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EXISTENCE OF PERIODIC SOLUTIONS FOR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. Using the coincidence degree theory by Mawhin, we prove the existence of periodic solutions for the second-order delay differential equations with impulses

$$\begin{gathered} x''(t)+f(t,x'(t))+g(x(t-\tau(t))=p(t), \quad t\geq 0,\ t\neq t_k,\\ \Delta x(t_k)=I_k(x(t_k),x'(t_k)),\\ \Delta x'(t_k)=J_k(x(t_k),x'(t_k)). \end{gathered}$$

We obtain new existence results and illustrated them by an example.

1. Introduction

This article concerns the existence of periodic solutions for the second-order delay differential equations with impulses

$$\begin{gathered} x''(t)+f(t,x'(t))+g(x(t-\tau(t))=p(t), t\geq 0, t\neq t_k,\\ \Delta x(t_k)=I_k(x(t_k),x'(t_k)),\\ \Delta x'(t_k)=J_k(x(t_k),x'(t_k)) \end{gathered} \tag{1.1}$$

where $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $x(t_k^+)=\lim_{t\to t_k^+}x(t)$, $x(t_k^-)=\lim_{t\to t_k^-}x(t)$ and $x(t_k^-)=x(t_k)$; also $\Delta x'(t_k)=x'(t_k^+)-x'(t_k^-)$, $x'(t_k^+)=\lim_{t\to t_k^+}x'(t)$, $x'(t_k^-)=\lim_{t\to t_k^-}x'(t)$ and $x'(t_k^-)=x'(t_k)$.

We assume that the following conditions:

- (H1) $f\in C(\mathbb{R}^2,\mathbb{R})$ and $f(t+T,x)=f(t,x)$, $g\in C(\mathbb{R},\mathbb{R})$, $p,\tau\in C(\mathbb{R},\mathbb{R})$ with $\tau(t+T)=\tau(t)$, $p(t+T)=p(t)$;
- (H2) $\{t_k\}$ satisfies $t_k<t_{k+1}$ and $\lim_{k\to\pm\infty}t_k=\pm\infty$, $k\in Z$, $I_k(x,y),J_k(x,y)\in C(\mathbb{R}^2,\mathbb{R})$, and there is a positive n such that $\{t_k\}\cap[0,T]=\{t_1,t_2,\dots,t_n\}$, $t_{k+n}=t_k+T, I_{k+n}(x,y)=I_k(x,y), J_{k+n}(x,y)=J_k(x,y)$.

Impulsive differential equations are mathematical apparatus for simulations of process and phenomena observed in control theory,physics,chemistry, population dynamics, biotechnologies, industrial robotics, economics, etc. So there have been

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quite a few results on properities of their solutions in recent years [1, 2, 5, 7, 15]. In paiticular, the existence of periodic solutions for first order differential equations with impulses has been studied in [14, 17]. Li and Shen [15] have studied the existence of periodic solutions for duffing equations with delays and impulses. In present paper, by using Mawhin's continuation theorem, we will establish some theorems on the existence of periodic solutions of (1.1). The results is related to not only $f(t,x)$ and $g(y)$ but also the impulses $I_k(x,y)$ and $J_k(x,y)$ and the delay $\tau(t)$. In addition, we give an example to illustrate our new results.

For background material on periodic solutions of first or second order differential equations without impulses, the references [3, 6, 9, 10, 11, 12, 13, 16] may be consulted.

2. Preliminaries

We establish the theorems of existence of periodic solution based on the following Mawhin's continuation theorem.

Let $PC(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ be continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)\}$, $PC^1(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k)\}$. Let $X = \{x(t) \in PC^1(\mathbb{R},\mathbb{R}), x(t+T) = x(t)\}$ with norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$, where $|x|_\infty = \sup_{t\in[0,T]} |x(t)|$, $Y = PC(\mathbb{R},\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^n$, with norm $\|y\| = \max\{|u|_\infty, |c|\}$, where $u \in PC(\mathbb{R},\mathbb{R}), c = (c_1, \dots c_{2n}) \in R^n \times \mathbb{R}^n$, $|c| = \max_{1\le k\le 2n}\{|c_k|\}$. Then X and Y are Banach spaces. $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero, where $D(L)$ denotes the domain of L. $P : X \to X, Q : Y \to Y$ are projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{D(L)\cap \ker P} : D(L) \cap \ker P \to \operatorname{Im} L$$

is invertible and we define the inverse of that map by K_p. Let Ω be an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1 ([5]). *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

- (i) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;
- (ii) $QNx \neq 0$, *for all* $x \in \partial\Omega \cap \ker L$;
- (iii) $\deg\{JQNx, \Omega \bigcap \ker L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \ker L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap D(L)$.

