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EXISTENCE OF SOLUTIONS FOR QUASILINEAR PARABOLIC EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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ABSTRACT. We prove the existence of a generalized solution a quasilinear parabolic equation with nonlocal boundary conditions, using the Faedo-Galerkin approximation.

1. Introduction

In this paper, we are concerned with the existence of a generalized solution of the following quasilinear parabolic equation with nonlocal boundary conditions:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|u|^{p-2}\frac{\partial u}{\partial x_i}) + |u|^{p-2}u = f(x,t), \quad x \in \Omega,\ t \in [0,T] \tag{1.1}$$

$$u(x,t) = \int_\Omega k(x,y)u(y,t)dy, \quad x \in \Gamma \tag{1.2}$$

$$u(x,0) = u_0(x). \tag{1.3}$$

As a physical motivation, problem (1.1)–(1.3) arises from the study of quasi-static thermoelasticity. The main difficulty of this problem is related to the presence of both quasilinear term in (1.1) and nonlocal boundary condition (1.2). Literatures to this type of problem are very limited. We only found [4] in which the authors study a quasilinear parabolic equation with nonlocal boundary conditions different from (1.2).

The quasilinear term in (1.1) makes it difficult to apply classical methods like semi-group method or method of upper and lower solutions. However, we found that Faedo-Galerkin method serves as a convenient tool for this type of problem. We proved the existence of a generalized solution of problem (1.1)–(1.3) by constructing approximate solution using Faedo-Galerkin method and applying weak convergence and compactness arguments.

It is well known that Faedo-Galerkin method is used to prove the existence of solutions for linear parabolic equations in [6]. In [5], Faedo-Galerkin method is coupled with contraction mapping theorems to prove the existence of weak solutions of semilinear wave equations with dynamic boundary conditions. Bouziani et al. use Faedo-Galerkin method to show the existence of a unique weak solution for a linear

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parabolic equation with nonlocal boundary conditions. Lion's book [7, Chapter 1], collects the work of Dubinskii and Raviart, in which they use Faedo-Galerkin method to prove the existence and uniqueness of weak solution for a quasilinear parabolic equation with homogeneous boundary condition.

Problem (1.1)–(1.3) is the extension of the problem in [7, p. 140] in which the boundary conditions are homegeneous.

This article is organized as follows: in section 2, we give the definition of the generalized solution of problem (1.1)–(1.3) and introduce the function spaces related to the generalized solution. In section 3, we demonstrate the construction of an approximation solution by Faedo-Galerkin method and derive a priori estimates for the approximation solution. Section 4 is devoted to the proof of existence of the generalized solution by compactness arguments.

2. Preliminaries

In this article, we use the following notation:
Ω: regular and bounded domain of R^n; Γ: boundary of Ω;
$(\cdot,\cdot)$: usual inner product in $L^2(\Omega)$;
$W^{k,p}(\Omega)$: Sobelev space on Ω; $H^r(\Omega)$: Sobelev space $W^{r,2}(\Omega)$;
$L^p(\Omega)$: L^p space defined on Ω; $|\cdot|_p$: norm in $L^p(\Omega)$; $|\cdot|_{p,\Gamma}$: norm in $L^p(\Gamma)$;
$H^{-r}(\Omega)$: dual space of $H^r(\Omega)$; $|\cdot|_{H^{-r}(\Omega)}$: norm in $H^{-r}(\Omega)$;
c: nonzero constant which may take different values on each occurrence;
C: nonnegative constant which may take different values on each occurrence;
$\hookrightarrow$: continuous embedding;
$K(x)$: norm of $k(x,y)$ in $L^q(\Omega)$ with respect to y,
i.e., $K(x) = (\int_\Omega |k(x,y)|^q dy)^{1/q}$;
$K_i(x)$: norm of $D_i k(x,y)$ in $L^q(\Omega)$ with respect to y,
i.e., $K_i(x) = (\int_\Omega |\frac{\partial k(x,y)}{\partial x_i}|^q dy)^{1/q}$.

In this article, we make the following assumptions:

- (A1) $n \geq 2$, $p > n$, $r > \frac{n}{2} + 2$;
- (A2) $\frac{1}{p} + \frac{1}{q} = 1$;
- (A3) $f \in L^q(0,T; L^q(\Omega))$ and $u_0 \in L^\infty(\Omega)$;
- (A4) For any $x \in \Gamma$, $K(x) < \infty$, $K_i(x) < \infty$;
- (A5) $\sum_{i=1}^n \int_\Gamma K(x)^{p-1} K_i(x) d\Gamma < 1 - \frac{1}{p}$.

