{\tau+\beta}}\Big(\int_{\{x:|x-y|\geq 1\}}\frac{|f(x)|}{|x-y|^{n+2\alpha}}dx\Big)\,dy\\&\leq\|h\|_q\|(|\cdot|^{-n-2\alpha}\chi_{\{|\cdot|\geq 1\}})*|f|\|_p\Big(\int_{\{y:|y|\geq\frac12\}}|y|^{-(\tau+\beta)\frac{pq}{pq-p-q}}\,dy\Big)^{1-\frac1p-\frac1q}\\&\leq C\|h\|_q\|f\|_p\Big(\int_{\{y:|y|\geq\frac12\}}|y|^{-(\tau+\beta)\frac{pq}{pq-p-q}}\,dx\Big)^{1-\frac1p-\frac1q}.\end{aligned}$$

In the same way, $n-(\tau+\beta)\frac{pq}{pq-p-q}<0$ if and only if $\tau+\beta>n(1-\frac1p-\frac1q)$, and then

$$I_2^{12}\leq C\|f\|_p\|h\|_q.$$

Using the Fubini theorem and Young's inequality, we obtain

$$
\begin{aligned}
I_2^2 &\le \int_{\{y:|y|\le\frac12\}} \frac{|h(y)|}{|y|^{\beta+\tau}} \int_{\{x:|x|\ge\frac12,|x-y|\ge1\}} \frac{|f(x)|}{|x-y|^{n+2\alpha}}\,dx\,dy\\
&\le \|h\|_q \|(|\cdot|^{-n-2\alpha}\chi_{\{|\cdot|\ge1\}})*|f|\|_{\frac{np}{n-2\alpha p}} \Big(\int_{\{y:|y|\le\frac12\}} |y|^{-\frac{\tau+\beta}{1-\frac1p-\frac1q+\frac{2\alpha}{n}}}\,dy\Big)^{1-\frac1p-\frac1q+\frac{2\alpha}{n}}\\
&\le C\|h\|_q\|f\|_p \Big(\int_{\{x:|x|\ge1\}} \frac{1}{|x|^{\frac{(n+2\alpha)n}{n-2\alpha}}}\,dx\Big)^{\frac{n-2\alpha}{n}}\\
&\quad\times \Big(\int_{\{y:|y|\le\frac12\}} |y|^{-\frac{\tau+\beta}{1-\frac1p-\frac1q+\frac{2\alpha}{n}}}\,dx\Big)^{1-\frac1p-\frac1q+\frac{2\alpha}{n}}.
\end{aligned}
$$

Since $n-\frac{\tau+\beta}{1-\frac1p-\frac1q+\frac{2\alpha}{n}}>0$ if and only if $\beta+\tau<n(1-\frac1p-\frac1q+\frac{2\alpha}{n})$, it follows that

$$I_2^2 \le C\|f\|_p\|h\|_q.$$

Secondly, we estimate J. There holds

$$
\begin{aligned}
J &= \int_{\{x:|x|\ge2\}} \int_{\{y:|x-y|\le1\}} \frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n-2\alpha}|y|^\beta}\,dy\,dx\\
&\quad+ \int_{\{x:|x|\le2\}} \int_{\{y:|x-y|\le1\}} \frac{|f(x)|h(y)|}{|x|^\tau|x-y|^{n-2\alpha}|y|^\beta}\,dy\,dx := J_1+J_2.
\end{aligned}
\tag{2.6}
$$

Note that if $|x|\ge2, |y-x|\le1$, then $|y|\ge|x|-|x-y|\ge1\ge|x-y|$ and $|x|>|x-y|$. By Young's inequality,

$$
\begin{aligned}
J_1 &= \int_{\{x:|x|\ge2\}} f(x) \int_{\{y:|x-y|\le1\}} \frac{|h(y)|}{|x-y|^{n-2\alpha+\beta+\tau}}\,dy\,dx\\
&\le \|f\|_p\|h*(\frac{\chi_{\{|\cdot|\le1\}}}{|\cdot|^{n-2\alpha+\beta+\tau}})\|_{p'}\\
&\le \|f\|_p\|h\|_q\Big(\int_{\{y:|y|\le1\}} \frac{1}{|y|^{(n-2\alpha+\beta+\tau)l}}\,dy\Big)^{1/l},
\end{aligned}
\tag{2.7}
$$

where $\frac{1}{p'}=\frac1q+\frac1l-1$, that is $\frac1l=2-\frac1p-\frac1q$. Thus, $n-(n-2\alpha+\beta+\tau)l\ge0$ if and only if $\beta+\tau<n(1-\frac1p-\frac1q+\frac{2\alpha}{n})$, it follows that

$$J_1\le C\|f\|_p\|h\|_q.$$

To estimate J_2, we decompose it as follows.

$$
\begin{aligned}
J_2 &= \int_{\{x:|x|\le2\}} \int_{\{y:|x-y|\le1,|y|\ge|x|\}} \frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n-2\alpha}|y|^\beta}\,dy\,dx\\
&\quad+ \int_{\{x:|x|\le2\}} \int_{\{y:|x-y|\le1,|y|\le|x|\}} \frac{|f(x)||h(y)|}{|x|^\tau|x-y|^{n-2\alpha}|y|^\beta}\,dy\,dx := J_2^1+J_2^2.
\end{aligned}
\tag{2.8}
$$

Now we estimate J_2^1 and J_2^2 respectively.

$$
\begin{aligned}
J_2^1 &\le \int_{\{x:|x|\le 2\}} \frac{|f(x)|}{|x|^{\tau+\beta}} \Big(\int_{\{y:|x-y|\le 1\}} \frac{|h(y)|}{|x-y|^{n-2\alpha}}\, dy \Big)\, dx \\
&\le \int_{\{x:|x|\le 2\}} \frac{|f(x)|}{|x|^{\tau+\beta}} I_\alpha(|h|)(x)\, dx \\
&\le \|f\|_p \|I_\alpha(|h|)\|_{\frac{nq}{n-2\alpha q}} \Big(\int_{\{x:|x|\le 2\}} \frac{1}{|x|^{\frac{\tau+\beta}{1-1/p-1/q+2\alpha/n}}}\, dx \Big)^{1-1/p-1/q+2\alpha/n} \\
&\le C\|f\|_p \|h\|_q \Big(\int_{\{x:|x|\le 2\}} \frac{1}{|x|^{\frac{\tau+\beta}{1-1/p-1/q+2\alpha/n}}}\, dx \Big)^{1-1/p-1/q+2\alpha/n}.
\end{aligned} \tag{2.9}
$$

Since $n - \frac{\tau+\beta}{1-1/p-1/q+2\alpha/n} > 0$ if and only if $\tau + \beta < n(1 - 1/p - 1/q + 2\alpha/n)$, it yields

$$J_2^1 \le C\|f\|_p \|h\|_q.$$

In the same way,

$$
\begin{aligned}
J_2^2 &\le \int_{\{x:|x|\le 2\}} \int_{\{y:|y|\le|x|\}} \frac{|f(x)||h(y)|}{|x|^{\tau}|x-y|^{n-2\alpha}|y|^{\beta}}\, dy\, dx \\
&\le \int_{\{y:|y|\le 2\}} \frac{|h(y)|}{|y|^{\tau+\beta}} \Big(\int_{\{x:|x|\ge|y|\}} \frac{|f(x)|}{|x-y|^{n-2\alpha}}\, dx \Big)\, dy \\
&\le \int_{\{y:|y|\le 2\}} \frac{|h(y)|}{|y|^{\tau+\beta}} I_\alpha(|f|)(y)\, dy \\
&\le \|h\|_q \|I_\alpha(|f|)\|_{\frac{np}{n-2\alpha p}} \Big(\int_{\{y:|y|\le 2\}} \frac{1}{|y|^{\frac{\tau+\beta}{1-1/p-1/q+2\alpha/n}}} \Big)^{1-1/p-1/q+2\alpha/n} \\
&\le C\|f\|_p \|h\|_q \Big(\int_{\{y:|y|\le 2\}} \frac{1}{|y|^{\frac{\tau+\beta}{1-1/p-1/q+2\alpha/n}}} \Big)^{1-1/p-1/q+2\alpha/n}.
\end{aligned} \tag{2.10}
$$

The inequality $\tau + \beta < n(1 - 1/p - 1/q + 2\alpha/n)$ implies

$$J_2^2 \le C\|f\|_p \|h\|_q.$$

The proof is complete. $\square$

3. L^∞-bound of solutions

In this section, we show that any solution of (1.1) in $L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ actually belongs to $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$. To this purpose, we will use the regularity lifting method developed in [2], which we will state as follows.

