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A JUMPING PROBLEM FOR A QUASILINEAR EQUATION INVOLVING THE 1-LAPLACE OPERATOR

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Abstract. We prove the existence of solutions for a jumping problem of a functional whose principal part is the total variation. Our main tool is a nonsmooth variational method.

1. Introduction

The expression *jumping problems* appeared first in the celebrated paper by Ambrosetti and Prodi [1]. They studied a semilinear elliptic PDE of the form

$$\begin{gathered} -\Delta u + g(x,u) = w \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $w \in W^{-1,2}(\Omega)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{s\to-\infty} \frac{g(x,s)}{s} = -\alpha, \quad \lim_{s\to+\infty} \frac{g(x,s)}{s} = -\beta\,.$$

Under suitable assumptions on g and the condition $\beta < \lambda_1 < \alpha < \lambda_2$, where λ_1 and λ_2 are the first two eigenvalues of the operator $-\Delta$, the authors provided a precise description of the number of solutions u, of (1.1), in dependence of w.

The result has been extended in various directions, also by means of variational methods applied to the associated functional $f : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined as

$$f(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \int_\Omega G(x,u) - \langle w, u\rangle\,,$$

where $G(x,s) = \int_0^s g(x,t)\,dt$. Let us mention, in particular, the case in which $f : W_0^{1,2}(\Omega) \to \mathbb{R}$ is the defined as

$$f(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u + \int_\Omega G(x,u) - \langle w, u\rangle\,,$$

considered in [10], or, more generally, the case in which $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ is defined as

$$f(u) = \int_\Omega L(x,u,\nabla u) + \int_\Omega G(x,u) - \langle w, u\rangle \tag{1.2}$$

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with

$$\lim_{s\to-\infty}\frac{g(x,s)}{|s|^{p-2}s}=-\alpha,\quad \lim_{s\to+\infty}\frac{g(x,s)}{|s|^{p-2}s}=-\beta\,,$$

considered in [21]. In these extensions, the feature is that the functional f is continuous, but not locally Lipschitz, so that the nonsmooth variational methods of [12, 15, 18] are used. Further developments in this direction are contained in [11, 22].

Information about the number of solutions in dependence of w are given, in the case of (1.2), by setting

$$w=t\varphi_1^{p-1}+w_0\,,$$

where φ_1 is a positive first eigenfunction of the p-Laplace operator and $w_0\in W^{-1,p'}(\Omega)$. From a variational point of view, this is equivalent to study the functional

$$f_t(u)=\int_\Omega L(x,u,\nabla u)+\int_\Omega G(x,u)-t\int_\Omega\varphi_1^{p-1}u-\langle w_0,u\rangle\,,$$

in which the "exploring term"

$$u\mapsto\int_\Omega\varphi_1^{p-1}u$$

has lower order at infinity with respect to the principal part of the functional, as $p>1$. Under the assumption that $\beta<\lambda_1<\alpha$, the main result asserts that there exist $\underline{t}\le\bar t$ such that the problem has at least two solutions for $t\ge\bar t$ and no solution for $t\le\underline{t}$.

In this article we are interested in a corresponding result for the case $p=1$. At a naive level, we would consider the functional $f_t:W_0^{1,1}(\Omega)\to\mathbb{R}$ defined as

$$f_t(u)=\int_\Omega|\nabla u|+\int_\Omega G(x,u)-t\int_\Omega\varphi H(u)\,,$$

where $\varphi>0$ in Ω and $H(s)=\int_0^s\frac{1}{\sqrt{1+t^2}}\,dt$. The choice of

$$u\mapsto\int_\Omega\varphi H(u)$$

as "exploring term" is related to the need of considering a lower order term at infinity with respect to the principal part of the functional.

It is well known that direct variational methods do not work properly in $W_0^{1,1}(\Omega)$, so that we will actually consider a "relaxed" functional defined on $BV_0(\Omega)$, which is the subspace of $BV(\mathbb{R}^n)$ made by functions vanishing outside Ω. Even after this first extension, (nonsmooth) critical point theory cannot be directly applied, as the functional does not satisfy the Palais-Smale condition. This is due to the fact that such a condition requires a norm convergence, which is almost impossible in BV, because of the lack strict convexity of the principal part of the functional. For this reason, as in [17, 19], we further extend the relaxed functional to $L^{n/(n-1)}(\mathbb{R}^n)$ with value $+\infty$ outside $BV_0(\Omega)$. In this way the functional becomes only lower semicontinuous, but the nonsmooth critical point theory of [12, 15, 18] can be successfully applied.

