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FRACTIONAL DIFFERENTIAL EQUATION WITH THE FUZZY INITIAL CONDITION

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Abstract. In this paper we study the existence and uniqueness of the solution for a class of fractional differential equation with fuzzy initial value. The fractional derivatives are considered in the Riemann-Liouville sense.

1. Introduction and preliminaries

Fractional calculus is a generalization of differentiation and integration to an arbitrary order. First works, devoted exclusively to the subject of fractional calculus, are the books [35, 41]. Many recently developed models in areas like rheology, viscoelasticity, electrochemistry, diffusion processes, etc. are formulated in terms of fractional derivatives or fractional integrals. The books [20, 22, 30] and [36] presents the theory of the fractional differential equations and their applications. Some theoretical aspects on the existence and uniqueness results for fractional differential equations have been considered recently by many authors [2, 4, 6, 7, 8, 10, 15, 23, 25, 26, 27, 33, 40]. A differential and integral calculus for fuzzy-valued mappings was developed in papers of Hukuhara [16], Dubois and Prade [11, 12, 13] and Puri and Ralescu [38, 39]. For significant results from the theory of fuzzy differential equations and their applications we refer to the books [9, 28] and the papers [3, 5, 14, 17, 19, 21, 28, 31, 37, 45]. The concept of fuzzy fractional differential equation was introduced by Agarwal, Lakshmikantham and Nieto [1] and [44].

The aim of this paper is to study the existence and uniqueness solution of fuzzy fractional differential equation with fuzzy initial value.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded α-level intervals. This means that if $u \in E$ then the α-level set, $[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\}$, $0 < \alpha \leq 1$, is a closed bounded interval denoted by $[u]^\alpha = [u_1^\alpha,\ u_2^\alpha]$ and there exist a $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$. Two fuzzy numbers u and v are called equal, $u = v$, if $u(x) = v(x)$ for all $x \in \mathbb{R}$. It follows that $u = v$ if and only if $[u]^\alpha = [v]^\alpha$ for all $\alpha \in (0, 1]$. The following arithmetic operations on

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fuzzy numbers are well known and frequently used below. If $u, v \in E$ then

$$[u+v]^\alpha = [u_1^\alpha + v_1^\alpha, u_2^\alpha + v_2^\alpha],$$
$$[u-v]^\alpha = [u_1^\alpha - v_2^\alpha, u_2^\alpha - v_1^\alpha],$$
$$[\lambda u]^\alpha = \lambda [u]^\alpha = \begin{cases} [\lambda u_1^\alpha, \lambda u_2^\alpha] & \text{if } \lambda \geq 0 \\ [\lambda u_2^\alpha, \lambda u_1^\alpha] & \text{if } \lambda < 0, \end{cases} \quad \lambda \in \mathbb{R},$$

Lemma 1.1 ([32])**.** *If $u \in E$ then the following properties hold:*

(i) $[u]^\beta \subset [u]^\alpha$ *if* $0 < \alpha \leq \beta \leq 1$*;*
(ii) *If* $\{\alpha_n\} \subset (0,1]$ *is a nondecreasing sequence which converges to α then* $[u]^\alpha = \bigcap_{n\geq 1} [u]^{\alpha_n}$ *(i.e., u_1^α and u_2^α are left-continuous with respect to α.*

Conversely, if $A_\alpha = \{[u_1^\alpha, u_2^\alpha]; \alpha \in (0,1]\}$ is a family of closed real intervals verifying (i) and (ii), then $\{A_\alpha\}$ defined a fuzzy number $u \in E$ such that $[u]^\alpha = A_\alpha$.