Here we give an example of a function $k(x,y)$ which satisfies assumptions (A4) and (A5): When $n = 2$, $p = 3$ and Ω is an unit square, let $k(x,y) = x_1 x_2 (y_1 y_2)^{2/3}$. It is easy to verify that $K(x) = (\int_\Omega |k(x,y)|^q dy)^{1/q}$ and $K_i(x) = (\int_\Omega |\frac{\partial k(x,y)}{\partial x_i}|^q dy)^{1/q}$ satisfy assumptions (A4) and (A5).

With assumption (A1), using Sobelev embedding theorems, see [1], we have

$$H^r(\Omega) \hookrightarrow W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^2(\Omega).$$

Define a space V:

$$V = \{v \in H^r(\Omega):\ v(x) = \int_\Omega k(x,y) v(y) dy,\ for\ x \in \Gamma\} \tag{2.1}$$

It is easy to see that V is a subspace of $H^r(\Omega)$.

Definition 2.1. Define a generalized solution of problem (1.1)–(1.3) as a function u, such that

(i) $u \in L^\infty(0,T;L^2(\Omega)) \cap C([0,T],H^{-r}(\Omega))$;
(ii) $\frac{du}{dt} \in L^q(0,T;H^{-r}(\Omega))$;
(iii) $u(x,0) = u_0(x)$;
(iv) for all $v \in V$ and a.e. $t \in [0,T]$,

$$(\frac{du}{dt}, v) - (\sum_{i=1}^n \frac{\partial}{\partial x_i}(|u|^{p-2}\frac{\partial u}{\partial x_i}), v) + (|u|^{p-2}u, v) = (f, v)\,. \tag{2.2}$$

Remark 2.2. From the proof of existence theorem in section 4, we will see that each inner product in the identity (2.2) is a function of t in $L^q(0,T)$, hence the identity holds for a.e. $t \in [0,T]$. On the other hand, since $u(t) \in V$, the boundary condition (1.2) is satisfied.

3. Construction of an approximate solution and a priori estimates

Since V is a subspace of $H^r(\Omega)$, which is separable. We can choose a countable set of distinct basis elements w_j, $j = 1, 2, \cdots$, which generate V and are orthonormal in $L^2(\Omega)$. Let V_m be the subspace of V generated by the first m elements: $w_1, w_2, \cdots, w_m$. We construct the approximate solution of the form:

$$u_m(x,t) = \sum_{j=1}^m g_{jm}(t) w_j(x), \quad (x,t) \in \Omega \times [0,T]. \tag{3.1}$$

where $(g_{jm}(t))_{j=1}^m$ remains to be determined.

Denote the orthogonal projection of u_0 on V_m as $u_m^0 = P_{V_m} u_0$, then $u_m^0 \to u_0$ in V, as $m \to \infty$. Let $(g_{jm}^0)_{j=1}^m$ be the coordinate of u_m^0 in the basis $(w_j)_{j=1}^m$ of V_m; i.e., $u_m^0 = \sum_{j=1}^m g_{jm}^0 w_j$, let $g_{jm}(0) = g_{jm}^0$.

We need to determine $(g_{jm}(t))_{j=1}^m$ to satisfy

$$({u_m}', w_j) - (\sum_{i=1}^n \frac{\partial}{\partial x_i}(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}), w_j) + (|u_m|^{p-2}u_m, w_j) = (f, w_j), \quad 1 \le j \le m.$$

Integrating by parts on the second term of left-hand side, we have

$$\begin{aligned} &({u_m}', w_j) + \sum_{i=1}^n \int_\Omega (|u_m|^{p-2} D_i u_m)(D_i w_j)\,dx \\ &- \sum_{i=1}^n \int_\Gamma (|u_m|^{p-2} D_i u_m) w_j d\Gamma + (|u_m|^{p-2}u_m, w_j) = (f, w_j), \quad 1 \le j \le m. \end{aligned} \tag{3.2}$$

The above system is a system of ordinary differential equations in $(g_{jm}(t))_{j=1}^m$. By Caratheodory theorem [3], there exists solution $(g_{jm}(t))_{j=1}^m$, $t \in [0, t_m)$.

We need a priori estimates that permit us to extend the solution to the whole domain $[0,T]$.