The conditions involving α and β will be expressed again as $\beta<\lambda_1<\alpha$, where λ_1 is the first eigenvalue of the 1-Laplace operator as defined e.g. in [23], and

$$\liminf_{s\to-\infty}\frac{G(x,s)}{s}\ge\alpha\,,\quad\liminf_{s\to+\infty}\frac{G(x,s)}{s}\ge-\beta\,.$$

We will prove that there exists $\bar{t}$ such that the problem has at least two solutions, provided that $t \geq \bar{t}$. A model case, which is covered in our result, is given by

$$G(x,s) = -\beta s^+ - \alpha s^- .$$

As G is only locally Lipschitz, the concept of solution is given in terms of "hemi-variational inequality" as in [19].

The content of the paper runs as follows: in Section 2 we recall some basic tools about functions with bounded variation and non-smooth critical point theory and in Sections 3 and 4 we state and prove our main result.

2. Notation and preliminary results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary.

2.1. **BV functions.** We denote by $BV(\Omega)$ the subspace of $L^1(\Omega)$ made by functions whose distributional gradient Du is a vector-valued Radon measure with bounded variation. For $u \in BV(\Omega)$, $|Du|$ denotes the total variation of Du. We write ∇u for the *approximate differential* of u defined as in [4, Definition 3.70]. Denoted by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$, we can decompose Du as: $Du = D^a u + D^s u$, where $D^a u$ and $D^s u$ are the absolute continuous part and the singular part of Du with respect to $\mathcal{L}^n$. It turns out that $D^a u = \nabla u \, d\mathcal{L}^n$. For $u \in W^{1,1}(\Omega)$, the approximate differential of u coincides with the distributional gradient of u and we have: $Du = \nabla u \, d\mathcal{L}^n$.

Remark 2.1. $BV(\Omega)$ is endowed with the norm:

$$\|u\|_{BV(\Omega)} = |Du|(\Omega) + \int_\Omega |u| \, d\mathcal{L}^n .$$

Definition 2.2 (Trace)**.** Let $u \in BV(\Omega)$. For $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$, there exists $u^{\partial\Omega}(x)$ such that:

$$\lim_{\rho \to 0^+} \frac{1}{\rho^n} \int_{\Omega \cap B_\rho(x)} |u(y) - u^{\partial\Omega}(x)| \, d\mathcal{L}^n(y) = 0 ,$$

where $B_\rho(x)$ is the open ball of radius ρ and center x.

Definition 2.3. We set: $BV_0(\Omega) := \{u \in BV(\mathbb{R}^n) : u = 0 \ \mathcal{L}^n \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$.

Of course, $BV_0(\Omega)$ is a closed linear subspace of $BV(\mathbb{R}^n)$. Moreover, for every $u \in BV_0(\Omega)$ it turns out that

$$|Du|(\overline{\Omega}) = |Du|(\Omega) + \int_{\partial\Omega} |u^{\partial\Omega}| d\mathcal{H}^{n-1} .$$

Theorem 2.4 (Sobolev-type inequality in BV)**.** *There exists $S > 0$ such that, for every $u \in BV(\Omega)$, we have*

$$S \, \|u\|_{L^{\frac{n}{n-1}}(\Omega)} \leq |Du|(\Omega) + \int_{\partial\Omega} |u^{\partial\Omega}| d\mathcal{H}^{n-1} .$$

Remark 2.5. Theorem 2.4 implies that, in $BV_0(\Omega)$, $|Du|(\overline{\Omega})$ is a norm equivalent to the canonical norm of $BV(\mathbb{R}^n)$.

Theorem 2.6. *Let p belong to the interval $[1, \frac{n}{n-1}[$. Then the inclusion of $BV_0(\Omega)$ in $L^p(\mathbb{R}^n)$ is compact.*

The following theorem states the existence of the first eigenvalue of the total variation.