Let $\mathbf{Z}$ be a given vector space, $\|\cdot\|_{\mathbf{X}}$ and $\|\cdot\|_{\mathbf{Y}}$ be two norms on $\mathbf{Z}$. Define a new norm $\|\cdot\|_{\mathbf{Z}}$ by

$$\|\cdot\|_{\mathbf{Z}} = \sqrt[p]{\|\cdot\|_{\mathbf{X}}^p + \|\cdot\|_{\mathbf{Y}}^p}.$$

Suppose that $\mathbf{Z}$ is complete with respect to the norm $\|\cdot\|_{\mathbf{Z}}$. Let $\mathbf{X}$ and $\mathbf{Y}$ be the completion under $\|\cdot\|_{\mathbf{X}}$ and $\|\cdot\|_{\mathbf{Y}}$, respectively. Here one can choose p such that $1 \le p \le \infty$. According to what one needs, it is easy to see that $\mathbf{Z} = \mathbf{X} \cap \mathbf{Y}$. The following regularity lifting theorem was obtained in [2].

Lemma 3.1 (Regularity Lifting I). *Let T be a contracting map from $\mathbf{X}$ into itself and from $\mathbf{Y}$ into itself. Assume that $f \in \mathbf{X}$ and that there exists a function $g \in \mathbf{Z}$ such that $f = Tf + g$, then f also belongs to $\mathbf{Z}$.*

Proof of Theorem 1.4. Let $(u,v)\in L^{p+1}(\mathbb{R}^n)\times L^{q+1}(\mathbb{R}^n)$ be a pair of solution to integral systems (1.1). We first show by Lemma 3.1 that (u_ξ,v_ξ), a cut-off function of (u,v) defined below, belongs to $L^{\tilde p}(\mathbb{R}^n)\times L^{\tilde q}(\mathbb{R}^n)$ for $\tilde p>\frac{np}{2\alpha-\beta}$, $\tilde q>\frac{nq}{2\alpha-\beta}$ and $\frac{1}{\tilde p}-\frac{1}{\tilde q}=\frac{1}{p+1}-\frac{1}{q+1}$, then we prove that $(u,v)\in L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$.

For any sufficient large positive real number ξ, define

$$u_\xi(x)=\begin{cases}u(x), & \text{if } |u(x)|\ge\xi \text{ or } |x|>\xi,\\ u_\xi(x)=0, & \text{otherwise.}\end{cases} \tag{3.1}$$

Similarly, we define v_ξ. Let

$$T_1^\xi g(x)=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|v_\xi|^{q-1}g(y)}{|y|^\beta}\,dy,\quad T_2^\xi f(x)=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|u_\xi|^{p-1}f(y)}{|y|^\beta}\,dy$$

and

$$T_\xi(f,g)=(T_1^\xi g,T_2^\xi f).$$

Let $\tilde u_\xi(x)=u(x)-u_\xi(x)$, and $E_\xi^u=\{x\in\mathbb{R}^n:|u(x)|\ge\xi \text{ or } |x|>\xi\}$. Similarly, we define $\tilde v_\xi$ and E_ξ^v. By (1.1), we have

$$\begin{aligned}u(x)&=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|v(y)|^{q-1}v(y)}{|y|^\beta}\,dy\\&=\int_{E_\xi^v}\frac{K_\alpha(x-y)|v(y)|^{q-1}v(y)}{|y|^\beta}\,dy+\int_{\mathbb{R}^n\setminus E_\xi^v}\frac{K_\alpha(x-y)|v(y)|^{q-1}v(y)}{|y|^\beta}\,dy\\&=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|v_\xi(y)|^{q-1}v_\xi(y)}{|y|^\beta}\,dy+\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|\tilde v_\xi(y)|^{q-1}\tilde v_\xi(y)}{|y|^\beta}\,dy.\end{aligned} \tag{3.2}$$

Moreover,

$$u_\xi(x)=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|v_\xi(y)|^{q-1}v_\xi(y)}{|y|^\beta}\,dy+M_1(x), \tag{3.3}$$

where

$$M_1(x)=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|\tilde v_\xi(y)|^{q-1}\tilde v_\xi(y)}{|y|^\beta}\,dy-\tilde u_\xi(x).$$

Similarly,

$$v_\xi(x)=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|u_\xi(y)|^{p-1}u_\xi(y)}{|y|^\beta}\,dy+M_2(x), \tag{3.4}$$

where

$$M_2(x)=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|\tilde u_\xi(y)|^{p-1}\tilde u_\xi(y)}{|y|^\beta}\,dy-\tilde v_\xi(x).$$

It yields

$$(u_\xi,v_\xi)=T_\xi(u_\xi,v_\xi)+(M_1(x),M_2(x)),$$

where $T_\xi(u_\xi,v_\xi)=(T_1^\xi v_\xi,T_2^\xi u_\xi)$.

We claim that $M_1(x),M_2(x)\in L^\infty(\mathbb{R}^n)\cap L^s(\mathbb{R}^n)$ for $s>1$. Obviously, $\tilde u_\xi,\tilde v_\xi\in L^\infty(\mathbb{R}^n)\cap L^s(\mathbb{R}^n)$. So it suffices to show that $H_1,H_2\in L^\infty(\mathbb{R}^n)\cap L^s(\mathbb{R}^n)$, where

$$\begin{aligned}H_1(x)&=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|\tilde v_\xi(y)|^{q-1}\tilde v_\xi(y)}{|y|^\beta}\,dy,\\H_2(x)&=\int_{\mathbb{R}^n}\frac{K_\alpha(x-y)|\tilde u_\xi(y)|^{p-1}\tilde u_\xi(y)}{|y|^\beta}\,dy.\end{aligned}$$

Now, we estimate H_1, the estimation for H_2 can be obtained in the same way. By the definition of $\tilde{v}_\xi(x)$, for $x \in \mathbb{R}^n$, we obtain

$$
\begin{aligned}
|H_1(x)| &\le C \int_{\{y:|y|\le\xi\}} \frac{K_\alpha(x-y)}{|y|^\beta}\,dy \\
&\le C \int_{\{y:|y|\le\xi,|x-y|\ge 1\}} \frac{K_\alpha(x-y)}{|y|^\beta}\,dy + C \int_{\{y:|y|\le\xi,|x-y|\le 1\}} \frac{K_\alpha(x-y)}{|y|^\beta}\,dy \\
&\le C \int_{\{y:|y|\le\xi,|x-y|\ge 1\}} \frac{1}{|x-y|^{n+2\alpha}|y|^\beta}\,dy \\
&\quad + C \int_{\{y:|y|\le\xi,|x-y|\le 1\}} \frac{1}{|y|^\beta|x-y|^{n-2\alpha}}\,dy \\
&= A(x) + B(x),
\end{aligned}
$$

where $C > 0$ depends on ξ. Since $0 \le \beta < 2\alpha < n$,

$$
A(x) \le C \int_{\{y:|y|\le\xi\}} \frac{1}{|y|^\beta}\,dy \le C.
$$

Similarly,

$$
\begin{aligned}
&B(x) \\
&\le \Big(\int_{\{y:|y|\le\xi,|x-y|\le 1,|x-y|\ge|y|\}} + \int_{\{y:|y|\le\xi,|x-y|\le 1,|x-y|\le|y|\}} \Big) \frac{C}{|y|^\beta|x-y|^{n-2\alpha}}\,dy \\
&\le \int_{\{y:|y|\le\xi\}} \frac{C}{|y|^{n-2\alpha+\beta}}dy + \int_{\{y:|x-y|\le 1\}} \frac{C}{|x-y|^{n-2\alpha+\beta}}\,dy \le C.
\end{aligned}
$$

As a result, $H_1 \in L^\infty(\mathbb{R}^n)$. On the other hand, by Theorem 1.3, for $r > \frac{ns}{2\alpha s+n-s\beta}$, we have

$$
\|H_1\|_{L^s(\mathbb{R}^n)} \le \|\tilde{v}_\xi^q\|_{L^r(B_\xi(0))} \le C;
$$

that is, $H_1 \in L^s(\mathbb{R}^n)$. The claim follows.