We derive a priori estimates for the approximate solution as follows: Multiply (3.2) by $g_{jm}(t)$, then sum over j from 1 to m, we have

$$\begin{aligned} &({u_m}', u_m) + \sum_{i=1}^n \int_\Omega (|u_m|^{p-2} D_i u_m)(D_i u_m) dx \\ &- \sum_{i=1}^n \int_\Gamma (|u_m|^{p-2} D_i u_m) u_m d\Gamma + (|u_m|^{p-2}u_m, u_m) = (f, u_m), \quad 1 \le j \le m. \end{aligned}$$

which gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|_2^2+\frac{4}{p^2}\sum_{i=1}^n\int_\Omega (D_i(|u_m|^{\frac{p-2}{2}}u_m))^2dx+|u_m(t)|_p^p\\ &=(f,u_m)+\sum_{i=1}^n\int_\Gamma (|u_m|^{p-2}D_iu_m)u_m d\Gamma. \end{aligned} \tag{3.3}$$

Integrating with respect to t from 0 to T on both sides, we obtain

$$\begin{aligned} &\frac{1}{2}|u_m(T)|_2^2+\int_0^T\frac{4}{p^2}\sum_{i=1}^n\int_\Omega (D_i(|u_m|^{\frac{p-2}{2}}u_m))^2dxdt+\int_0^T|u_m(t)|_p^pdt\\ &=\int_0^T(f,u_m)dt+\int_0^T\sum_{i=1}^n\int_\Gamma (|u_m|^{p-2}D_iu_m)u_m d\Gamma dt+\frac{1}{2}|u_m(0)|_2^2. \end{aligned} \tag{3.4}$$

This gives

$$\begin{aligned} &\frac{1}{2}|u_m(T)|_2^2+\int_0^T\frac{4}{p^2}\sum_{i=1}^n\int_\Omega (D_i(|u_m|^{\frac{p-2}{2}}u_m))^2dxdt+\int_0^T|u_m(t)|_p^pdt\\ &\le\int_0^T|(f,u_m)|dt+\int_0^T\sum_{i=1}^n\int_\Gamma |(|u_m|^{p-2}D_iu_m)u_m| d\Gamma dt+\frac{1}{2}|u_m(0)|_2^2. \end{aligned} \tag{3.5}$$

The first term in the right-hand side of (3.5) can be estimated as follows:

$$\begin{aligned} \int_0^T|(f,u_m)|dt &=\int_0^T\int_\Omega|fu_m|dxdt\\ &\le\int_0^T|f|_q|u_m|_pdt \quad \text{(hölder's inequality)}\\ &\le\int_0^T(\frac{1}{p}|u_m|_p^p+\frac{p-1}{p}|f|_q^{\frac{p}{p-1}})dt. \quad \text{(Young's inequality)} \end{aligned} \tag{3.6}$$

Next, we estimate second term in the right-hand side of (3.5): For $x\in\Gamma$, we have

$$|u_m(x,t)|=\int_\Omega|k(x,y)u_m(y,t)|dy\le|k(x,y)|_q|u_m|_p.$$

Then we have $|u_m(x,t)|\le K(x)|u_m|_p$ for $x\in\Gamma$. Similarly, we have $|D_iu_m(x,t)|\le K_i(x)|u_m|_p$ for $x\in\Gamma$.

Then using hölder's inequality and assumptions (A4) and (A5), we have

$$\begin{aligned} &\int_0^T\Big|\sum_{i=1}^n\int_\Gamma (|u_m|^{p-2}D_iu_m)u_m d\Gamma\Big|dt\\ &\le\int_0^T\sum_{i=1}^n\int_\Gamma K(x)^{p-1}|u_m|_p^{p-1}K_i(x)|u_m|_pd\Gamma dt\\ &\le\int_0^T\Big(\sum_{i=1}^n\int_\Gamma K(x)^{p-1}K_i(x)d\Gamma\Big)|u_m|_p^pdt\\ &=C\int_0^T|u_m|_p^pdt \end{aligned} \tag{3.7}$$

where
$$C=\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p-1}K_i(x)d\Gamma<1-\frac{1}{p}\,.$$
With the above estimates and (3.5), we have
$$\begin{aligned}&\frac{1}{2}|u_m(T)|_2^2+\int_0^T\frac{4}{p^2}\sum_{i=1}^{n}\int_{\Omega}(D_i(|u_m|^{\frac{p-2}{2}}u_m))^2dxdt+\int_0^T(1-\frac{1}{p}-C)|u_m(t)|_p^pdt\\&\le\int_0^T(\frac{p-1}{p}|f|_q^{\frac{p}{p-1}})dt+\frac{1}{2}|u_m(0)|_2^2.\end{aligned}$$
which holds for any finite $T>0$.