Theorem 2.7. *There exists*

$$\lambda_1 = \min_{u \in BV_0(\Omega)\setminus\{0\}} \frac{|Du|(\overline{\Omega})}{\int_\Omega |u|\, d\mathcal{L}^n}\,.$$

For more details on BV-functions and their properties see for instance [4, 23]. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that there exists $M > 0$ for which

$$|p| \le \psi(p) \le M(1 + |p|)\,.$$

The functional

$$F(u) = \int_\Omega \psi(\nabla u) d\mathcal{L}^n \tag{2.1}$$

is lower-semicontinuous in $W^{1,1}(\Omega)$ but, since this space is not reflexive, it has not good properties for the compactness. Therefore, when one deals with functionals with linear growth in the gradient, it is usual to extend F on the larger space $L^{n/(n-1)}(\Omega)$ in such a way to have that this extension is lower-semicontinuous. This procedure is called *relaxation* and on past years it was widely investigated for functionals of the type in (2.1) and with more complicated integrands ψ, depending also on x and u and eventually also under trace constraints (see [3, 4, 7, 8]). For F defined in (2.1) its relaxation, denoted by $\overline{F}$, takes the form

$$\overline{F}(u) = \begin{cases} \int_\Omega \psi(\nabla u) d\mathcal{L}^n + \int_\Omega \psi^\infty\big(\frac{dD^s u}{d|D^s u|}\big) d|D^s u| & \text{if } u \in BV(\Omega)\,, \\ +\infty & \text{if } u \in L^{n/(n-1)}(\Omega) \setminus BV(\Omega)\,, \end{cases} \tag{2.2}$$

where

$$\psi^\infty(p) = \lim_{t \to +\infty} \frac{\psi(tp)}{t}\,.$$

Remark 2.8. If $\psi(p) = |p|$, F is the Diriclet functional with linear growth

$$F(u) = \int_\Omega |\nabla u| d\mathcal{L}^n$$

and an easy computation gives

$$\overline{F}(u) = \int_\Omega |\nabla u| d\mathcal{L}^n + |D^s u|(\Omega) \equiv |Du|(\Omega) \quad \text{if } u \in BV(\Omega)\,.$$

If F is defined only on $W_0^{1,1}(\Omega)$, then the relaxed functional can be naturally identified with $\overline{F} : L^{n/(n-1)}(\mathbb{R}^n) \to [0, +\infty]$ defined as

$$\overline{F}(u) = \begin{cases} |Du|(\overline{\Omega}) & \text{if } u \in BV_0(\Omega)\,, \\ +\infty & \text{if } u \in L^{n/(n-1)}(\mathbb{R}^n) \setminus BV_0(\Omega)\,. \end{cases}$$

Remark 2.9. In view of Remark 2.8 it is usual to refer to the number λ_1 given in Theorem 2.7 as the first eigenvalue of the 1-Laplace operator with homogeneous Dirichlet condition.

In [6] it is computed the first variation for functionals involving a term of the type in (2.2) with a ψ depending also on x. For convex functionals of the same type and involving also u, a characterization of the subdifferential is performed in [5]. In Section 3 we consider non-convex functionals containing also a non locally

Lipschitz term. In this case the critical points can be characterized by means of hemivariational inequalities, using subdifferential calculus and nonsmooth analysis (see [9, 19]).

2.2. **Nonsmooth critical point theory.** Let X be a real Banach space and let X^* be its dual space. First of all, let us recall from [13] some basic notions.

Definition 2.10. If $f : X \to \mathbb{R}$ is a locally Lipschitz function, we set, for every $u, v \in X$,

$$f^\circ(u;v) := \limsup_{z\to u,\, w\to v,\, t\to 0^+} \frac{f(z+tw)-f(z)}{t}\,. \tag{2.3}$$

The real number $f^\circ(u;v)$ is called the *generalized directional derivative* of f at u with respect to the direction v. For every $u \in X$, let also

$$\partial f(u) = \{u^* \in X^* :\ \langle u^*, v\rangle \le f^\circ(u;v)\,,\ \forall v \in X\}\,. \tag{2.4}$$

The set $\partial f(u)$ is called the *subdifferential* of f at u.

It turns out that f° is positively one-homogeneous with respect to the second variable.

Assume now that $f : X \to]-\infty, +\infty]$ is a lower semicontinuous function and set

$$\operatorname{epi}(f) = \{(u,\lambda) \in X \times \mathbb{R} :\ f(u) \le \lambda\}\,.$$

Definition 2.11. For every $u \in X$ with $f(u) < +\infty$, we denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist a neighborhood W of $(u, f(u))$ in $\operatorname{epi}(f)$, $\delta > 0$ and a continuous map $\mathcal{H} : W \times [0,\delta] \to X$ satisfying

$$\|\mathcal{H}((w,\mu),t) - w\| \le t\,, \quad f(\mathcal{H}((w,\mu),t)) \le \mu - \sigma t\,,$$

whenever $(w,\mu) \in W$ and $t \in [0,\delta]$.