We define the operators $L : D(L) \subset X \to Y$ by

$$Lx = (x'', \Delta x(t_1), \dots, \Delta x(t_n), \Delta x'(t_1), \dots, \Delta x'(t_n)), \tag{2.1}$$

and $N : X \to Y$ by

$$\begin{aligned} Nx = (&-f(t,x'(t)) - g(x(t-\tau(t))) + p(t), \\ &I_1(x(t_1)), \dots, I_n(x(t_n)), J_1(x'(t_1)), \dots, J_n(x'(t_n))). \end{aligned} \tag{2.2}$$

It is easy to see that (1.1) can be converted into the abstract equation $Lx = Nx$.

Lemma 2.2 ([8])**.** *L is a Fredholm operator of index zero with*

$$\ker L = \{x(t) = c, t \in R\}, \tag{2.3}$$

and

$$\begin{aligned} \operatorname{Im} L = \{&(y, a_1, \dots, a_n, b_1, \dots, b_n) \in Y : \\ &\int_0^T y(s)ds + \sum_{k=1}^n b_k(T - t_k) + \sum_{k=1}^n a_k + x'(0)T = 0\}. \end{aligned} \tag{2.4}$$

Furthermore, let the linear continuous projector operator $P : X \to X$ and $Q : Y \to Y$ be defined by

$$Px = x(0), \tag{2.5}$$

and

$$\begin{aligned} &Q(y, a_1, \dots, a_n, b_1, \dots, b_n) \\ &= \frac{2}{T^2}[\int_0^T (T - s)y(s)ds + \sum_{k=1}^n b_k(T - t_k) + \sum_{k=1}^n a_k + x'(0)T], 0, \dots, 0). \end{aligned} \tag{2.6}$$

Then the linear operator $K_p : \operatorname{Im} L \to D(L) \cap \ker P$ can be written as

$$\begin{aligned} &K_p(y, a_1, \dots, a_n, b_1, \dots, b_n) \\ &= \int_0^T (T - s)y(s)ds + \sum_{k=1}^n b_k(T - t_k) + \sum_{k=1}^n a_k + x'(0)T. \end{aligned} \tag{2.7}$$

Lemma 2.3. *Suppose $\Omega \subset X$ is bounded open set,then N is L-compact in $\overline{\Omega}$.*

Proof. It is easy to see that $QN(\overline{\Omega})$ is bound. By using the Ascoli-Arzela theorem, we can prove that $K_p(I - Q)Nx$ is compact. Thus N is L-compact in $\overline{\Omega}$. □

Lemma 2.4 ([10])**.** *Suppose $\alpha > 0$, $x(t) \in PC^1(\mathbb{R}, \mathbb{R})$ with $x(t + T) = x(t)$, Then*

$$\int_0^T \int_{t-\alpha}^t |x'(s)|^2 \, ds \, dt = \alpha \int_0^T |x'(t)|^2 dt \tag{2.8}$$

and

$$\int_0^T \int_t^{t+\alpha} |x'(s)|^2 \, ds \, dt = \alpha \int_0^T |x'(t)|^2 dt. \tag{2.9}$$

Let

$$\begin{gathered} A_1(t, \alpha) = \sum_{t-\alpha \le t_k \le t} a_k, \quad A_2(t, \alpha) = \sum_{t \le t_k \le t+\alpha} a_k, \\ B_1(t, \alpha) = \sum_{t-\alpha \le t_k \le t} a'_k, \quad B_2(t, \alpha) = \sum_{t \le t_k \le t+\alpha} a'_k, \\ A(\alpha) = \Big(\int_0^T A_1^2(t, \alpha)dt\Big)^{1/2} + \Big(\int_0^T A_2^2(t, \alpha)dt\Big)^{1/2}, \\ B(\alpha) = \Big(\int_0^T B_1^2(t, \alpha)dt\Big)^{1/2} + \Big(\int_0^T B_2^2(t, \alpha)dt\Big)^{1/2}, \\ C(\alpha) = \int_0^T A_1^2(t, \alpha)dt + \int_0^T A_2^2(t, \alpha)dt, \\ D(\alpha) = \int_0^T A_1(t, \alpha)B_1(t)dt + \int_0^T A_2(t, \alpha)B_2(t)dt, \end{gathered}$$