For a real inteval $I = [0, a]$, a mapping $u : I \to E$ is called a fuzzy function. We denote $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$, for $t \in I$ and $0 < \alpha \leq 1$. the derivative $u'(t)$ of a fuzzy function u is defined by (see [37])

$$[u'(t)]^\alpha = [(u_1^\alpha)'(t), (u_2^\alpha)'(t)], \quad \alpha \in (0,1], \tag{1.1}$$

provided that is equation defines a fuzzy number $u'(t) \in E$. The fuzzy integral $\int_a^b u(t)dt$, $a, b \in T$, is defined by (see [11])

$$\Big[\int_a^b u(t)dt\Big]^\alpha = \Big[\int_a^b u_1^\alpha(t)dt, \int_a^b u_2^\alpha(t)dt\Big] \tag{1.2}$$

provided that the Lebesgue integrals on the right exist. Suppose that $u_1^\alpha, u_2^\alpha \in C((0,a], \mathbb{R}) \cap L^1((0,a), \mathbb{R})$ for all $\alpha \in [0,1]$. Then for $q > 0$, we put

$$A_\alpha =: \frac{1}{\Gamma(q)}\Big[\int_0^t (t-s)^{q-1} u_1^\alpha(s)ds, \int_0^t (t-s)^{q-1} u_2^\alpha(s)ds\Big]. \tag{1.3}$$

Lemma 1.2. *The family $\{A_\alpha; \alpha \in [0,1]\}$, given by (1.3), defined a fuzzy number $x \in E$ such that $[u]^\alpha = A_\alpha$.*

Proof. Since $u \in E$ then, for $\alpha \leq \beta$, we have that $u_1^\alpha(s) \leq u_1^\beta(s)$ and $u_2^\alpha(t) \geq u_2^\beta(t)$. It follows that $A_\alpha \supseteq A_\beta$. Since $u_1^0(t) \leq u_1^{\alpha_n}(t) \leq u_1^1(t)$, we have

$$|(t-s)^{q-1} u_i^{\alpha_n}(s)| \leq \max\{a^{q-1}|u_i^0(s)|, a^{q-1}|u_i^1(s)|\} =: g_i(s)$$

for $\alpha_n \in (0,1]$ and $i = 1, 2$. Obviously, g_i is Lebesgue integrable on $[0, a]$. Therefore, if $\alpha_n \uparrow \alpha$ then by the Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_0^t (t-s)^{q-1} u_i^{\alpha_n}(s)ds = \int_0^t (t-s)^{q-1} u_i^\alpha(s)ds, \ i = 1, 2.$$

From Lemma 1.1, the proof is complete. □

Let $u \in C((0,a], E) \cap L^1((0,a), E)$. Define the *fuzzy fractional primitive of order* $q > 0$ of u,

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)ds,$$

by

$$[I^q u(t)]_\alpha = \left[\int_0^t (t-s)^{q-1} u_1^\alpha(t)dt, \int_0^t (t-s)^{q-1} u_2^\alpha(t)dt\right].$$

For $q = 1$ we obtain $I^1 u(t) = \int_0^t u(s)ds$; that is, the integral operator.

Let $u \in C((0,a],E) \cap L^1((0,a),E)$ be a given function such that $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$ for all $t \in (0,a]$ and $\alpha \in (0,1]$. We define the fuzzy fractional derivative of order $0 < q < 1$ of u,

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} u(s)ds,$$

by

$$[D^q u(t)]^\alpha =: \frac{1}{\Gamma(1-q)} \Big[\frac{d}{dt} \int_0^t (t-s)^{-q} u_1^\alpha(s)ds, \frac{d}{dt} \int_0^t (t-s)^{-q} u_2^\alpha(s)ds\Big],$$

provided that equation defines a fuzzy number $D^q u(t) \in E$. In fact,

$$[D^q u(t)]^\alpha := [D^q u_1^\alpha(t), D^q u_2^\alpha(t)].$$

Obviously, $D^q u(t) = \frac{d}{dt} I^{1-q} u(t)$ for $t \in (0,a]$.