The extended real number $|df|(u)$ is called *the weak slope* of f at u.

The above notion has been introduced in [18], following an equivalent approach. The version we have recalled here is taken from [9]. According to [16], we also define a function $\mathcal{G}_f : \operatorname{epi}(f) \to \mathbb{R}$ by $\mathcal{G}_f(u,\lambda) = \lambda$.

Definition 2.12. A point $u \in X$ is said to be a *lower critical point* of f, if $f(u) < +\infty$ and $|df|(u) = 0$.

Definition 2.13 (Cerami-Palais-Smale condition)**.** Let $c \in \mathbb{R}$. A sequence (u_k) in X is said to be a *Cerami-Palais-Smale sequence at level* c for f ($(CPS)_c$-sequence, for short), if $f(u_k) \to c$ and $(1 + \|u_k\|)|df|(u_k) \to 0$. We say that f satisfies the *Cerami-Palais-Smale condition at level* c ($(CPS)_c$-condition, for short), if every $(CPS)_c$-sequence for f admits a strongly convergent subsequence in X.

Definition 2.14. Let $c \in \mathbb{R}$. We say that f satisfies condition $(epi)_c$, if there exists $\varepsilon > 0$ such that

$$\inf\{|d\mathcal{G}_f|(u,\lambda) :\ f(u) < \lambda,\ |\lambda - c| < \varepsilon\} > 0\,.$$

The next result is an extension of the celebrated Mountain pass theorem [2] to our setting. For the proof, when the usual Palais-Smale condition is assumed, we refer the reader to [15, 20]. We also refer to [14] for the fact that the Cerami-Palais-Smale condition can be reduced to the Palais-Smale condition by a change of metric.

Theorem 2.15 (Mountain pass theorem). *Let $u_0, u_1, \overline{u} \in X$ and $r > 0$ be such that:*

$$\|u_0 - \overline{u}\| < r, \quad \|u_1 - \overline{u}\| > r \tag{2.5}$$

and

$$\inf\{f(u) : u \in X, \|u - \overline{u}\| = r\} \geq \max\{f(u_0), f(u_1)\}. \tag{2.6}$$

Set

$$\Gamma := \{\gamma \in C([0,1]; X) : \gamma(0) = u_0, \gamma(1) = u_1\}$$

and define

$$c_1 := \inf_{u \in \overline{B_r(\overline{u})}} f(u), \quad c_2 = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} f(\gamma(t)),$$

so that

$$c_1 \leq \inf\{f(u) : u \in X, \|u - \overline{u}\| = r\} \leq c_2.$$

Assume that $c_1 > -\infty$, $c_2 < +\infty$ and that f satisfies $(CPS)_c$ and $(epi)_c$ for $c = c_1, c_2$.

Then f admits two distinct lower critical points w_1, w_2 with $f(w_1) = c_1$ and $f(w_2) = c_2$.

3. Statement of the main result

We consider:

- $\varphi \in L^n(\Omega)$ such that $\varphi > 0$ a.e. in Ω;
- $G : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$G(\cdot, s) \text{ is measurable for any } s \in \mathbb{R}; \tag{3.1}$$

for every $t > 0$ there exists $a_t \in L^1(\Omega)$ such that

$$|G(x, s_1) - G(x, s_2)| \leq a_t(x)|s_1 - s_2| \tag{3.2}$$

for a.e. $x \in \Omega$ and every $s_1, s_2 \in \mathbb{R}$ with $|s_1| \leq t$, $|s_2| \leq t$; there exist $\alpha, \beta \in \mathbb{R}$ such that $\beta < \lambda_1 < \alpha$ and

$$\liminf_{s \to -\infty} \frac{G(x,s)}{s} \geq \alpha, \quad \liminf_{s \to +\infty} \frac{G(x,s)}{s} \geq -\beta \quad \text{for a.e. } x \in \Omega. \tag{3.3}$$

From (3.2) it follows that $G(x, \cdot)$ is locally Lipschitz for a.e. $x \in \Omega$. We denote by $G^\circ(x, s; t)$ the generalized directional derivative with respect to the second variable. Then we also assume that