Next, we show that $T_\xi(f,g)$ is a contraction map from $L^{\tilde{p}}(\mathbb{R}^n) \times L^{\tilde{q}}(\mathbb{R}^n)$ into $L^{\tilde{p}}(\mathbb{R}^n) \times L^{\tilde{q}}(\mathbb{R}^n)$ for $\tilde{q}, \tilde{p} > 1$ satisfying

$$
\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} = \frac{1}{p+1} - \frac{1}{q+1}. \tag{3.5}
$$

We may verify by the fact $p, q > 1$, (1.2) and (3.5) that

$$
\tilde{q} > \frac{n\tilde{p}}{n+(2\alpha-\beta)\tilde{p}}, \quad \tilde{p} > \frac{n\tilde{q}}{n+(2\alpha-\beta)\tilde{q}}.
$$

Choosing d_1 such that

$$
\frac{1}{d_1} = \frac{1}{\tilde{q}} + \frac{q-1}{q+1}, \tag{3.6}
$$

we verify by (1.2) that

$$
\tilde{q} > d_1 > \frac{n\tilde{p}}{n+(2\alpha-\beta)\tilde{p}}. \tag{3.7}
$$

By Theorem 1.3, we find

$$
\|T_1^\xi g\|_{\tilde{p}} \le C\||v_\xi|^{q-1} g\|_{d_1}. \tag{3.8}
$$

This and Hölder's inequality yield

$$
\|T_1^\xi g\|_{\tilde{p}} \le C\|v_\xi\|_{q+1}^{q-1}\|g\|_{\tilde{q}}. \tag{3.9}
$$

In the same way, choosing $\frac{1}{d_2} = \frac{1}{\tilde{p}} + \frac{p-1}{p+1}$, we obtain

$$\|T_2^\xi f\|_{\tilde{q}} \le C\||u_\xi|^{p-1} f\|_{d_2} \le C\|u_\xi\|_{p+1}^{p-1}\|f\|_{\tilde{p}}. \tag{3.10}$$

Since $(u,v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$, one can choose ξ sufficiently large so that

$$\|T_1^\xi g\|_{\tilde{p}} \le \frac{1}{2}\|g\|_{\tilde{q}}, \quad \|T_2^\xi f\|_{\tilde{q}} \le \frac{1}{2}\|f\|_{\tilde{p}}. \tag{3.11}$$

Therefore,

$$\begin{aligned} \|T_\xi(f,g)\|_{\tilde{p}\times\tilde{q}} &= \|(T_1^\xi g, T_2^\xi f)\|_{\tilde{p}\times\tilde{q}} = \|T_1^\xi g\|_{\tilde{p}} + \|T_2^\xi f\|_{\tilde{q}} \\ &\le \frac{1}{2}\|g\|_{\tilde{q}} + \frac{1}{2}\|f\|_{\tilde{p}} = \frac{1}{2}\|(f,g)\|_{\tilde{p}\times\tilde{q}}. \end{aligned} \tag{3.12}$$

In other words, $T_\xi(f,g)$ is a contraction map from $L^{\tilde{p}}(\mathbb{R}^n) \times L^{\tilde{q}}(\mathbb{R}^n)$ into itself for $\tilde{p}, \tilde{q} > 1$, $\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} = \frac{1}{p+1} - \frac{1}{q+1}$. In particular, for $\tilde{p} = p+1$ and $\tilde{q} = q+1$, we see that $T_\xi(f,g)$ is also a contraction map from $L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$ into itself. Choosing $\tilde{p}, \tilde{q}$ large enough such that $\tilde{p} > \frac{np}{2\alpha-\beta}, \tilde{q} > \frac{nq}{2\alpha-\beta}, \frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} = \frac{1}{p+1} - \frac{1}{q+1}$, by Lemma 3.1, we conclude that $(u_\xi, v_\xi) \in (L^{\tilde{p}}(\mathbb{R}^n) \times L^{\tilde{q}}(\mathbb{R}^n)) \cap (L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n))$.

Finally, we show that $u, v \in L^\infty(\mathbb{R}^n)$. Since $u(x) = u_\xi(x) + \tilde{u}_\xi(x), v(x) = v_\xi(x) + \tilde{v}_\xi(x)$ and $\tilde{u}_\xi, \tilde{v}_\xi \in L^\infty(\mathbb{R}^n)$, we only need to verify $u_\xi, v_\xi \in L^\infty(\mathbb{R}^n)$. By (3.3),(3.4) and $M_1, M_2 \in L^\infty(\mathbb{R}^n)$, it is sufficient to verify that $I_1, I_2 \in L^\infty(\mathbb{R}^n)$, where

$$I_1(x) = \int_{R^n} \frac{K_\alpha(x-y)|v_\xi|^{q-1}v_\xi(y)}{|y|^\beta}\,dy, \quad I_2(x) = \int_{R^n} \frac{K_\alpha(x-y)|u_\xi|^{p-1}u_\xi(y)}{|y|^\beta}\,dy.$$

There holds

$$\begin{aligned} |I_1(x)| &\le \int_{\{y:|y|\le\xi\}} \frac{K_\alpha(x-y)|v_\xi|^q}{|y|^\beta}\,dy + \int_{\{y:|y|\ge\xi\}} \frac{K_\alpha(x-y)|v_\xi|^q}{|y|^\beta}\,dy \\ &:= J(x) + G(x). \end{aligned} \tag{3.13}$$

If $x \in \mathbb{R}^n \backslash B_{2\xi}(0)$, $y \in B_\xi(0)$, then $|x-y| > |x| - |y| > \xi > |y|$. Thus,

$$\begin{aligned} J(x) &\le C\int_{\{y:|y|\le\xi\}} \frac{|v_\xi|^q}{|x-y|^{n+2\alpha}|y|^\beta}\,dy \le C\int_{\{y:|y|\le\xi\}} \frac{|v_\xi|^q}{|y|^\beta}\,dy \\ &\le \Big(\int_{\{y:|y|\le\xi\}} |v_\xi|^{q+1}\,dy\Big)^{q/(q+1)}\Big(\int_{\{y:|y|\le\xi\}} \frac{1}{|y|^{(q+1)\beta}}\,dy\Big)^{1/(q+1)} \le C \end{aligned} \tag{3.14}$$

since $q < (n-\beta)/\beta$. If $x \in B_{2\xi}(0)$, we have

$$\begin{aligned} J(x) &\le \int_{\{y:|y|\le\xi,|x-y|\ge1\}} \frac{K_\alpha(x-y)|v_\xi|^q}{|y|^\beta}\,dy + \int_{\{y:|y|\le\xi,|x-y|\le1\}} \frac{K_\alpha(x-y)|v_\xi|^q}{|y|^\beta} dy \\ &\le C\int_{\{y:|y|\le\xi,|x-y|\ge1\}} \frac{|v_\xi|^q}{|x-y|^{n+2\alpha}|y|^\beta}\,dy \\ &\quad + C\int_{\{y:|y|\le\xi,|x-y|\le1\}} \frac{|v_\xi|^q}{|x-y|^{n-2\alpha}|y|^\beta}\,dy := CJ_1(x) + CJ_2(x). \end{aligned}$$