$$E(\alpha)=\int_0^T B_1^2(t,\alpha)dt+\int_0^T B_2^2(t,\alpha)dt$$

The following Lemma is crucial for us to establish theorems related to the delay $\tau(t)$ and $I_k(x,y)$.

Lemma 2.5. *Suppose $\tau(t)\in C(\mathbb{R},\mathbb{R})$ with $\tau(t+T)=\tau(t)$ and $\tau(t)\in[-\alpha,\alpha]$ for all $t\in[0,T]$, $x(t)\in PC^1(\mathbb{R},\mathbb{R})$ with $x(t+T)=x(t)$ and there is a positive n such that $\{t_k\}\cap[0,T]=\{t_1,t_2,\dots,t_n\}$, $\Delta x(t_k)=\lambda I_k(x(t_k),x'(t_k))$ for all $\lambda\in(0,1)$ and $t_{k+n}=t_k+T, I_{k+n}(x,y)=I_k(x,y)$. Furthermore there exist nonnegative constants a_k,a_k such that $|I_k(x,y)|\le a_k|x|+a_k'$. Then*

$$\begin{aligned}
&\int_0^T |x(t)-x(t-\tau(t))|^2dt\\
&\le 2\alpha^2\int_0^T |x'(t)|^2dt+2\alpha A(\alpha)|x(t)|_\infty\Big(\int_0^T |x'(t)|^2dt\Big)^{1/2}\\
&\quad +2\alpha B(\alpha)\Big(\int_0^T |x'(t)|^2dt\Big)^{1/2}+C(\alpha)|x(t)|_\infty^2+D(\alpha)|x(t)|_\infty+E(\alpha).
\end{aligned}\tag{2.10}$$

Proof. If $\tau(t)\in[0,\alpha]$, then for all $t\in[0,T]$, using Schwarz inequality, we obtain

$$\begin{aligned}
&|x(t)-x(t-\tau(t))|^2\\
&=|\int_{t-\tau(t)}^t x'(s)ds+\lambda\sum_{t-\tau(t)\le t_k<t} I_k(x(t_k)|^2\\
&\le\Big(\int_{t-\alpha}^t |x'(s)|ds\Big)^2+2\lambda\Big(\int_{t-\alpha}^t |x'(s)|ds\Big)\sum_{t-\alpha\le t_k<t}|I_k(x(t_k)|\\
&\quad+\Big(\lambda\sum_{t-\alpha\le t_k<t}|I_k(x(t_k))|\Big)^2\\
&\le\alpha\int_{t-\alpha}^t |x'(s)|^2ds+2\int_{t-\alpha}^t |x'(s)|ds\sum_{t-\alpha\le t_k<t}[a_k|x(t)|_\infty+a_k']\\
&\quad+\Big[\sum_{t-\alpha\le t_k<t}(a_k|x(t)|_\infty+a_k')\Big]^2.
\end{aligned}$$