- there exist $b_0 \in L^1(\Omega)$ and $b_1 \in L^n(\Omega)$ such that:

$$|G(x, s)| \leq b_1(x)|s| \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}, \tag{3.4}$$

$$G^\circ(x, s; -s) \leq b_0(x) + b_1(x)|s| \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}, \tag{3.5}$$

$$G^\circ(x, s; s) \leq b_0(x) + G(x, s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } s \leq 0. \tag{3.6}$$

From (3.4) it follows that the functional

$$\{u \mapsto \int_\Omega G(x, u)\}$$

is continuous on $L^{n/(n-1)}(\Omega)$, although it is not locally Lipschitz, as there is no upper bound for $G^\circ(x, s; s)$ when $s > 0$. In particular, the assumptions on G are satisfied in the model case

$$G(x, s) = -\beta s^+ - \alpha s^-.$$

We are interested in solutions $u \in BV_0(\Omega)$ of the hemivariational inequality (see [19])

$$|Dv|(\overline{\Omega}) - |Du|(\overline{\Omega}) + \int_\Omega G^\circ(x,u;v-u) \geq t\int_\Omega \varphi\frac{1}{\sqrt{1+u^2}}(v-u)\,, \quad \forall v \in BV_0(\Omega) \tag{3.7}$$

associated with the lower semicontinuous functional $f_t : L^{n/(n-1)}(\mathbb{R}^n) \to \mathbb{R}\cup\{+\infty\}$ defined as:

$$f_t(u) = \begin{cases} |Du|(\overline{\Omega}) + \int_\Omega G(x,u) - t\int_\Omega \varphi H(u) & \text{if } u \in BV_0(\Omega)\,, \\ +\infty & \text{if } u \in L^{n/(n-1)}(\mathbb{R}^n) \setminus BV_0(\Omega)\,, \end{cases} \tag{3.8}$$

where

$$H(s) = \int_0^s \frac{1}{\sqrt{1+t^2}}\,dt\,.$$

If $G(x,\cdot)$ is of class C^1 for a.e. $x \in \Omega$ and we set $g(x,s) = D_sG(x,s)$, then assumptions (3.5) and (3.6) are equivalent to

$$sg(x,s) \geq -b_0(x) - b_1(x)|s|\,,$$
$$sg(x,s) \leq b_0(x) + G(x,s) \quad \text{for } s \leq 0\,.$$

Remark 3.1. In view of Remark 2.8 we recall that f_t is achieved by relaxation of the functional defined on $W_0^{1,1,}(\Omega)$ as

$$\int_\Omega |\nabla u| + \int_\Omega G(x,u) - t\int_\Omega \varphi H(u)\,.$$

Theorem 3.2. *There exists $\bar{t} \in \mathbb{R}$ such that, for every $t \geq \bar{t}$, there exists at least two solutions of* (3.7).

4. Proof of the main result

4.1. Compactness properties.

Proposition 4.1. *For every $t > 0$ and $c \in \mathbb{R}$, the functional f_t satisfies $(CPS)_c$.*

Proof. First of all, by (3.4) we have that $G(x,0) = 0$. Let (u_k) be a sequence in $BV_0(\Omega)$ satisfying

$$\lim_{k\to+\infty} f_t(u_k) = c\,, \tag{4.1}$$

$$\lim_{k\to+\infty}(1 + \|u_k\|_{n/(n-1)})|df_t|(u_k) = 0\,. \tag{4.2}$$

By [19, Theorems 2.16, 3.6 and 4.1], there exists a sequence (w_k) in $L^n(\mathbb{R}^n)$ such that

$$\lim_{k\to+\infty}(1 + \|u_k\|_{n/(n-1)})\|w_k\|_n = 0\,, \tag{4.3}$$

$$\begin{aligned} |Dv|(\overline{\Omega}) \geq{}& |Du_k|(\overline{\Omega}) - \int_\Omega G^\circ(x,u_k;v-u_k) \\ &+ t\int_\Omega \varphi\frac{1}{\sqrt{1+u_k^2}}(v-u_k) + \int_\Omega w_k(v-u_k) \quad \forall v \in BV_0(\Omega). \end{aligned} \tag{4.4}$$