Now we estimate $J_1(x), J_2(x)$ respectively. By Hölder's inequality, we have

$$\begin{aligned} J_1(x) &\le C\Big(\int_{\{y:|y|\le\xi\}} |v_\xi|^{q+1}\,dy\Big)^{q/(q+1)}\Big(\int_{\{y:|y|\le\xi\}} \frac{1}{|y|^{(q+1)\beta}}\,dy\Big)^{1/(q+1)} \\ &\le C \end{aligned} \tag{3.15}$$

and because of $\tilde{q} > nq/(2\alpha-\beta)$, we deduce

$$\begin{aligned}
&J_2(x)\\
&\le \int_{\{y:|y|\le\xi,|x-y|\le 1,|x-y|\ge|y|\}} \frac{|v_\xi|^q}{|x-y|^{n-2\alpha}|y|^\beta}\,dy\\
&\quad+\int_{\{y:|y|\le\xi,|x-y|\le 1,|x-y|\le|y|\}} \frac{|v_\xi|^q}{|x-y|^{n-2\alpha}|y|^\beta}\,dy\\
&\le \int_{\{y:|y|\le\xi\}} \frac{|v_\xi|^q}{|y|^{n-2\alpha+\beta}}\,dy+\int_{\{y:|x-y|\le 1,|y|\le\xi\}} \frac{|v_\xi|^q}{|x-y|^{n-2\alpha+\beta}}\,dy \qquad (3.16)\\
&\le \Big(\int_{\{y:|y|\le\xi\}} |v_\xi|^{\tilde{q}}\,dy\Big)^{q/\tilde{q}}\Big(\int_{\{y:|y|\le\xi\}} \frac{1}{|y|^{\frac{\tilde{q}}{\tilde{q}-q}(n-2\alpha+\beta)}}\,dy\Big)^{(\tilde{q}-q)/\tilde{q}}\\
&\quad+\Big(\int_{\{y:|y|\le\xi\}} |v_\xi|^{\tilde{q}}\,dy\Big)^{q/\tilde{q}}\Big(\int_{\{y:|x-y|\le 1\}} \frac{1}{|x-y|^{\frac{\tilde{q}}{\tilde{q}-q}(n-2\alpha+\beta)}}\,dy\Big)^{(\tilde{q}-q)/\tilde{q}}\\
&\le C.
\end{aligned}$$

Inequalities (3.14),(3.15) and (3.16) imply that $J \in L^\infty(\mathbb{R}^n)$. Now we estimate $G(x)$. For any $x \in \mathbb{R}^n$,

$$\begin{aligned}
G(x) &\le \int_{\{y:|y|\ge\xi,|x-y|\ge 1\}} \frac{K_\alpha(x-y)|v_\xi|^q}{|y|^\beta}\,dy+\int_{\{y:|y|\ge\xi,|x-y|\le 1\}} \frac{K_\alpha(x-y)|v_\xi|^q}{|y|^\beta}\,dy\\
&\le C\int_{\{y:|y|\ge\xi,|x-y|\ge 1\}} \frac{|v_\xi|^q}{|x-y|^{n+2\alpha}|y|^\beta}\,dy\\
&\quad+C\int_{\{y:|y|\ge\xi,|x-y|\le 1\}} \frac{|v_\xi|^q}{|x-y|^{n-2\alpha}|y|^\beta}\,dy\\
&\le C\int_{\{y:|y|\ge\xi,|x-y|\ge 1\}} \frac{|v_\xi|^q}{|x-y|^{n+2\alpha}|y|^\beta}\,dy+C\int_{\{y:|x-y|\le 1\}} \frac{|v_\xi|^q}{|x-y|^{n-2\alpha+\beta}}\,dy\\
&:= CG_1(x)+CG_2(x).
\end{aligned}$$

By Hölder's inequality,

$$G_2(x) \le \Big(\int_{\mathbb{R}^n} |v_\xi|^{\tilde{q}}dy\Big)^{q/\tilde{q}}\Big(\int_{\{y:|x-y|\le 1\}} \frac{1}{|x-y|^{\frac{\tilde{q}}{\tilde{q}-q}(n-2\alpha+\beta)}}dy\Big)^{(\tilde{q}-q)/\tilde{q}} \le C.$$

Now we estimate $G_1(x)$. Since $\tilde{q} > \frac{nq}{2\alpha-\beta} > \frac{nq}{n-\beta}$, we can choose an r such that $1 < r < \frac{n\tilde{q}}{n\tilde{q}-nq-\tilde{q}\beta}$. Hence, Hölder's inequality implies that

$$\begin{aligned}
G_1(x) &\le \Big(\int_{\{y:|y|\ge\xi\}} |v_\xi|^{\tilde{q}}dy\Big)^{q/\tilde{q}}\Big(\int_{\{|x-y|\ge 1\}} \frac{1}{y:|x-y|^{(n+2\alpha)r}}dy\Big)^{1/r}\\
&\quad\times\Big(\int_{\{y:|y|\ge\xi\}} \frac{1}{|y|^{\frac{\beta}{1-q/\tilde{q}-1/r}}}dy\Big)^{1-q/\tilde{q}-1/r} \le C.
\end{aligned} \qquad (3.17)$$

Consequently, both J and G belong to $L^\infty(\mathbb{R}^n)$, so is I_1. Similarly, we have $I_2 \in L^\infty(\mathbb{R}^n)$. Therefore, $u, v \in L^\infty(\mathbb{R}^n)$. The proof of Theorem 1.3 is completed. $\square$

4. Regularity of solutions to integral systems

In this section, we show that the solution of (1.1) is Hölder continuous. We recall the regularity lifting theorem II given in [2]. Let V be a Hausdorff topological vector space. Suppose there are two extended norms defined on V,

$$\|\cdot\|_X, \|\cdot\|_Y : V \to [0,\infty].$$

Let

$$X := \{v \in V : \|v\|_X < \infty\}, \quad Y := \{v \in V : \|v\|_Y < \infty\}.$$

We also assume that X is complete and that the topology in V is weaker than the topology of X and the weak topology of Y, which means that the convergence in X or weak convergence in Y will imply convergence in V. The pair of spaces (X, Y) described as above is called an $XY-$pair, if whenever the sequence $\{u_n\} \subset X$ with $u_n \to u$ in X and $\|u_n\|_Y < C$ will imply $u \in Y$.

From [2, Remark 3.3.5], we know that if $X = L^p(U)$ for $1 \le p \le \infty, Y = C^{0,\gamma}(U)$ for $0 < \gamma \le 1$, and V is the space of distributions, where U can be any subset of $\mathbb{R}^n$ or $\mathbb{R}^n$ itself, then (X, Y) is an $XY-$pair.

Lemma 4.1 (Regularity Lifting II). *Suppose that Banach spaces X, Y are an XY-pair, both contained in some larger topological space V satisfying properties described above. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be closed subsets of X and Y respectively. Suppose that $T : \mathfrak{X} \to X$ is a contraction:*

$$\|Tf - Tg\|_X \le \eta\|f - g\|_X, \quad \forall f, g \in \mathfrak{X} \text{ and for some } 0 < \eta < 1;$$

and $T : \mathfrak{Y} \to Y$ is shrinking:

$$\|Tg\|_Y \le \theta\|g\|_Y, \quad \forall g \in \mathfrak{Y} \text{ and for some } 0 < \theta < 1;$$

Define

$$Sf = Tf + F \quad \text{for some } F \in \mathfrak{X} \cap \mathfrak{Y}.$$

Moreover, assume that $S : \mathfrak{X} \cap \mathfrak{Y} \to \mathfrak{X} \cap \mathfrak{Y}$. Then there exists a unique solution u of the equation

$$u = Tf + F \quad \text{in } \mathfrak{X},$$

and more importantly, $u \in Y$.