2. Main result

Let $0 < q < 1$. We shall consider the initial value problem

$$D^q u(t) = f(t,u(t)), \quad \lim_{t\to 0^+} t^{1-q} u(t) = v_0 \tag{2.1}$$

where f is a continuous mapping from $[0,a] \times \mathbb{R}$ into $\mathbb{R}$ and v_0 is a fuzzy number with α-level intervals $[v_0]^\alpha = [v_{01}^\alpha, v_{02}^\alpha]$, $0 < \alpha \le 1$. The extension principle of Zadeh leads to the following definition of $f(t,u)$ when u is a fuzzy number

$$f(t,u)(y) = \sup\{u(x) : y = f(t,x)\}, \quad x \in \mathbb{R}.$$

It follows that

$$[f(t,u)]^\alpha = [\min\{f(t,x) : x \in [u_1^\alpha, u_2^\alpha]\}, \quad \max\{f(t,x) : x \in [u_1^\alpha, u_2^\alpha]\}]$$

for $u \in E$ with α-level sets $[u]^\alpha = [u_1^\alpha, u_2^\alpha]$, $0 < \alpha \le 1$. We call $u : (0,a] \to E$ a fuzzy solution of (2.1), if

$$\begin{aligned} D^q u_1^\alpha(t) &= \min\{f(t,x) : x \in [u_1^\alpha(t), u_2^\alpha(t)]\}, \quad \lim_{t\to 0^+} t^{1-q} u_1^\alpha(t) = v_{01}^\alpha \\ D^q u_2^\alpha(t) &= \max\{f(t,x) : x \in [u_1^\alpha(t), u_2^\alpha(t)]\}, \quad \lim_{t\to 0^+} t^{1-q} u_2^\alpha(t) = v_{02}^\alpha \end{aligned} \tag{2.2}$$

for $t \in (0,a]$ and $0 < \alpha \le 1$. Denote $\widetilde{f} = (f_1, f_2)$, $f_1(t,u) = \min\{f(t,x) : x \in [u_1, u_2]\}$ and $f_2(t,u) = \max\{f(t,x) : x \in [u_1,u_2]\}$ where $u = (u_1,u_2) \in \mathbb{R}^2$. Thus for fixed α, we have an initial value problem in $\mathbb{R}^2$:

$$\begin{aligned} D^q u_1^\alpha(t) &= \widetilde{f}(t, u_1^\alpha(t), u_2^\alpha(t)), \quad \lim_{t\to 0^+} t^{1-q} u_1^\alpha(t) = v_{01}^\alpha \\ D^q u_2^\alpha(t) &= \widetilde{f}(t, u_1^\alpha(t), u_2^\alpha(t)), \quad \lim_{t\to 0^+} t^{1-q} u_2^\alpha(t) = v_{02}^\alpha \end{aligned} \tag{2.3}$$

If we can solve it (uniquely), we have only to verify that the intervals $[u_1^\alpha(t), u_2^\alpha(t)]$, $0 < \alpha \le 1$, define a fuzzy number $u(t)$ in E. Since f is assumed continuous, the initial value problem (2.3) is equivalent to the following fractional integral equation

$$u(t) = v_0(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \widetilde{f}(s,u(s))ds, \quad 0 \le t \le a, \tag{2.4}$$

where $v_0(t) = t^{q-1} v_0/\Gamma(q)$.

Theorem 2.1. *Assume that*

(a) *$f \in C([0,a] \times \mathbb{R}, \mathbb{R})$ and $|f(t,u)| \le M_0$ on $[0,a] \times [0,b]$;*
(b) *$g \in C([0,a] \times [0,b], \mathbb{R}_+)$, $g(t,r) \le M_1$ on $[0,a] \times [0,b]$, $g(t,0) \equiv 0$, $g(t,r)$ is nondecreasing in r for each $t$and $r(t) \equiv 0$ is the only solution of*

$$D^q r(t) = g(t, r(t)), \quad t \in (0,a] \tag{2.5}$$

with the initial condition $\lim_{t\to 0^+} t^{1-q} r(t) = 0$;
(c)

$$|f(t,u) - f(t,\overline{u})| \le g(t, |u - \overline{u}|), \quad t \ge 0,\ u, \overline{u} \in \mathbb{R}, \tag{2.6}$$

(d) *solutions $r(t, r_0)$ of (2.5) are continuous with respect to the initial condition $r_0 = \lim_{t\to 0^+} t^{1-q} r(t)$.*

Then the initial value problem (2.1) has a unique fuzzy solution.