In particular, the choice $v = u_k - u_k^-$ yields

$$\begin{aligned} &|Du_k|(\overline{\Omega}) + |Du_k^-|(\overline{\Omega}) \\ &\geq |D(u_k - u_k^-)|(\overline{\Omega}) \end{aligned}$$

$$\geq |Du_k|(\overline{\Omega}) - \int_\Omega G^\circ(x, -u_k^-; -u_k^-) - t\int_\Omega \varphi \frac{1}{\sqrt{1+u_k^2}}\, u_k^- - \int_\Omega w_k u_k^- \,,$$

hence

$$|Du_k^-|(\overline{\Omega}) \geq - \int_\Omega G^\circ(x, -u_k^-; -u_k^-) - t\int_\Omega \varphi \frac{1}{\sqrt{1+u_k^2}}\, u_k^- - \int_\Omega w_k u_k^- \,,$$

which by (3.6) implies

$$\begin{aligned} f_t(u_k) &\geq |Du_k^+|(\overline{\Omega}) + \int_\Omega G(x, u_k^+) + \int_\Omega [G(x, -u_k^-) - G^\circ(x, -u_k^-; -u_k^-)] \\ &\quad - t\int_\Omega \varphi H(u_k) - t\int_\Omega \varphi \frac{1}{\sqrt{1+u_k^2}}\, u_k^- - \int_\Omega w_k u_k^- \\ &\geq |Du_k^+|(\overline{\Omega}) + \int_\Omega G(x, u_k^+) - \int_\Omega b_0 - t\int_\Omega \varphi H(u_k) - t\int_\Omega \varphi \\ &\quad - \int_\Omega w_k u_k^- . \end{aligned} \tag{4.5}$$

By [19, Theorems 3.12 and 4.1], it sufficies to prove that (u_k) is bounded in $L^{n/(n-1)}(\mathbb{R}^n)$.

We first show that $(|Du_k^+|(\overline{\Omega}))$ is bounded. Assume, for a contradiction, that $\lim_{k\to+\infty} |Du_k^+|(\overline{\Omega}) = +\infty$ and consider the sequence $v_k := u_k^+/\varrho_k$, where $\varrho_k = |Du_k^+|(\overline{\Omega})$. Then, up to a subsequence, (v_k) is convergent to some $v \in BV_0(\Omega)$ with $|Dv|(\overline{\Omega}) \leq 1$ weakly in $L^{n/(n-1)}(\mathbb{R}^n)$ and a.e. in Ω. Moreover, we also have that $u_k^+(x) \to +\infty$ a.e. on $\{v(x) \neq 0\}$. Since $H(u_k) \leq H(u_k^+)$, from (4.5) we infer that

$$\begin{aligned} \frac{f_t(u_k)}{\varrho_k} &\geq 1 + \int_\Omega \frac{G(x, \varrho_k v_k)}{\varrho_k} - \frac{1}{\varrho_k}\int_\Omega b_0 - t\int_\Omega \varphi \frac{H(\varrho_k v_k)}{\varrho_k} \\ &\quad - \frac{t}{\varrho_k}\int_\Omega \varphi - \frac{1}{\varrho_k}\int_\Omega w_k u_k^- \,. \end{aligned} \tag{4.6}$$

On the other hand, by (3.4) and (3.3) we have

$$\begin{gathered} \frac{G(x, \varrho_k v_k)}{\varrho_k} - t\varphi \frac{H(\varrho_k v_k)}{\varrho_k} \geq -b_1(x) v_k - t\varphi v_k \,, \\ \liminf_{k\to+\infty} \Big(\frac{G(x, \varrho_k v_k)}{\varrho_k} - t\varphi \frac{H(\varrho_k v_k)}{\varrho_k} \Big) \geq -\beta v \,. \end{gathered}$$

From Fatou's lemma, (4.1), (4.3) and (4.6) we infer that

$$0 \geq 1 - \beta \int_\Omega v \geq 1 - \frac{\beta}{\lambda_1} |Dv|(\overline{\Omega}) \geq 1 - \frac{\beta}{\lambda_1} \,.$$

Since $\beta < \lambda_1$, a contradiction follows. Therefore, $(|Du_k^+|(\overline{\Omega}))$ is bounded.