Proof of Theorem 1.5. Since (u, v) is a solution pair for (1.1), we have

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} K_\alpha(|x-y|)\frac{v^q(y)}{|y|^\beta}\,dy \\ &= -\int_{\mathbb{R}^n}\int_{|x-y|}^\infty K'_\alpha(s)\frac{v^q(y)}{|y|^\beta}\,ds\,dy \\ &= -\int_0^\infty\int_{B_s(x)} K'_\alpha(s)\frac{v^q(y)}{|y|^\beta}\,ds\,dy \end{aligned}$$

and

$$v(x) = -\int_0^\infty\int_{B_s(x)} K'_\alpha(s)\frac{u^p(y)}{|y|^\beta}\,ds\,dy.$$

For any $\Omega \subset\subset \mathbb{R}^n \setminus \{0\}$, denote $d = \operatorname{dist}(0, \Omega) > 0$. Let $X = L^\infty(\Omega)$ and $Y = C^{0,\gamma}(\Omega)$. By Theorem 1.4, $u, v \in L^\infty(\mathbb{R}^n)$, we define

$$\begin{aligned} \mathfrak{X} &= \{w \in X \big| \|w\|_{L^\infty} \le 2\|u\|_{L^\infty} + 2\|v\|_{L^\infty}\}, \\ \mathfrak{Y} &= \{w \in Y \big| \|w\|_{L^\infty} \le 2\|u\|_{L^\infty} + 2\|v\|_{L^\infty}\}. \end{aligned}$$

For every $\varepsilon > 0$ such that $0 < \varepsilon < \frac{d}{2}$, we define

$$T^q_\varepsilon \hat{v}(x) = -\int_0^\varepsilon\int_{B_s(x)} K'_\alpha(s)\frac{\hat{v}^q(y)}{|y|^\beta}\,ds\,dy,$$

$$T_\varepsilon^p\hat u(x)=-\int_0^\varepsilon\int_{B_s(x)}K_\alpha'(s)\frac{\hat u^p(y)}{|y|^\beta}\,ds\,dy,\quad T_\varepsilon(\hat u,\hat v)=(T_\varepsilon^q\hat v,T_\varepsilon^p\hat u).$$

Furthermore, we define

$$F^qv(x)=-\int_\varepsilon^\infty\int_{B_s(x)}K_\alpha'(s)\frac{v^q(y)}{|y|^\beta}\,ds\,dy,$$

$$F^pu(x)=-\int_\varepsilon^\infty\int_{B_s(x)}K_\alpha'(s)\frac{u^p(y)}{|y|^\beta}\,ds\,dy,\quad F=(F^pv,F^qu).$$

Obviously, a solution (u,v) of (1.1) is a solution of the equation

$$(\hat u,\hat v)=T_\varepsilon(\hat u,\hat v)+F.$$

Write $S_\varepsilon(\hat u,\hat v)=T_\varepsilon(\hat u,\hat v)+F$. We will show for $\varepsilon>0$ small that T_ε is a contracting operator from $\mathfrak{X}\times\mathfrak{X}$ to $X\times X$, and also is a shrinking operator from $\mathfrak{Y}\times\mathfrak{Y}$ to $Y\times Y$. Furthermore, $F\in(\mathfrak{X}\times\mathfrak{X})\cap(\mathfrak{Y}\times\mathfrak{Y})$, and $S_\varepsilon:(\mathfrak{X}\times\mathfrak{X})\cap(\mathfrak{Y}\times\mathfrak{Y})\to(\mathfrak{X}\times\mathfrak{X})\cap(\mathfrak{Y}\times\mathfrak{Y})$. This then will yields $(u,v)\in Y\times Y$ by Lemma 4.1.

We first show that T_ε is a contracting operator from $\mathfrak{X}\times\mathfrak{X}$ to $X\times X$. For any $f,g\in\mathfrak{X}$, we denote here and below that $f^p=f_+^p$. By the mean value theorem, we have

$$\begin{aligned}\big|T_\varepsilon^qf(x)-T_\varepsilon^qg(x)\big|&\le\int_0^\varepsilon\int_{B_s(x)}|f^q(y)-g^q(y)|\frac{|K_\alpha'(s)|}{|y|^\beta}\,ds\,dy\\&\le C\max\{\|f\|_{L^\infty}^{q-1},\|g\|_{L^\infty}^{q-1}\}\|f-g\|_{L^\infty}\int_0^\varepsilon s^{n-\beta}|K_\alpha'(s)|\,ds.\end{aligned}$$

By (2.1), we obtain

$$\int_0^\varepsilon s^{n-\beta}|K_\alpha'(s)|\,ds\le O(\varepsilon^{2\alpha-\beta})$$

as $\varepsilon\to0$. Hence, for $\varepsilon>0$ small,

$$\big|T_\varepsilon^qf(x)-T_\varepsilon^qg(x)\big|\le C\max\{\|f\|_{L^\infty}^{q-1},\|g\|_{L^\infty}^{q-1}\}\|f-g\|_{L^\infty}\varepsilon^{2\alpha-\beta}.\tag{4.1}$$

Choosing $\varepsilon>0$ small so that $C\max\{\|f\|_{L^\infty}^{q-1},\|g\|_{L^\infty}^{q-1}\}\varepsilon^{2\alpha-\beta}\le1/4$, we see that T_ε^q is a contracting operator from $\mathfrak{X}$ to X. Similarly, T_ε^p is also a contracting operator from $\mathfrak{X}$ to X. Therefore, T_ε is a contracting operator from $\mathfrak{X}\times\mathfrak{X}$ to $X\times X$.

Next, we verify that T_ε is a shrinking operator from $\mathfrak{Y}\times\mathfrak{Y}$ to $Y\times Y$. We only show it for T_ε^q, it can be done in the same way for T_ε^p. Assume $f\in\mathfrak{Y}$. Then for any $x,z\in\Omega$, we have

$$\begin{aligned}|T_\varepsilon^qf(x)-T_\varepsilon^qf(z)|&=\Big|\int_0^\varepsilon\Big\{\int_{B_s(x)}\frac{f^q(y)}{|y|^\beta}\,dy-\int_{B_s(z)}\frac{f^q(y)}{|y|^\beta}\,dy\Big\}K_\alpha'(s)\,ds\Big|\\&=\Big|\int_0^\varepsilon\int_{B_s(x)}\Big[\frac{f^q(y)}{|y|^\beta}-\frac{f^q(y+z-x)}{|y+z-x|^\beta}\Big]K_\alpha'(s)\,dy\,ds\Big|\\&\le\Big|\int_0^\varepsilon\int_{B_s(x)}\Big[f^q(y)\Big(\frac{1}{|y|^\beta}-\frac{1}{|y+z-x|^\beta}\Big)\\&\quad+\Big(\frac{f^q(y)-f^q(y+z-x)}{|y+z-x|^\beta}\Big)\Big]K_\alpha'(s)\,dy\,ds\Big|.\end{aligned}$$

For $y\in B_s(x)$, $0<s<\varepsilon$, we have $|y|\ge|x|-s\ge d-\frac d2=\frac d2>0$ and $|y+z-x|\ge|z|-|y-x|\ge d-s\ge\frac d2$. So both $\frac{1}{|y|^\beta}$ and $\frac{1}{|y+z-x|^\beta}$ are regular in $B_s(x)$ for

$0 < s < \varepsilon$. In particular, there exists $C > 0$ such that $|\frac{1}{|y|^\beta} - \frac{1}{|y+z-x|^\beta}| \le C|x-z|$. Hence,

$$\begin{aligned}
&|\int_0^\varepsilon \int_{B_s(x)} f^q(y)\big(\frac{1}{|y|^\beta} - \frac{1}{|y+z-x|^\beta}\big) K'_\alpha(s)\,dy\,ds| \\
&\le C\|f\|^q_{L^\infty}|x-z|\,|\int_0^\varepsilon s^n K'_\alpha(s)\,ds| \\
&\le C\|f\|^q_{L^\infty}|x-z|\varepsilon^{2\alpha} \\
&\le C\|f\|^{q-1}_{L^\infty}\|f\|_{C^{0,\gamma}}|x-z|\varepsilon^{2\alpha}.
\end{aligned}$$

If $|x-z| \le 1$, $|x-z| \le |x-z|^\gamma$; if $|x-z| > 1$, $|x-z| \le (diam\Omega)^{1-\gamma}|x-z|^\gamma$. Therefore,

$$\begin{aligned}
&|\int_0^\varepsilon \int_{B_s(x)} f^q(y)\big(\frac{1}{|y|^\beta} - \frac{1}{|y+z-x|^\beta}\big) K'_\alpha(s)\,dy\,ds| \\
&\le C\|f\|^{q-1}_{L^\infty}\|f\|_{C^{0,\gamma}}|x-z|^\gamma\varepsilon^{2\alpha}.
\end{aligned}$$