By the Schwarz inequality and Lemma 2.4, we obtain

$$\begin{aligned}
&\int_0^T |x(t)-x(t-\tau(t))|^2dt\\
&\le\alpha\int_0^T\int_{t-\alpha}^t |x'(s)|^2\,ds\,dt\\
&\quad+2|x(t)|_\infty\int_0^T A_1(t,\alpha)\int_{t-\alpha}^t |x'(s)|\,ds\,dt+2\int_0^T B_1(t,\alpha)\int_{t-\alpha}^t |x'(s)|\,ds\,dt\\
&\quad+|x(t)|_\infty^2\int_0^T A_1^2(t,\alpha)dt+|x(t)|_\infty\int_0^T A_1(t,\alpha)B_1(t,\alpha)dt+\int_0^T B_1^2(t,\alpha)dt\\
&\le\alpha\int_0^T\int_{t-\alpha}^t |x'(s)|^2\,ds\,dt+2|x(t)|_\infty
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\int_0^T A_1^2(t,\alpha)dt\Big)^{1/2}\Big(\int_0^T\Big(\int_{t-\alpha}^t |x'(s)|ds\Big)^2 dt\Big)^{1/2}\\
&+2\Big(\int_0^T B_1^2(t,\alpha)dt\Big)^{1/2}\Big(\int_0^T\Big(\int_{t-\alpha}^t |x'(s)|ds\Big)^2 dt\Big)^{1/2}\\
&+|x(t)|_\infty^2\int_0^T A_1^2(t,\alpha)dt+|x(t)|_\infty\int_0^T[A_1(t,\alpha)B_1(t,\alpha)]dt+\int_0^T B_1^2(t,\alpha)dt\\
\le{}& \alpha^2\int_0^T|x'(t)|^2dt+2\alpha|x(t)|_\infty\Big(\int_0^T A_1^2(t,\alpha)dt\Big)^{1/2}\Big(\int_0^T|x'(t)|^2dt\Big)^{1/2}\\
&+2\alpha\Big(\int_0^T B_1^2(t,\alpha)dt\Big)^{1/2}(\int_0^T|x'(t)|^2dt)^{1/2}\\
&+|x(t)|_\infty^2\int_0^T A_1^2(t,\alpha)dt+|x(t)|_\infty\int_0^T A_1(t,\alpha)B_1(t,\alpha)dt+\int_0^T B_1^2(t,\alpha)dt.
\end{aligned}
$$

If $\tau(t)\in[-\alpha,0]$, then for all $t\in[0,T]$, similarly, we obtain

$$
\begin{aligned}
&\int_0^T|x(t)-x(t-\tau(t))|^2dt\\
&\le \alpha^2\int_0^T|x'(t)|^2dt+2\alpha|x(t)|_\infty\Big(\int_0^T A_2^2(t,\alpha)dt\Big)^{1/2}\Big(\int_0^T|x'(t)|^2dt\Big)^{1/2}\\
&\quad+2\alpha(\int_0^T B_2^2(t,\alpha)dt)^{1/2}(\int_0^T|x'(t)|^2dt)^{1/2}\\
&\quad+|x(t)|_\infty^2\int_0^T A_2^2(t,\alpha)dt+|x(t)|_\infty\int_0^T A_2(t,\alpha)B_2(t,\alpha)dt+\int_0^T B_2^2(t,\alpha)dt.
\end{aligned}
$$

Let $\Delta_1=\{t: t\in[0,T],\tau(t)\ge 0\}$, $\Delta_2=\{t: t\in[0,T],\tau(t)<0\}$. Then for for all $t\in[0,T]$,

$$
\begin{aligned}
&\int_0^T|x(t)-x(t-\tau(t))|^2dt\\
&=\int_{\Delta_1}|x(t)-x(t-\tau(t))|^2dt+\int_{\Delta_2}|x(t)-x(t-\tau(t))|^2dt\\
&\le 2\alpha^2\int_0^T|x'(t)|^2dt+2\alpha A(\alpha)|x(t)|_\infty(\int_0^T|x'(t)|^2dt)^{1/2}\\
&\quad+2\alpha B(\alpha)\Big(\int_0^T|x'(t)|^2dt\Big)^{1/2}+C(\alpha)|x(t)|_\infty^2+D(\alpha)|x(t)|_\infty+E(\alpha).
\end{aligned}
$$

□

3. Main results

For the next theorem we use the following conditions:

(H3) There are constants $\sigma,\beta\ge 0$ such that

$$|f(t,x)|\le\sigma|x|,\quad \forall(t,x)\in[0,T]\times\mathbb{R}, \tag{3.1}$$

$$xf(t,x)\ge\beta|x|^2,\quad \forall(t,x)\in[0,T]\times\mathbb{R}; \tag{3.2}$$

(H4) there are constants $\beta_i\ge 0$ ($i=1,2,3$) such that

$$|g(x)|\ge\beta_1+\beta_2|x|, \tag{3.3}$$

$$|g(x) - g(y)| \le \beta_3 |x - y|; \tag{3.4}$$

(H5) there are constants $\gamma_i > 0$ $(i = 1, 2, 3)$, such that $|\int_x^{x+\lambda I_k(x,y)} g(s)ds| \le |I_k(x,y)|(\gamma_1 + \gamma_2|x| + \gamma_3|I_k(x,y)|)$, $\quad \forall \lambda \in (0,1)$;

(H6) there are constants $a_k, a_k' \ge 0$ such that $|I_k(x,y)| \le a_k|x| + a_k'$;

(H7) $yJ_k(x,y) \le 0$ and there are constants $b_k \ge 0$ such that $|J_k(x,y)| \le b_k$.