Under assumption (A1)-(A5), we have the following a priori estimates:

(B) u_m is bounded in $L^{\infty}(0,T;\ L^2(\Omega))$;
(C) $|u_m|^{\frac{p-2}{2}}|u_m|$ is bounded in $L^2(0,T;H^1(\Omega))$;
(D) u_m is bounded in $L^p(0,T;L^p(\Omega))$.

Since T is an arbitrary positive number, we have
$$|u_m|_p^p<\infty\quad\text{a.e. } t$$

4. Existence of a generalized solution

To prove the existence of a generalized solution, we first prove the following lemma:

Lemma 4.1. *Let u_m, constructed in* (3.1)*, be the approximate solution of* (1.1)–(1.3) *in the sense of Definition 2.1. Then u_m' is bounded in $L^q(0,T;\ H^{-r}(\Omega))$.*

Proof. For $v\in V\subset H^r$, from (3.2), we have
$$\begin{aligned}&(u_m{}',v)+(\sum_{i=1}^{n}\int_{\Omega}(|u_m|^{p-2}D_iu_m)(D_iv)dx\\&-\sum_{i=1}^{n}\int_{\Gamma}(|u_m|^{p-2}D_iu_m)vd\Gamma+(|u_m|^{p-2}u_m,v)=(f,v).\end{aligned}\tag{4.1}$$
The last term in the left-hand side can be estimated as in [7]:
$$\begin{aligned}|(|u_m|^{p-2}u_m,v)|&\le|\ |u_m|^{p-1}|_q|v|_p\\&\le(|u_m|_p^p)^{1/q}|v|_p\\&\le(|u_m|_p^p)^{1/q}C|v|_{H^r},\end{aligned}$$
since $H^r\hookrightarrow L^p$. Hence $|\,|u_m|^{p-2}u_m|_{H^{-r}(\Omega)}\le C(|u_m|_p^p)^{1/q}<\infty$. The norm of $|u_m|^{p-2}u_m$ in $L^q(0,T;H^{-r}(\Omega))$ is bounded by
$$\Big(\int_0^T(C(|u_m|_p^p)^{1/q})^qdt\Big)^{1/q}=\Big(\int_0^TC^q|u_m|_p^pdt\Big)^{1/q}<\infty.$$
Therefore, $|u_m|^{p-2}u_m$ is bounded in $L^q(0,T;H^{-r}(\Omega))$.

Next, we consider the term $\sum_{i=1}^n\int_{\Gamma}(|u_m|^{p-2}D_iu_m)vd\Gamma$ in the left-hand side of (4.1):
$$v\to\sum_{i=1}^{n}\int_{\Gamma}(|u_m|^{p-2}D_iu_m)vd\Gamma=(a(u_m),v).$$

We have

$$
\begin{aligned}
&\sum_{i=1}^n \int_\Gamma (|u_m|^{p-2} D_i u_m) v d\Gamma \\
&\leq \sum_{i=1}^n |\ (|u_m|^{p-2} D_i u_m)|_{q,\Gamma} |v|_{p,\Gamma} \\
&= \sum_{i=1}^n \Big| \Big(|\int_\Omega k(x,y) u_m(y,t) dy|^{p-2} \int_\Omega D_i k(x,y) u_m(y,t) dy \Big) \Big|_{q,\Gamma} \\
&\quad \times \Big| \int_\Omega k(x,y) v(y,t) dy \Big|_{p,\Gamma} \\
&\leq \sum_{i=1}^n |(K(x)^{p-2} K_i(x) |u_m|_p^{p-1})|_{q,\Gamma} |(K(x)|v|_p)|_{p,\Gamma} \\
&\leq \sum_{i=1}^n |K(x)^{p-2} K_i(x)|_{q,\Gamma} |K(x)|_{p,\Gamma} |u_m|_p^{p-1} |v|_p \\
&\leq \sum_{i=1}^n |K(x)^{p-2} K_i(x)|_{q,\Gamma} |K(x)|_{p,\Gamma} |u_m|_p^{p-1} C |v|_{H^r}.
\end{aligned}
$$