Proof. It can be shown that (2.6) implies

$$\|\widetilde{f}(t,u) - \widetilde{f}(t,\overline{u})\| \le g(t, \|u - \overline{u}\|), \quad t \ge 0,\ u, \overline{u} \in \mathbb{R}^2 \tag{2.7}$$

where the $\|\cdot\|$ is defined by $\|u\| = \max\{|u_1|, |u_2|\}$. It is well known that (2.7) and the assumptions on g [23, Theorems 2.1 and 2.2] guarantee the existence, uniqueness and continuous dependence on initial value of a solution to

$$D^q u(t) = \widetilde{f}(t, u(t)), \quad \lim_{t\to 0^+} t^{1-q} u(t) = v_0 \in \mathbb{R}^2 \tag{2.8}$$

and that for any continuous function $u_0 : R_+ \to R^2$ the successive approximations

$$u_{n+1}(t) = v_0(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \widetilde{f}(s, u_n(s)) ds, \quad n = 0, 1, \dots, \tag{2.9}$$

converge uniformly on closed subintervals of $\mathbb{R}_+$ to the solution of (2.8) [23, Theorem 2.1]. By choosing $v_0 = (v_{01}^\alpha, v_{02}^\alpha)$ in (2.8) we get a unique solution $u^\alpha(t) = (u_1^\alpha(t), u_2^\alpha(t))$ to (2.2) for each $\alpha \in (0,1]$. Next we will show that the intervals $[u_1^\alpha(t), u_2^\alpha(t)], 0 < \alpha \le 1$, define a fuzzy number $u(t) \in E$ for each $t \ge 0$; i.e., that u is a fuzzy solution to (2.1). The successive approximations $u_0(t) = v_0 \in E$,

$$u_{n+1}(t) = v_0(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \widetilde{f}(s, u_n(s)) ds, \quad t \ge 0,\ n = 0, 1, \dots,$$

where the integral is the fuzzy integral, define a sequence of fuzzy numbers $u_n(t) \in E$ for each $t \ge 0$. Hence

$$[u_n(t)]^\alpha \supset [u_n(t)]^\beta \quad \text{if } 0 < \alpha \le \beta \le 1,$$

which implies that

$$[u_1^\alpha(t), u_2^\alpha(t)] \supset [u_1^\beta(t), u_2^\beta(t)], \quad 0 < \alpha \le \beta \le 1,$$

since by the convergence of sequence (2.9), the end points of $[u_n(t)]_\alpha$ converge to $u_1^\alpha(t)$ and $u_2^\alpha(t)$ respectively. Thus the inclusion property (i) of Lemma 1.1 holds for the intervals $[u_1^\alpha(t), u_2^\alpha(t)], 0 < \alpha \le 1$. For the proof of continuity property (ii) of Lemma 1.1, let (α_k) be a nondecreasing sequence in $(0,1]$ converging to α. Then $v_{01}^{\alpha_k} \to v_{01}^\alpha$ and $v_{02}^{\alpha_k} \to v_{02}^\alpha$ because $v_0 \in E$. But then by the continuous dependence on the initial value of the solution of (2.8), $u_1^{\alpha_k}(t) \to u_1^\alpha(t)$ and $u_2^{\alpha_k}(t) \to u_2^\alpha(t)$, i.e. (ii) holds for the intervals $[u_1^\alpha(t), u_2^\alpha(t)]$, $0 < \alpha \le 1$. Hence, by Lemma 1.1, $u(t) \in E$ and so u is a fuzzy solution of (2.1). The uniqueness follows from the uniqueness of the solution of (2.8). □