Theorem 3.1. *Suppose* (H1)–(H7) *hold. Then* (1.1) *has at least one T-periodic solution provided the following two conditions hold*

$$\sum_{k=1}^{n} a_k < 1, \tag{3.5}$$

$$\begin{aligned} &\Big[\gamma_2(\sum_{k=1}^{n} a_k) + \gamma_3(\sum_{k=1}^{n} a_k^2)\Big] M^2 + \beta_3 \Big[2|\tau(t)|_\infty^2 \\ &+ 2|\tau(t)|_\infty A(|\tau(t)|_\infty) M + C(|\tau(t)|_\infty) M^2\Big]^{1/2} < \beta, \end{aligned} \tag{3.6}$$

where

$$M = \frac{1}{1 - \sum_{k=1}^{n} a_k} \Big(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2}\Big).$$

Proof. Consider the equation $Lx = \lambda Nx$, with $\lambda \in (0,1)$, where L and N are defined by (2.1) and (2.2). Let

$$\Omega_1 = \{x \in D(L) : \ker L, Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

For $x \in \Omega_1$, we have

$$\begin{gathered} x''(t) + \lambda f(t, x'(t)) + \lambda g(t, x(t - \tau(t)) = \lambda p(t), \quad t \ne t_k, \\ \Delta x(t_k) = \lambda I_k(x(t_k), x'(t_k)), \\ \Delta x'(t_k) = \lambda J_k(x(t_k), x'(t_k)). \end{gathered} \tag{3.7}$$

Integrating them on $[0, T]$, using Schwarz inequality, we have

$$\begin{aligned} &|\int_0^T g(x(t - \tau(t))dt| \\ &= |\int_0^T p(t)dt - \int_0^T f(t, x'(t))dt + \sum_{k=1}^{n} J_k(x(t_k), x'(t_k))| \\ &\le T|p(t)|_\infty + \sigma \int_0^T |x'(t)|dt + \sum_{k=1}^{n} b_k \\ &\le \sigma T^{1/2} \Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + T|p(t)|_\infty + \sum_{k=1}^{n} b_k. \end{aligned}$$

From the above formula, there is a $t_0 \in [0, T]$ such that

$$|g(x(t_0 - \tau(t_0))| \le \frac{\sigma}{T^{1/2}} \Big(\int_0^T |x'(t)|^2 dt)^{1/2} + |p(t)|_\infty + \frac{1}{T} \sum_{k=1}^{n} b_k.$$

It follows from (3.3) that

$$\beta_1 + \beta_2 |x(t_0 - \tau(t_0))| \le \frac{\sigma}{T^{1/2}} \Big(\int_0^T |x'(t)|^2 dt)^{1/2} + |p(t)|_\infty + \frac{1}{T} \sum_{k=1}^{n} b_k.$$

Thus

$$|x(t_0-\tau(t_0))| \le \frac{\sigma}{\beta_2 T^{1/2}}\Big(\int_0^T |x'(t)|^2dt\Big)^{1/2} + d,$$

where $d = \big(||p(t)|_\infty + \frac{1}{T}\sum_{k=1}^n b_k - \beta_1|\big)/\beta_2$. So there must be an integer m and a point $t_1 \in [0,T]$ such that $t_0 - \tau(t_0) = mT + t_1$. Hence

$$|x(t_1)| = |x(t_0-\tau(t_0))| \le \frac{\sigma}{\beta_2 T^{1/2}}\Big(\int_0^T |x'(t)|^2dt\Big)^{1/2} + d,$$

which implies

$$x(t) = x(t_1) + \int_{t_1}^t x'(s)ds + \sum_{t_1\le t_k<t} I_k(x(t_k),x'(t_k)).$$