On the other hand, by the mean value theorem,

$$\begin{aligned}
&|\int_0^\varepsilon \int_{B_s(x)} \frac{f^q(y) - f^q(y+z-x)}{|y+z-x|^\beta} K'_\alpha(s)\,dy\,ds| \\
&\le |\int_0^\varepsilon \int_{B_s(x)} |w(\xi)|^{q-1}\frac{|f(y) - f(y+z-x)|}{|y+z-x|^\beta} K'_\alpha(s)\,dy\,ds| \\
&\le C\|f\|^{q-1}_{L^\infty}\|f\|_{C^{0,\gamma}}|x-z|^\gamma|\int_0^\varepsilon s^n K'_\alpha(s)\,ds| \\
&\le C\|f\|^{q-1}_{L^\infty}\|f\|_{C^{0,\gamma}}|x-z|^\gamma\varepsilon^{2\alpha},
\end{aligned}$$

where w is valued between $f(y)$ and $f(y+z-x)$. Consequently,

$$|T^q_\varepsilon f(x) - T^q_\varepsilon f(z)| \le C\|f\|_{C^{0,\gamma}}|x-z|^\gamma\varepsilon^{2\alpha}.$$

Choosing $\varepsilon > 0$ sufficiently small, we obtain

$$\sup_{x\ne z}\frac{|T^q_\varepsilon f(x) - T^q_\varepsilon f(z)|}{|x-z|} \le \frac{1}{4}\|f\|_{C^{0,\gamma}}.$$

We may derive in the same way as (4.1) that

$$|T^q_\varepsilon f(x)| \le C\|f\|_{L^\infty}\varepsilon^{2\alpha-\beta} \le \frac{1}{4}\|f\|_{C^{0,\gamma}}.$$

Therefore, for any $f \in \mathfrak{Y}$,

$$\|T^q_\varepsilon f(x)\|_{C^{0,\gamma}} \le \frac{1}{2}\|f\|_{C^{0,\gamma}};$$

that is, T^q_ε is a shrinking operator from $\mathfrak{Y}$ to Y.

Now, we show that $F^q v(x)$ and $F^p u(x)$ are Hölder continuous for $u, v \in \mathfrak{Y}$. We only deal with $F^q v(x)$. For $F^p u(x)$, it can be shown similarly. We write

$$\begin{aligned}
F^q v(x) &= -\int_\varepsilon^1 \int_{B_s(x)} K'_\alpha(s)\frac{v^q(y)}{|y|^\beta}\,ds\,dy - \int_1^\infty \int_{B_s(x)} K'_\alpha(s)\frac{v^q(y)}{|y|^\beta}\,ds\,dy \\
&:= F_1(x) + F_2(x).
\end{aligned}$$

For $x, z \in \Omega$, we have

$$\begin{aligned} |F_1(x)-F_1(z)| &= \Big|\int_\varepsilon^1 \Big(\int_{B_s(x)} \frac{v^q(y)}{|y|^\beta}\,dy - \int_{B_s(x)} \frac{v^q(y)}{|y|^\beta}\,dy\Big) K_\alpha'(s)\,ds\Big| \\ &\le \|v\|_{L^\infty}^q \int_\varepsilon^1 \Big(\int_{(B_s(x)\setminus B_s(z))\cup(B_s(z)\setminus B_s(x))} \frac{1}{|y|^\beta}\,dy\Big)|K_\alpha'(s)|\,ds. \end{aligned}$$

Denote by A^* the symmetric rearrangement of the set A, and f^* the symmetric-decreasing rearrangement of a function f. It is known that for the characteristic function χ_A of a set A, $\chi_A^* = \chi_{A^*}$. Moreover, for nonnegative functions f and g, there holds

$$\int_{\mathbb{R}^n} fg\,dx \le \int_{\mathbb{R}^n} f^*g^*\,dx.$$

If $|x-z| \ge 2s$, then

$$\begin{aligned} \int_{(B_s(x)\setminus B_s(z))\cup(B_s(z)\setminus B_s(x))} \frac{1}{|y|^\beta}\,dy &= \int_{B_s(x)} \frac{1}{|y|^\beta}\,dy + \int_{B_s(z)} \frac{1}{|y|^\beta}\,dy \\ &= \int_{\mathbb{R}^n} \chi_{B_s(x)} \frac{1}{|y|^\beta}\,dy + \int_{\mathbb{R}^n} \chi_{B_s(z)} \frac{1}{|y|^\beta}\,dy \\ &\le \int_{\mathbb{R}^n} \chi_{B_s(0)} \frac{1}{|y|^\beta}\,dy + \int_{\mathbb{R}^n} \chi_{B_s(0)} \frac{1}{|y|^\beta}\,dy \\ &\le Cs^{n-\beta} \\ &\le Cs^{n-\beta-1}|x-z|^\gamma. \end{aligned}$$

If $|x-z| < 2s$, we have

$$(B_s(x)\setminus B_s(z))\cup(B_s(z)\setminus B_s(x)) \subset (B_s(x)\cup B_s(z))\setminus B_{s-\frac{|x-z|}{2}}(\frac{x+z}{2}).$$

Let $r = \big(s^n - \big(s-\frac{|x-z|}{2}\big)^n\big)^{1/n}$. Noting $0<s<1$ and reasoning in the same way, we obtain

$$\begin{aligned} \int_{(B_s(x)\setminus B_s(z))\cup(B_s(z)\setminus B_s(x))} \frac{1}{|y|^\beta}\,dy &\le \int_{(B_s(x)\cup(B_s(z))\setminus B_{s-\frac{|x-z|}{2}}(\frac{x+z}{2})} \frac{1}{|y|^\beta}\,dy \\ &\le 2\int_{B_r(0)} \frac{1}{|y|^\beta}\,dy \\ &\le C\Big(s^n - \big(s-\frac{|x-z|}{2}\big)^n\Big)^{\frac{n-\beta}{n}} \\ &\le Cs^{n-1-\beta}|x-z|^\gamma. \end{aligned} \tag{4.2}$$

As a result,

$$|F_1(x)-F_1(z)| \le -\|v\|_{L^\infty}^q|x-z|^{1-\frac{\beta}{n}} \int_\varepsilon^1 s^{n-\beta-1}K_\alpha'(s)\,ds \le C(\varepsilon)\|v\|_{L^\infty}^q|x-z|^\gamma;$$

that is,

$$\sup_{x\ne z} \frac{|F_1(x)-F_1(z)|}{|x-z|^\gamma} \le C(\varepsilon). \tag{4.3}$$

Now we estimate the Hölder norm of F_2. For $x, z \in \Omega$, by (4.2),

$$
\begin{aligned}
|F_2(x) - F_2(z)| &= \Big| \int_1^\infty \Big(\int_{B_s(x)} \frac{v^q(y)}{|y|^\beta}\,dy - \int_{B_s(x)} \frac{v^q(y)}{|y|^\beta}\,dy \Big) K'_\alpha(s)\,ds \Big| \\
&\le \|v\|_{L^\infty}^q \int_1^\infty \Big(\int_{(B_s(x)\setminus B_s(z))\cup(B_s(z)\setminus B_s(x))} \frac{1}{|y|^\beta}\,dy \Big) |K'_\alpha(s)|\,ds \\
&\le -\|v\|_{L^\infty}^q |x-z|^\gamma \int_1^\infty s^{n-\beta-1} K'_\alpha(s)\,ds \\
&\le -\|v\|_{L^\infty}^q |x-z|^\gamma \int_1^\infty s^{n-\beta} K'_\alpha(s)\,ds.
\end{aligned}
$$

We may verify that

$$
\begin{aligned}
-\int_1^\infty s^{n-\beta-1} K'_\alpha(s) &= K_\alpha(1) + (n-\beta) \int_1^\infty s^{n-\beta-1} K_\alpha(s)\,ds \\
&\le K_\alpha(1) + C(n-\beta) \int_1^\infty s^{n-\beta-1} s^{-n-2\alpha}\,ds \\
&\le K_\alpha(1) + C.
\end{aligned}
$$

Thus,

$$
|F_2(x) - F_2(z)| \le C\|v\|_{L^\infty}^q |x-z|^\gamma;
$$

that is,

$$
\sup_{x\neq z} \frac{|F_2(x) - F_2(z)|}{|x-z|^\gamma} \le C. \tag{4.4}
$$

Inequalities (4.3) and (4.4) yield

$$
\sup_{x\neq z} \frac{|F^q v(x) - F^q v(z)|}{|x-z|^\gamma} \le C(\varepsilon). \tag{4.5}
$$