We now show that also $(|Du_k^-|(\overline{\Omega}))$ is bounded. Again, we assume by contradiction that $\lim_{k\to+\infty} |Du_k^-|(\overline{\Omega}) = +\infty$ and consider the sequence $\tilde{v}_k := u_k^-/\tilde{\varrho}_k$, where $\tilde{\varrho}_k = |Du_k^-|(\overline{\Omega})$. Then, up to a subsequence, $(\tilde{v}_k)$ is convergent to some $\tilde{v} \in BV_0(\Omega)$ with $|D\tilde{v}|(\overline{\Omega}) \leq 1$ weakly in $L^{n/(n-1)}(\mathbb{R}^n)$ and a.e. in Ω. Moreover, we have that $u_k^-(x) \to +\infty$ a.e. in $\{\tilde{v}(x) \neq 0\}$. On the other hand, from (4.5) we also infer that

$$f_t(u_k) \geq |Du_k^+|(\overline{\Omega}) + \int_\Omega G(x, u_k^+) - \int_\Omega b_0 - t\int_\Omega \varphi H(u_k^+)$$

$$+ t\int_\Omega \varphi H(u_k^-) - t\int_\Omega \varphi - \int_\Omega w_k u_k^- .$$

Since $(|Du_k^+|(\overline{\Omega}))$ is bounded, from (4.1), (4.3) and (3.4) we deduce that $(\varphi H(u_k^-))$ is bounded in $L^1(\Omega)$. By Fatou's lemma it follows that $\tilde{v} = 0$ a.e. in Ω. By (3.4) we have

$$\frac{f_t(u_k)}{\tilde{\varrho}_k} \geq \frac{1}{\tilde{\varrho}_k}\,|Du_k^+|(\overline{\Omega}) + 1 + \frac{1}{\tilde{\varrho}_k}\int_\Omega G(x,u_k^+) - \int_\Omega b_1 v_k - t\int_\Omega \varphi v_k .$$

Passing to the limit as $k \to \infty$ and taking into account (4.1), we infer that $0 \geq 1$ and a contradiction follows. Therefore also $(|Du_k^-|(\overline{\Omega}))$ is bounded. In particular, (u_k) is bounded in $L^{n/(n-1)}(\mathbb{R}^n)$ and the assertion follows. $\square$

4.2. **Geometric conditions.** To apply Theorem 2.15 we check that the functional f_t satisfies the property stated in the two following propositions.

Proposition 4.2. *Set*

$$X^+ := \{u \in L^{n/(n-1)}(\mathbb{R}^n) : u \geq 0 \text{ a.e. in } \mathbb{R}^n,$$
$$B_r(v) := \{u \in L^{n/(n-1)}(\mathbb{R}^n) : \|u-v\|_{n/(n-1)} < r\} .$$

Then there exists $\overline{t} > 0$ such that, for every $t \geq \overline{t}$, there exist $u_0 \in X^+ \cap BV_0(\Omega)$ and $r > 0$ such that $f_t(u) \geq f_t(u_0)$ for every $u \in \overline{B_r(u_0)}$.

Proof. From assumptions (3.3), (3.4) and the fact that $\varphi \in L^n(\Omega)$, we infer that f_t is coercive on X^+, although it does not on the whole $L^{n/(n-1)}(\mathbb{R}^n)$. Moreover by assumptions on G it is also lower semi-continuous and therefore it immediately follows that there exists a minimum point u_0 of f_t on X^+ with $u_0 \in BV_0(\Omega)$ and $f_t(u_0) \leq 0$.

Observe that for $t > 0$,

$$\{u \in BV_0(\Omega) : |Du|(\overline{\Omega}) \leq 1\,,\ u \geq 0 \text{ a.e. in } \Omega\,,\ 1 - \int_\Omega b_1 u + t\int_\Omega \varphi u \leq 0\}$$

is a decreasing family of weakly compact subsets of $L^{n/(n-1)}(\mathbb{R}^n)$ with empty intersection, as $\varphi > 0$ a.e. in Ω. Therefore there exists $\overline{t} > 0$ such that

$$\{u \in BV_0(\Omega) : |Du|(\overline{\Omega}) \leq 1\,,\ u \geq 0 \text{ a.e. in } \Omega\,,\ 1 - \int_\Omega b_1 u + \overline{t}\int_\Omega \varphi u \leq 0\} = \emptyset\,. \tag{4.7}$$