This yields

$$\begin{aligned}
|x(t)|_\infty &\le |x(t_1)| + \int_{t_1}^t |x'(s)|ds + \sum_{t_1\le t_k<t} |I_k(x(t_k))| \\
&\le \frac{\sigma}{\beta_2 T^{1/2}}(\int_0^T |x'(t)|^2dt)^{1/2} + d + \int_0^T |x'(t)|dt + \sum_{k=1}^n a_k|x|_\infty + \sum_{k=1}^n a'_k \\
&\le |x|_\infty \sum_{k=1}^n a_k + (\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2})\Big(\int_0^T |x'(t)|^2dt\Big)^{1/2} + d + \sum_{k=1}^n a'_k.
\end{aligned}$$

It follows that

$$\begin{aligned}
|x(t)|_\infty &\le \frac{d+\sum_{k=1}^n a'_k}{1-\sum_{k=1}^n a_k} + \frac{1}{1-\sum_{k=1}^n a_k}(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2})(\int_0^T |x'(t)|^2dt)^{1/2} \\
&= u_1 + M(\int_0^T |x'(t)|^2dt)^{1/2},
\end{aligned} \tag{3.8}$$

where u_1 is a positive constant. On the other hand, multiplying both side of (3.7) by $x'(t)$, we have

$$\begin{aligned}
&\int_0^T x''(t)x'(t)dt + \lambda\int_0^T f(t,x'(t))x'(t)dt \qquad +\lambda\int_0^T g(t,x(t-\tau(t))x'(t)dt \\
&= \lambda\int_0^T p(t)x'(t)dt.
\end{aligned}$$

Since

$$\int_0^T x''(t)x'(t)dt = -\frac{1}{2}\sum_{i=1}^n[(x'(t_k^+))^2 - (x'(t_k))^2],$$

it follows from assumption (H7) that

$$\begin{aligned}
&(x'(t_k^+))^2 - (x'(t_k))^2 \\
&= (x'(t_k^+) + x'(t_k))(x'(t_k^+) - (x'(t_k)) \\
&= \Delta x'(t_k)(2x'(t_k) + \Delta x'(t_k)) \\
&= \lambda J_k(x(t_k),x'(t_k))(2x'(t_k) + \lambda J_k(x(t_k),x'(t_k)) \\
&= 2\lambda J_k(x(t_k),x'(t_k))x'(t_k) + [\lambda J_k(x(t_k),x'(t_k))]^2 \le b_k^2.
\end{aligned}$$

In view of (3.2), by Schwarz inequality, we obtain

$$
\begin{aligned}
&\beta \int_0^T |x'(t)|^2 dt \\
&\leq -\int_0^T g(x(t-\tau(t))x'(t)dt + \int_0^T p(t)x'(t)dt + \frac{1}{2}\sum_{k=1}^n b_k^2 \\
&= \int_0^T [g(x(t)-g(x(t-\tau(t))]x'(t)dt - \int_0^T g(x(t))x'(t)dt \\
&\quad + \int_0^T p(t)x'(t)dt + \frac{1}{2}\sum_{i=1}^n b_k^2 \\
&\leq \int_0^T |g(x(t))-g(x(t-\tau(t))||x'(t)|dt + |p(t)|_\infty \int_0^T |x'(t)|dt \\
&\quad + |\int_0^T g(x(t))x'(t)dt| + \frac{1}{2}\sum_{i=1}^n b_k^2 \\
&\leq \Big[\Big(\int_0^T |g(x(t))-g(x(t-\tau(t)))|^2 dt\Big)^{1/2} + |p(t)|_\infty T^{1/2}\Big]\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} \\
&\quad + |\int_0^T g(x(t))x'(t)dt| + \frac{1}{2}\sum_{i=1}^n b_k^2.
\end{aligned} \tag{3.9}
$$

From (H5) and (H6), we have

$$
\begin{aligned}
&|\int_0^T g(x(t))x'(t)dt| \\
&= |\int_{x(0)}^{x(t_1)} g(s)ds + \int_{x(t_1^+)}^{x(t_2)} g(s)ds + \dots + \int_{x(t_n^+)}^{x(T)} g(s)ds| \\
&= |\int_{x(0)}^{x(T)} g(s)ds - \sum_{k=1}^n \int_{x(t_k)}^{x(t_k^+)} g(s)ds| \\
&\leq \sum_{k=1}^n |\int_{x(t_k)}^{x(t_k)+\lambda I_k(x(t_k),x'(t_k))} g(s)ds| \\
&\leq \sum_{k=1}^n [|I_k(x(t_k),x'(t_k))|(\gamma_1 + \gamma_2|x(t_k)| + \gamma_3|I_k(x(t_k),x'(t_k))|)] \\
&\leq [\gamma_2(\sum_{k=1}^n a_k) + \gamma_3(\sum_{k=1}^n a_k^2)]|x(t)|_\infty^2 + u_2|x(t)|_\infty + u_3,
\end{aligned}
$$