From the definition of $F_1(x)$ and $F_2(x)$, we have

$$
\begin{aligned}
|F_1(x)| &\le \|v\|_{L^\infty}^q \int_\varepsilon^1 \Big(\int_{B_s(x)} \frac{1}{|y|^\beta}\,dy \Big) |K'_\alpha(s)|\,ds \\
&\le C\|v\|_{L^\infty}^q \int_\varepsilon^1 s^{n-\beta} |K'_\alpha(s)|\,ds \\
&\le C(\varepsilon)\|v\|_{L^\infty}^q,
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
|F_2(x)| &\le \|v\|_{L^\infty}^q \int_1^\infty \Big(\int_{B_s(x)} \frac{1}{|y|^\beta}\,dy \Big) |K'_\alpha(s)|\,ds \\
&\le C\|v\|_{L^\infty}^q \int_1^\infty s^{n-\beta} |K'_\alpha(s)|\,ds \\
&\le C\|v\|_{L^\infty}^q.
\end{aligned} \tag{4.7}
$$

It follows from (4.6) and (4.7) that

$$
|F^q v|_{L^\infty} \le C(\varepsilon)\|v\|_{L^\infty}^q. \tag{4.8}
$$

Inequalities (4.6) and (4.8) imply that $F^q v$ is Hölder continuous, and this together with (4.8) imply $F^q v \in \mathfrak{X} \cap \mathfrak{Y}$.

Finally, we show that S_ε maps $(\mathfrak{X}\times\mathfrak{X})\cap(\mathfrak{Y}\times\mathfrak{Y})$ to itself. We need only to verify that if $\|w\|_{L^\infty}\le 2\|u\|_{L^\infty}+2\|v\|_{L^\infty}$, then

$$\|T_\varepsilon^q w\|_{L^\infty}\le 2\|u\|_{L^\infty}+2\|v\|_{L^\infty}. \tag{4.9}$$

In the same way, we can prove

$$\|T_\varepsilon^p w\|_{L^\infty}\le 2\|u\|_{L^\infty}+2\|v\|_{L^\infty}. \tag{4.10}$$

Now, we verify (4.9). Indeed,

$$\begin{aligned}
|T_\varepsilon^q w(x)| &= -\int_0^\varepsilon\int_{B_s(x)} K_\alpha'(s)\frac{w^q(y)}{|y|^\beta}\,ds\,dy\\
&\le (2\|u\|_{L^\infty}+2\|v\|_{L^\infty})^q\int_0^\varepsilon\int_{B_s(x)}\frac{1}{|y|^\beta}|K_\alpha'(s)|\,dyds\\
&\le C(2\|u\|_{L^\infty}+2\|v\|_{L^\infty})^q\int_0^\varepsilon s^{n-\beta}|K_\alpha'(s)|ds\\
&\le C(2\|u\|_{L^\infty}+2\|v\|_{L^\infty})^q\varepsilon^{2\alpha-\beta}.
\end{aligned}$$

Therefore, choosing ε sufficiently small, but independent of w, we obtain (4.9). This completes the proof of the theorem. $\square$

5. Symmetry of solutions

In this section, we show that positive solutions of (1.1) are radially symmetric. For a given real number λ, we may define

$$\Sigma_\lambda=\{x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n|x_1\le\lambda\},\quad T_\lambda=\{x\in\mathbb{R}^n|x_1=\lambda\}.$$

For $x\in\Sigma_\lambda$, let $x_\lambda=(2\lambda-x_1,x_2,\cdots,x_n)$, and define

$$u_\lambda(x)=u(x_\lambda),\quad v_\lambda(x)=v(x_\lambda).$$

Lemma 5.1. *For any positive solution u of* (1.1), *we have*

$$u(x)-u_\lambda(x)=\int_{\Sigma_\lambda}\Big(K_\alpha(x-y)-K_\alpha(x_\lambda-y)\Big)\Big(\frac{v^q(y)}{|y|^\beta}-\frac{v_\lambda^q(y)}{|y_\lambda|^\beta}\Big)\,dy, \tag{5.1}$$

$$v(x)-v_\lambda(x)=\int_{\Sigma_\lambda}\Big(K_\alpha(x-y)-K_\alpha(x_\lambda-y)\Big)\Big(\frac{u^p(y)}{|y|^\beta}-\frac{u_\lambda^p(y)}{|y_\lambda|^\beta}\Big)\,dy. \tag{5.2}$$

Proof. Let $\Sigma_\lambda^c=\{x=(x_1,x_2,\cdots,x_n)\in\mathrm{R}^n|x_1>\lambda\}$. It follows from (1.1) that

$$\begin{aligned}
u(x)&=\int_{\Sigma_\lambda}\frac{K_\alpha(x-y)v^q(y)}{|y|^\beta}\,dy+\int_{\Sigma_\lambda^c}\frac{K_\alpha(x-y)v^q(y)}{|y|^\beta}dy\\
&=\int_{\Sigma_\lambda}\frac{K_\alpha(x-y)v^q(y)}{|y|^\beta}\,dy+\int_{\Sigma_\lambda}\frac{K_\alpha(x-y_\lambda)v^q(y_\lambda)}{|y_\lambda|^\beta}\,dy\\
&=\int_{\Sigma_\lambda}\frac{K_\alpha(x-y)v^q(y)}{|y|^\beta}\,dy+\int_{\Sigma_\lambda}\frac{K_\alpha(x_\lambda-y)v_\lambda^q(y)}{|y_\lambda|^\beta}\,dy.
\end{aligned}$$

Here we have used the fact that $|x-y_\lambda|=|x_\lambda-y|$ and the fact that K_α is radially symmetric in R^n. Substituting x by x_λ gives

$$u(x_\lambda)=\int_{\Sigma_\lambda}\frac{G_\alpha(x_\lambda-y)v^q(y)}{|y|^\beta}\,dy+\int_{\Sigma_\lambda}\frac{G_\alpha(x-y)v_\lambda^q(y)}{|y_\lambda|^\beta}\,dy.$$

Hence,

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} \Big(K_\alpha(x-y) - K_\alpha(x_\lambda - y)\Big)\Big(\frac{v^q(y)}{|y|^\beta} - \frac{v_\lambda^q(y)}{|y_\lambda|^\beta}\Big)\,dy.$$

Similarly, we have

$$v(x) - v_\lambda(x) = \int_{\Sigma_\lambda} \Big(K_\alpha(x-y) - K_\alpha(x_\lambda - y)\Big)\Big(\frac{u^p(y)}{|y|^\beta} - \frac{u_\lambda^p(y)}{|y_\lambda|^\beta}\Big)\,dy.$$

This completes the proof. □

Proof of Theorem 1.6. We use the moving plane method developed for integral equations in [6] to prove the result. First, we show for sufficiently negative λ that

$$u(x) \le u(x_\lambda), \quad v(x) \le v(x_\lambda), \quad \forall x \in \Sigma_\lambda. \tag{5.3}$$

Set

$$w_\lambda(x) = u(x) - u(x_\lambda), \quad z_\lambda(x) = v(x) - v(x_\lambda),$$
$$\Sigma_\lambda^{u,-} = \{x \in \Sigma_\lambda | u(x) > u(x_\lambda)\}, \quad \Sigma_\lambda^{v,-} = \{x \in \Sigma_\lambda | v(x) > v(x_\lambda)\}.$$

From Lemma 5.1, we deduce that

$$\begin{aligned} u(x) - u_\lambda(x) &= \int_{\Sigma_\lambda \setminus \Sigma_\lambda^{v,-}} \Big(K_\alpha(x-y) - K_\alpha(x_\lambda - y)\Big)\Big(\frac{v^q(y)}{|y|^\beta} - \frac{v_\lambda^q(y)}{|y_\lambda|^\beta}\Big)\,dy \\ &\quad + \int_{\Sigma_\lambda^{v,-}} \Big(K_\alpha(x-y) - K_\alpha(x_\lambda - y)\Big)\Big(\frac{v^q(y)}{|y|^\beta} - \frac{v_\lambda^q(y)}{|y_\lambda|^\beta}\Big)\,dy. \end{aligned}$$