Therefore,

$$
|a(u_m)|_{H^{-r}(\Omega)} \leq \sum_{i=1}^n |K(x)^{p-2} K_i(x)|_{q,\Gamma} |K(x)|_{p,\Gamma} |u_m|_p^{p-1} C < \infty.
$$

Then the norm of $a(u_m)$ in $L^q(0,T; H^{-r}(\Omega))$ is bounded by

$$
\Big(\int_0^T \sum_{i=1}^n (|K(x)^{p-2} K_i(x)|_{q,\Gamma} |K(x)|_{p,\Gamma} C)^q |u_m|_p^p dt \Big)^{1/q} < \infty.
$$

Hence, $a(u_m)$ is bounded in $L^q(0,T; H^{-r}(\Omega))$.

Next, we consider the second term in the left-hand side of (4.1). Integrating by parts gives

$$
\begin{aligned}
&\sum_{i=1}^n \int_\Omega (|u_m|^{p-2} D_i u_m)(D_i v) dx \\
&= \frac{1}{c} \Big(\sum_{i=1}^n \int_\Gamma |u_m|^{p-2} u_m D_i v d\Gamma - \int_\Omega |u_m|^{p-2} u_m \Delta v dx \Big).
\end{aligned} \tag{4.2}
$$

Consider $v \to \sum_{i=1}^n \int_\Gamma |u|^{p-2} u D_i v d\Gamma = (I_1(u), v)$, we have:

$$
\begin{aligned}
|(I_1(u), v)| &\leq \sum_{i=1}^n |\ |u|^{p-2} u|_{q,\Gamma} |D_i v|_{p,\Gamma} \\
&= \sum_{i=1}^n \Big| \Big(\int_\Omega k(x,y) u(y,t) dy \Big)^{p-1} \Big|_{q,\Gamma} \Big| \int_\Omega D_i k(x,y) v(y,t) dy \Big|_{p,\Gamma} \\
&\leq \sum_{i=1}^n |(K(x)^{p-1} |u|_p^{p-1})|_{q,\Gamma} |(K_i(x)|v|_p)|_{p,\Gamma}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |K(x)^{p-1}|_{q,\Gamma} |K_i(x)|_{p,\Gamma} |u|_p^{p-1} |v|_p \\
&\leq \sum_{i=1}^{n} |K(x)^{p-1}|_{q,\Gamma} |K_i(x)|_{p,\Gamma} |u|_p^{p-1} C |v|_{H^r}
\end{aligned}
$$

So we have

$$
|I_1(u_m)|_{H^{-r}(\Omega)} \leq \sum_{i=1}^{n} |K(x)^{p-1}|_{q,\Gamma} |K_i(x)|_{p,\Gamma} |u_m|_p^{p-1} C < \infty.
$$

With this, it is easy to see that norm of $I_1(u_m)$ in $L^q(0,T;H^{-r}(\Omega))$ is bounded.

Next, consider $v \to \int_\Omega |u|^{p-2} u \Delta v dx = (I_2(u), v)$. From the proof of [7, Theorem 12.2], we know $I_2(u_m)$ is bounded in $L^q(0,T;H^{-r}(\Omega))$. Since $f \in L^q(0,T;L^q(\Omega)) \subset L^q(0,T;H^{-r}(\Omega))$, from (4.1) and the above discussion, we have u_m' is bounded in $L^q(0,T;H^{-r}(\Omega))$. $\square$

With Lemma 4.1, we can use [7, Theorem 12.1]. We quote the theorem here.

Theorem 4.2. *Let B, B_1 be Banach spaces, and S be a set. Define*

$$
M(v) = (\sum_{i=1}^{n} \int_\Omega |v|^{p-2} (\frac{\partial v}{\partial x_i})^2 dx)^{1/p}
$$

on S with:

- (a) *$S \subset B \subset B_1$, and $M(v) \geq 0$ on S, $M(\lambda v) = |\lambda| M(v)$;*
- (b) *the set $\{v | v \in S, M(v) \leq 1\}$ is relatively compact in B.*

Define the set $F = \{v : v$ is locally summable on $[0,T]$ with value in B_1, $\int_0^T (M(v(t)))^{p_0} dt \leq C$, v' bounded in $L^{p_1}(0,T;B_1)\}$. Where $1 < p_i < \infty$, $i = 0, 1$. Then $F \subset L^{p_0}(0,T;B)$ and F is relatively compact in $L^{p_0}(0,T;B)$.