Let us show that, for every $t \geq \overline{t}$, u_0 is a local minimum of f_t in $L^{n/(n-1)}(\mathbb{R}^n)$. By contradiction, let (v_k) be a sequence convergent to u_0 with $f_t(v_k) < f_t(u_0)$. Since $G(x,0) = 0$ and H is an odd function, by [4, Theorem 3.99] and (3.4) we have:

$$\begin{aligned} f_t(v_k) &= |D(v_k^+)|(\overline{\Omega}) + |D(v_k^-)|(\overline{\Omega}) + \int_\Omega G(x,v_k^+) + \int_\Omega G(x,-v_k^-) \\ &\quad - t\int_\Omega \varphi H(v_k^+) + t\int_\Omega \varphi H(v_k^-) \\ &\geq f_t(u_0) + |D(v_k^-)|(\overline{\Omega}) - \int_\Omega b_1 v_k^- + t\int_\Omega \varphi H(v_k^-) \\ &> f_t(v_k) + |D(v_k^-)|(\overline{\Omega}) - \int_\Omega b_1 v_k^- + t\int_\Omega \varphi H(v_k^-) . \end{aligned}$$

It follows

$$|D(v_k^-)|(\overline{\Omega}) - \int_\Omega b_1 v_k^- + t\int_\Omega \varphi H(v_k^-) < 0 .$$

Since $v_k^- \to 0$ a.e. in Ω, if we define a sequence (η_k) in $L^\infty(\Omega)$ by

$$\eta_k = \begin{cases} H(v_k^-)/v_k^- & \text{where } v_k^- \neq 0\,, \\ 1 & \text{where } v_k^- = 0\,, \end{cases}$$

by Lebesgue theorem we infer that

$$\lim_k \varphi\eta_k = \varphi \quad \text{strongly in } L^n(\Omega)\,.$$

Let $w_k = v_k^-/|D(v_k^-)|(\overline{\Omega})$. Then, up to a subsequence, (w_k) is weakly convergent in $L^{n/(n-1)}(\mathbb{R}^n)$ to some $w \in BV_0(\Omega)$ satisfying $w \geq 0$, $|Dw|(\overline{\Omega}) \leq 1$ and

$$1 - \int_\Omega b_1 w + t\int_\Omega \varphi w \leq 0\,.$$

This fact contradicts (4.7) and the assertion follows. $\square$

Proposition 4.3. *Let $\psi \in BV_0(\Omega) \setminus \{0\}$ be a first eigenfunction of the total variation, so that $\lambda_1 \int_\Omega |\psi| = |D\psi|(\overline{\Omega})$, with $\psi \geq 0$ a.e. in Ω (see [23]). Then, for every $t \in \mathbb{R}$, there holds:*

$$\lim_{s\to+\infty} f_t(-s\psi) = -\infty\,. \tag{4.8}$$

Proof. By the definition of H, (3.3) and (3.4) we have

$$\limsup_{s\to+\infty} \frac{G(x,-s\psi(x)) - t\varphi(x)H(-s\psi(x))}{s} = -\psi(x)\liminf_{\sigma\to-\infty}\frac{G(x,\sigma)}{\sigma} \leq -\alpha\psi(x)\,,$$

$$\frac{G(x,-s\psi(x)) - t\varphi(x)H(-s\psi(x))}{s} \leq b_1(x)\psi(x) + |t|\varphi(x)\psi(x)\,.$$

From Fatou's lemma we infer that

$$\limsup_{s\to+\infty} \frac{\int_\Omega (G(x,-s\psi(x)) - t\varphi(x)H(-s\psi(x)))}{s} \leq -\alpha\int_\Omega \psi(x) = -\frac{\alpha}{\lambda_1}\,|D\psi|(\overline{\Omega})\,.$$

Since $\alpha > \lambda_1$, the assertion follows. $\square$

Conclusion. Let $\bar t > 0$ be as in Proposition 4.2, let $t \geq \bar t$ and let u_0 be a local minimum of f_t. If we set $\overline{u} = u_0$, by Proposition 4.3 we can find $r > 0$ and u_1 as in Theorem 2.15. By Proposition 4.1, the functional f_t satisfies $(CPS)_c$ for every $c \in \mathbb{R}$. Moreover, by [19, Theorems 3.11 and 4.1] also $(epi)_c$ holds for any $c \in \mathbb{R}$. From Theorem 2.15 we get the existence of at least two lower critical points of f_t. By [19, Theorems 2.16, 3.6 and 4.1], they are solutions of (3.7).

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