3. Example

Consider the crisp differential equation

$$D^q u(t) = -u(t) \tag{3.1}$$

with the fuzzy initial condition

$$\lim_{t\to 0^+} t^{1-q}u(t) = (1|2|3), \tag{3.2}$$

where $t \in (0, a]$, $0 < q \le 1$, and $v_0 = (1|2|3) \in E$ is a fuzzy triangular number, that is, $[v_0]^\alpha = [1+\alpha, 3-\alpha]$ for $\alpha \in (0,1]$. If we put $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$, then $[D^q u(t)]^\alpha = [D^q u_1^\alpha(t), D^q u_2^\alpha(t)]$. We obtain the system

$$\begin{gathered}
D^q u_1^\alpha(t) = -u_2^\alpha(t), \quad \lim_{t\to 0^+} t^{1-q}u_1^\alpha(t) = 1+\alpha \\
D^q u_2^\alpha(t) = -u_1^\alpha(t), \quad \lim_{t\to 0^+} t^{1-q}u_2^\alpha(t) = 3-\alpha,
\end{gathered}$$

or

$$D^q y(t) = Ay(t), \quad \lim_{t\to 0^+} t^{1-q}y(t) = c, \tag{3.3}$$

where

$$y(t) = \begin{bmatrix} u_1^\alpha(t) \\ u_2^\alpha(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1+\alpha \\ 3-\alpha \end{bmatrix}.$$

Using the same method that in [18], we obtain the solution of (3.3). It is given by

$$y(t) = t^{q-1}E_{q,q}(At^q)c = t^{q-1}E_{q,q}(At^q)\begin{bmatrix} 1+\alpha \\ 3-\alpha \end{bmatrix},$$

where

$$\begin{aligned}
E_{q,q}(At^q) &= \sum_{k=0}^{\infty} \frac{(At^q)^k}{\Gamma(q(k+1))} \\
&= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{2qn}}{\Gamma(q(2n+1))} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{t^{2qn}}{\Gamma(q(2n+1))} \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & -\sum_{n=0}^{\infty} \frac{t^{(2n+1)q}}{\Gamma(q(2n+2))} \\ -\sum_{n=0}^{\infty} \frac{t^{(2n+1)q}}{\Gamma(q(2n+1))} & 0 \end{bmatrix}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
u_1^\alpha(t) &= \sum_{n=0}^{\infty} \frac{t^{(2n+1)q-1}}{\Gamma(q(2n+1))}(1+\alpha) - \sum_{n=0}^{\infty} \frac{t^{(2n+2)q-1}}{\Gamma(q(2n+2))}(3-\alpha), \\
u_2^\alpha(t) &= \sum_{n=0}^{\infty} \frac{t^{(2n+1)q-1}}{\Gamma(q(2n+1))}(3-\alpha) - \sum_{n=0}^{\infty} \frac{t^{(2n+2)q-1}}{\Gamma(q(2n+2))}(1+\alpha).
\end{aligned}$$

It easy to see that $[u_1^\alpha(t), u_2^\alpha(t)]$ define the α-level intervals of a fuzzy number. So $[u(t)]^\alpha$ are the α-level intervals of the fuzzy solution of (3.1)-(3.2).

3.1. **Conclusion.** Using the Hukuhara derivative, we given a result for the existence and uniqueness of the solution for a class of fractional differential equations with fuzzy initial value. This approach based on Hukuhara derivative has the disadvantage that any solution of a fuzzy differential equation has increasing length of its support. Consequently, this approach cannot really reflect any of the rich behavior of ordinary differential equations [9]. Moreover, there exist simple fuzzy functions (e.g., $F(t) = cg(t)$, where c is a fuzzy number and $g : [a, b] \to [0, \infty)$ is a function with $g'(t) < 0$) which are not Hukuhara differentiable. Bede and Gal [5] (see also [14, 21, 42, 43]), solved the above mentioned shortcoming under strongly generalized differentiability of fuzzy-number-valued functions. In this case the derivative exists and the solution of a fuzzy differential equation may have decreasing length of the support, but the uniqueness is lost. Another approach consists in interpreting a fuzzy differential equations as a family of differential inclusions [17], [31]. The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-number-valued function. In future, the study of fuzzy fractional equations, using different approaches mentioned above can help to develop this theory.

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