We need Theorem 4.2 to prove the following lemma:

Lemma 4.3. *Let u_m, constructed as in (3.1), be the approximate solution of (1.1)–(1.3) in the sense of Definition 2.1, then $u_m \to u$ in $L^p(0,T;L^p(\Omega))$ strongly and almost everywhere.*

Proof. Let $S = \{v : |v|^{\frac{p-2}{2}} v \in H^1(\Omega)\}$. Since $H^1(\Omega)$ is also compactly embedded in $L^2(\Omega)$, the proof of [7, Proposition 12.1,p. 143] also works for $|v|^{\frac{p-2}{2}} v \in H^1(\Omega)$, then (b) holds.

Let $B = L^p(\Omega)$, $B_1 = H^{-r}(\Omega)$, $p_0 = p$, $p_1 = q$, we have

$$
\begin{aligned}
\int_0^T (M(u_m))^{p_0} dt &= \int_0^T (\sum_{i=1}^{n} \int_\Omega |u_m|^{p-2} (\frac{\partial u_m}{\partial x_i})^2 dx) dt \\
&= C \int_0^T \sum_{i=1}^{n} \int_\Omega (D_i(|u_m|^{\frac{p-2}{2}} u_m))^2 dx dt < \infty
\end{aligned}
$$

Now with Lemma 4.1 and a priori estimates, conclusion follows easily from application of Theorem 4.2. $\square$

Next, we prove that we can pass the limit in (4.1). Lemmas 4.4–4.7, below, show that we can pass the limit in each term in the left-hand side of (4.1).

Lemma 4.4. *Let u_m, as constructed in* (3.1)*, be the approximate solution of* (1.1)–(1.3) *in the sense of Definition 2.1, then* $(|u_m|^{p-2}u_m, v) \to (|u|^{p-2}u, v)$ *as* $m \to \infty$*.*

Proof. We need to show that $|u_m|^{p-2}u_m \rightharpoonup |u|^{p-2}u$ in $L^q(\Omega)$ weakly, this is a consequence of [7, Lemma 1.3]. □

Lemma 4.5. *Let u_m, constructed as in* (3.1)*, be the approximate solution of* (1.1)–(1.3) *in the sense of Definition 2.1, then* $\int_\Gamma (|u_m|^{p-2}D_i u_m)v d\Gamma \to \int_\Gamma (|u|^{p-2}D_i u)v d\Gamma$ *as* $m \to \infty$*.*

Proof. By a priori estimates, u_m is bounded in $L^p(\Omega)$ for almost every t, then there exists subsequence of u_m, still denoted as u_m, converges to u weak star in $L^p(\Omega)$ (Alaoglu's Theorem) for almost every $t \in [0, T]$.

Under the assumption that for fixed x, $|k(x, y)|_q = (\int |k(x, y)|^q dy)^{1/q} < \infty$; i.e., $k(x, y) \in L^q(\Omega)$ for fixed $x \in \Gamma$, we have

$$\int_\Omega k(x, y)u_m(y, t)dy \to \int_\Omega k(x, y)u(y, t)dy \quad \text{as } m \to \infty.$$

Similarly,

$$\int_\Omega D_i k(x, y)u_m(y, t)dy \to \int_\Omega D_i k(x, y)u(y, t)dy \quad \text{as } m \to \infty.$$

Therefore, for $x \in \Gamma$, we have $|u_m(x, t)|^{p-2}D_i u_m(x, t) \to |u(x, t)|^{p-2}D_i u(x, t)$ a.e.

Next, we prove that $|(|u_m(x, t)|^{p-2}D_i u_m(x, t))|_{q,\Gamma} < \infty$. For $x \in \Gamma$, we have

$$\begin{gathered} u_m(x, t) = \int_\Omega k(x, y)u_m(y, t)dy, \\ |u_m(x, t)| < |k(x, y)|_q |u_m|_p \le K(x)C \end{gathered}$$

Since $K(x) \in L^p(\Gamma)$, we have $|u_m|_{p,\Gamma} < \infty$. Similarly, we have $|D_i u_m|_{p,\Gamma} < \infty$ and $|v|_{p,\Gamma} < \infty$. Then

$$\begin{aligned} &\big|\, |u_m|^{p-2}D_i u_m\big|_{q,\Gamma} \\ &\le \big|\, |u_m|^{p-2}\big|_{\frac{p}{p-2},\Gamma} |D_i u_m|_{p,\Gamma} \quad (\text{since } \frac{1}{q} = \frac{p-2}{p} + \frac{1}{p},\ [1, \text{p25}]) \\ &= (|u_m|_{p,\Gamma})^{p-2} |D_i u_m|_{p,\Gamma} < \infty \end{aligned}$$