Since $|x-y| < |x_\lambda - y|$ and $|y| > |y_\lambda|$ in Σ_λ, taking into account that $K_\alpha(x)$ is decreasing as well as that t^q is convex, we obtain

$$\begin{aligned} u(x) - u_\lambda(x) &\le \int_{\Sigma_\lambda^{v,-}} \Big(K_\alpha(x-y) - K_\alpha(x_\lambda - y)\Big)\Big(\frac{v^q(y)}{|y|^\beta} - \frac{v_\lambda^q(y)}{|y_\lambda|^\beta}\Big)\,dy \\ &\le \int_{\Sigma_\lambda^{v,-}} \Big(K_\alpha(x-y) - K_\alpha(x_\lambda - y)\Big)\Big(\frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta}\Big)\,dy \\ &\le C \int_{\Sigma_\lambda^{v,-}} K_\alpha(x-y) \frac{v^{q-1}(v - v_\lambda)}{|y|^\beta}\,dy. \end{aligned}$$

We may derive as in the proof of (3.9) and (3.10) for

$$\frac{1}{\tilde p} - \frac{1}{\tilde q} = \frac{1}{p+1} - \frac{1}{q+1}$$

and

$$\frac{1}{d_1} = \frac{1}{\tilde q} + \frac{q-1}{q+1}, \quad \frac{1}{d_2} = \frac{1}{\tilde p} + \frac{p-1}{p+1}$$

that

$$\|w_\lambda\|_{L^{\tilde p}(\Sigma_\lambda^{u,-})} \le C\|v^{q-1} z_\lambda\|_{L^{d_1}(\Sigma_\lambda^{v,-})} \le C\|v\|^{q-1}_{L^{q+1}(\Sigma_\lambda^{v,-})} \|z_\lambda\|_{L^{\tilde q}(\Sigma_\lambda^{v,-})} \tag{5.4}$$

and

$$\|z_\lambda\|_{L^{\tilde q}(\Sigma_\lambda^{v,-})} \le C\|u^{p-1} w_\lambda\|_{L^{d_2}(\Sigma_\lambda^{u,-})} \le C\|u\|^{p-1}_{L^{p+1}(\Sigma_\lambda^{u,-})} \|w_\lambda\|_{L^{\tilde p}(\Sigma_\lambda^{u,-})}. \tag{5.5}$$

As a result,

$$\|w_\lambda\|_{L^{\tilde p}(\Sigma_\lambda^{u,-})} \le C\|u\|^{p-1}_{L^{p+1}(\Sigma_\lambda^{u,-})} \|v\|^{q-1}_{L^{q+1}(\Sigma_\lambda^{v,-})} \|w_\lambda\|_{L^{\tilde p}(\Sigma_\lambda^{u,-})}. \tag{5.6}$$

Since $(u,v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$, for sufficiently negative λ,

$$C\|u\|^{p-1}_{L^{p+1}(\Sigma_\lambda^{u,-})}\|v\|^{q-1}_{L^{q+1}(\Sigma_\lambda^{v,-})} \le \frac{1}{2}.$$

Hence,

$$\|w_\lambda\|_{L^{\tilde{p}}(\Sigma_\lambda^{u,-})} \le \frac{1}{2}\|w_\lambda\|_{L^{\tilde{p}}(\Sigma_\lambda^{u,-})}.$$

This implies $\Sigma_\lambda^{u,-}$ must be a set of measure zero. Similarly, the measure of $\Sigma_\lambda^{v,-}$ is zero. Consequently, (5.3) holds.

Next, we increase the value of λ continuously; that is, we move the plane T_λ to the right as long as the inequality (5.3) holds. We show that by moving T_λ in this way, it will not stop before the plane hitting the origin. Let

$$\lambda_0 = \sup\{\lambda | u(x) - u_\lambda(x) \le 0, v(x) - v_\lambda(x) \le 0, \forall x \in \Sigma_\lambda\}. \tag{5.7}$$

Obviously $\lambda_0 \le 0$, We claim that

$$\lambda_0 = 0. \tag{5.8}$$

In fact, if it were not the case, we would show that the plane could be moved further to the right by a small distance, and this would contradict with the definition of λ_0. Suppose by the contrary that $\lambda_0 < 0$, and that there exist some points x_0, x_1 in Σ_{λ_0} such that $u(x_0) = u_{\lambda_0}(x_0), v(x_1) = v_{\lambda_0}(x_1)$. By Lemma 5.1 and noting that $x_{\lambda_0} = (x_0)_{\lambda_0}$, we obtain

$$\begin{aligned} 0 &= u(x_0) - u_{\lambda_0}(x_0) \\ &= \int_{\Sigma_{\lambda_0}} \Big(K_\alpha(x_0 - y) - K_\alpha(x_{\lambda_0} - y)\Big)\Big(\frac{v^q(y)}{|y|^\beta} - \frac{v^q_{\lambda_0}(y)}{|y_{\lambda_0}|^\beta}\Big)\,dy. \end{aligned}$$

Since $|y| > |y_0|$ in Σ_{λ_0},

$$\frac{v^q(y)}{|y|^\beta} < \frac{v^q_{\lambda_0}(y)}{|y_{\lambda_0}|^\beta} \quad \text{in } \Sigma_{\lambda_0}.$$

Moreover, $|x_0 - y| < |x_{\lambda_0} - y|$ in Σ_{λ_0}, we infer that

$$v(x) \equiv v_{\lambda_0}(x) \equiv 0, \quad a.e.\ x \in \Sigma_{\lambda_0}.$$

This also implies that $v(x) \equiv 0$, which is a contradiction to the fact that v is positive. So we have

$$u(x) < u_{\lambda_0}(x), \quad a.e.\ x \in \Sigma_{\lambda_0}.$$

Similarly,

$$v(x) < v_{\lambda_0}(x), \quad \text{a.e. } x \in \Sigma_{\lambda_0}.$$

Since $(u,v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$, for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{R^n\setminus B_R(0)} u^{p+1}\,dx < \varepsilon, \quad \int_{R^n\setminus B_R(0)} v^{q+1}\,dx < \varepsilon.$$

By Lusin theorem, for any $\delta > 0$, there exists a closed set F_δ with $F_\delta \subset B_R(0) \cup \Sigma_{\lambda_0} = E$ and $m(E - F_\delta) < \delta$ such that $w_{\lambda_0}|F_\delta, z_{\lambda_0}|F_\delta$ is continuous.

As $w_{\lambda_0}, z_{\lambda_0} < 0$ in the interior of $\Sigma_{\lambda_0}, w_{\lambda_0}, z_{\lambda_0} < 0$ in F_δ. Choosing $\varepsilon_0 > 0$ sufficiently small so that for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_0)$, it holds that $w_\lambda, z_\lambda < 0$ in F_δ. For such a λ,

$$\begin{aligned} \Sigma_\lambda^{u,-} &\subset M^u := (R^n\setminus B_R(0)) \cup (E\setminus F_\delta) \cup [(\Sigma_\lambda\setminus\Sigma_{\lambda_0}^{u,-}) \cap B_R(0)], \\ \Sigma_\lambda^{v,-} &\subset M^v := (R^n\setminus B_R(0)) \cup (E\setminus F_\delta) \cup [(\Sigma_\lambda\setminus\Sigma_{\lambda_0}^{v,-}) \cap B_R(0)]. \end{aligned}$$

We may choose ε, δ and ε_0 small so that

$$C\|u\|^{p-1}_{L^{p+1}(\Sigma_\lambda^{u,-})}\|v\|^{q-1}_{L^{q+1}(\Sigma_\lambda^{v,-})} \leq \frac{1}{2}.$$

Hence,

$$\|w_\lambda\|_{L^{\tilde{p}}(\Sigma_\lambda^{u,-})} \leq \frac{1}{2}\|w_\lambda\|_{L^{\tilde{p}}(\Sigma_\lambda^{u,-})},$$

which implies that $\Sigma_\lambda^{u,-}$ must be of measure zero. Again, it contradicts the definition of λ_0. Equation (5.8) is proved.

On the other hand, we can also move the plane from positive infinite to zero by the similar procedure. Hence, $u(x), v(x)$ are symmetric and monotonic with respect to $x_1 = 0$. Moreover, since the x_1 direction can be chosen arbitrarily, $u(x), v(x)$ are radial symmetric and strictly monotonic with respect to the origin. Thus we have completed the proof of Theorem 1.6. □

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