where u_2, u_3 are positive constants. From (3.8), we have

$$
\begin{aligned}
&|\int_0^T g(x(t))x'(t)dt| \\
&\leq [\gamma_2(\sum_{k=1}^n a_k) + \gamma_3(\sum_{k=1}^n a_k^2)]M^2 \int_0^T |x'(t)|^2 dt + u_4(\int_0^T |x'(t)|^2 dt)^{1/2} + u_5,
\end{aligned} \tag{3.10}
$$

where u_4, u_5 are positive constants. Applying Lemma 2.5, we obtain

$$
\begin{aligned}
&\int_0^T |g(x(t) - g(x(t-\tau(t)))|^2 dt \\
&\le \beta_3^2 \int_0^T |x(t) - x(t-\tau(t))|^2 dt \\
&\le \beta_3^2 [2|\tau(t)|_\infty^2 \int_0^T |x'(t)|^2 dt + 2|\tau(t)|_\infty A(|\tau(t)|_\infty)|x(t)|_\infty \Big(\int_0^T |x'(t)|^2 dt \Big)^{1/2} \\
&\quad + 2|\tau(t)|_\infty B(|\tau(t)|_\infty) \Big(\int_0^T |x'(t)|^2 dt \Big)^{1/2} + C(|\tau(t)|_\infty)|x(t)|_\infty^2 \\
&\quad + D(|\tau(t)|_\infty)|x(t)|_\infty + E(|\tau(t)|_\infty)].
\end{aligned}
$$

Substituting (3.8) into the above inequality, we have

$$
\begin{aligned}
&\int_0^T |g(x(t) - g(x(t-\tau(t)))|^2 dt \\
&\le \beta_3^2 [2|\tau(t)|_\infty^2 + 2|\tau(t)|_\infty A(|\tau(t)|_\infty) M \\
&\quad + C(|\tau(t)|_\infty) M^2] \int_0^T |x'(t)|^2 dt + u_6 \Big(\int_0^T |x'(t)|^2 dt \Big)^{1/2} + u_7,
\end{aligned}
$$

where u_6, u_7 are positive constants. Using the inequality

$$
(a+b)^{1/2} \le a^{1/2} + b^{1/2} \quad for \quad a \ge 0, b \ge 0, \tag{3.11}
$$

we have

$$
\begin{aligned}
&\Big(\int_0^T |g(x(t)) - g(x(t-\tau(t)))|^2 dt \Big)^{1/2} \\
&\le \beta_3 [2|\tau(t)|_\infty^2 + 2|\tau(t)|_\infty A(|\tau(t)|_\infty) M \\
&\quad + C(|\tau(t)|_\infty) M^2]^{1/2} \Big(\int_0^T |x'(t)|^2 dt \Big)^{1/2} + u_6^{1/2} \Big(\int_0^T |x'(t)|^2 dt \Big)^{1/4} + u_7^{1/2}.
\end{aligned}
$$

Substituting the above formula and (3.10) in (3.9), we obtain

$$
\begin{aligned}
&\{ \beta - [\gamma_2 (\sum_{k=1}^n a_k) + \gamma_3 (\sum_{k=1}^n a_k^2)] M^2 - \beta_3 [2|\tau(t)|_\infty^2 \\
&\quad + 2|\tau(t)|_\infty A(|\tau(t)|_\infty) M + C(|\tau(t)|_\infty) M^2]^{1/2} \} \int_0^T |x'(t)|^2 dt \\
&\le u_8 (\int_0^T |x'(t)|^2 dt)^{\frac{3}{4}} + u_9 (\int_0^T |x'(t)|^2 dt)^{1/2} + u_{10},
\end{aligned}
$$

where u_8, u_9, u_{10} are positive constants. Then there is a constant $M_1 > 0$ such that