By Lemma [7, Lemma 1.3], we have: $|u_m|^{p-2}D_i u_m \rightharpoonup |u|^{p-2}D_i u$ weakly in $L^q(\Gamma)$ for a.e. $t \in [0, T]$. Since $|v|_{p,\Gamma} < \infty$, the proof is complete. □

Lemma 4.6. *Let u_m, as constructed in* (3.1)*, be the approximate solution of* (1.1)–(1.3) *in the sense of Definition 2.1, then*

$$\int_\Omega (|u_m|^{p-2}D_i u_m)(D_i v)dx \to \int_\Omega (|u|^{p-2}D_i u)(D_i v)dx.$$

Proof. From (4.2) we know, we need to prove:

(i) $\int_\Gamma |u_m|^{p-2}u_m D_i v d\Gamma \to \int_\Gamma |u|^{p-2}u D_i v d\Gamma$; and
(ii) $\int_\Omega |u_m|^{p-2}u_m \Delta v dx \to \int_\Omega |u|^{p-2}u \Delta v dx$.

(i) From the proof of Lemma 4.5, we have, for $x \in \Gamma$, $|u_m(x, t)|^{p-2}u_m(x, t) \to |u(x, t)|^{p-2}u(x, t)$ almost everywhere, and

$$|\, |u_m|^{p-2}u_m|_{q,\Gamma} = |u_m|_{p,\Gamma}^{p-1} < \infty.$$

Therefore, we can apply [7, Lemma 1.3] to conclude that $|u_m(x,t)|^{p-2}u_m(x,t) \rightharpoonup |u(x,t)|^{p-2}u(x,t)$ weakly in $L^q(\Gamma)$. Since $D_i v \in L^p(\Gamma)$, (i) is proved.

(ii) From Lemma 4.3, we have $|u_m(x,t)|^{p-2}u_m(x,t) \to |u(x,t)|^{p-2}u(x,t)$ almost everywhere, for $x \in \Omega$. Since $|\,|u_m|^{p-2}u_m|_q = |u_m|_p^{p-1} < \infty$, by [7, Lemma 1.3], we have: $|u_m|^{p-2}u_m \rightharpoonup |u|^{p-2}u$ weakly in $L^q(\Omega)$. Since $\Delta v \in L^p(\Omega)$, we complete the proof of (ii). $\square$

Lemma 4.7. *Let u_m, as constructed in* (3.1)*, be the approximate solution of* (1.1)–(1.3) *in the sense of Definition 2.1, then $(u'_m, v) \to (u', v)$ and $u(t)$ is continuous on $[0,T]$.*

Proof. Since u'_m is bounded in $L^q(0,T;H^{-r}(\Omega))$, by Alaoglu's theorem, there exists a subsequence, still denoted by u'_m, converging to χ weak star in $L^q(0,T;H^{-r}(\Omega))$. By slightly modifying the proof of [2, Theorem 1] (with the space $L^q(0,T;H^{-r}(\Omega))$, instead of $L^2(0,T;B_2^1(0,1))$.), we have $\chi = u'$ and $u(t)$ is continuous on $[0,T]$. $\square$

Based on the above discussion, we summarize the existence theorem as follows.

Theorem 4.8. *Under assumptions (A1)-(A5), there exists a generalized solution u of problem* (1.1)–(1.3)*, such that*

(1) $u \in L^\infty(0,T;L^2(\Omega)) \cap C([0,T],H^{-r}(\Omega))$;
(2) $|u|^{\frac{p-2}{2}}u$ *is bounded in* $L^2(0,T;H^1(\Omega))$.
(3) $\frac{du}{dt} \in L^q(0,T;H^{-r}(\Omega))$.
(4) $u(x,0) = u_0(x)$.
(5) *for all $v \in V$ and a.e. $t \in [0,T]$,*
$$(\frac{du}{dt}, v) - (\sum_{i=1}^n \frac{\partial}{\partial x_i}(|u|^{p-2}\frac{\partial u}{\partial x_i}), v) + (|u|^{p-2}u, v) = (f, v)\,.$$

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