$$
\int_0^T |x'(t)|^2 dt \le M_1. \tag{3.12}
$$

From (3.8), we have

$$
|x(t)|_\infty \le d + M(\int_0^T |x'(t)|^2 dt)^{1/2} \le d + M(M_1)^{1/2}.
$$

Then there is a constant $M_2 > 0$ such that $|x(t)|_\infty \le M_2$. Furthermore, integrating (3.7) on $[0,T]$, using Schwarz inequality, we obtain

$$\begin{aligned}
\int_0^T |x''(t)|dt &= \int_0^T |-f(t,x(t)) - g(x(t-\tau(t))) + p(t)|dt \\
&\le \int_0^T |f(t,x'(t))|dt + \int_0^T |g(x(t-\tau(t)))|dt + \int_0^T |p(t)|dt \\
&\le \sigma \int_0^T |x'(t)|dt + g_\delta T + T|p(t)|_\infty \\
&\le \sigma T^{1/2}(\int_0^T |x'(t)|^2 dt)^{1/2} + g_\delta T + T|p(t)|_\infty \\
&\le \sigma T^{1/2}(M_1)^{1/2} + g_\delta T + T|p(t)|_\infty,
\end{aligned}$$

where $h_\delta = \max_{|x|\le\delta} |g(x)|$. That is to say that there is a constant $M_3 > 0$ such that

$$\int_0^T |x''(t)|dt \le M_3. \tag{3.13}$$

From (3.12), it is easy to see that there are $t_2 \in [0,T]$ and $u_{11} > 0$ such that $|x'(t_2)| \le u_{11}$, then for $t \in [0,T]$

$$|x'(t)|_\infty \le |x'(t_2)| + \int_0^T |x''(t)|dt + \sum_{k=1}^n b_k. \tag{3.14}$$

Hence there is a constant $M_4 > 0$ such that

$$|x'(t)|_\infty \le M_4. \tag{3.15}$$

It follows that there is a constant $B > \max\{M_2, M_4\}$ such that $\|x\| \le B$, Thus Ω_1 is bounded.

Let $\Omega_2 = \{x \in \ker L, QNx = 0\}$. Suppose $x \in \Omega_2$, then $x(t) = c \in R$ and satisfies

$$QN(x,0) = (-\frac{2}{T^2}\int_0^T [f(t,0) + g(c) - p(t)]dt, 0, \dots, 0) = 0. \tag{3.16}$$

Then

$$\int_0^T [f(t,0) + g(c) - p(t)]dt = 0. \tag{3.17}$$

It follows from (3.17) that there must be a $t_0 \in [0,T]$ such that

$$g(c) = -f(t_0,0) + p(t_0). \tag{3.18}$$

From (3.18) and assumption (H3), (H4), we have

$$\beta_1 + \beta_2|c| \le |g(c)| \le |f(t_0,0)| + |p(t_0)| \le \sigma \times 0 + |p(t)|_\infty. \tag{3.19}$$

Thus

$$|c| \le \frac{||p(t)|_\infty - \beta_1|}{\beta_2} \tag{3.20}$$

which implies Ω_2 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$, where $\Omega_3 = \{x \in X : |x| < ||p(t)|_\infty - \beta_1|/\beta_2 + 1\}$. By Lemmas 2.2 and 2.3, we can see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument,

(i) $Lx \ne \lambda Nx$ for all $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;

(ii) $QNx \neq 0$ for all $x \in \partial\Omega \cap \ker L$.

At last we prove that (iii) of Lemma 2.1 is satisfied. We take $H(x,\mu) : \Omega \times [0,1] \to X$,

$$H(x,\mu) = \mu x + \frac{2(1-\mu)}{T^2} \int_0^T [-f(t,x'(t)) + g(x(t-\tau(t)) + p(t)]dt.$$

From assumptions (H3) and (H4), we can easily obtain $H(x,\mu) \neq 0$, for all $(x,\mu) \in \partial\Omega \cap \ker L \times [0,1]$, which results in

$$\begin{aligned} \deg\{JQNx, \Omega \cap \ker L, 0\} &= \deg\{H(x,0), \Omega \cap \ker L, 0\} \\ &= \deg\{H(x,1), \Omega \cap \ker L, 0\} \neq 0, \end{aligned}$$

where $J(x,0,\dots,0) = x$. Therefore, by Lemma 2.1, Equation (1.1) has at least one T-periodic solution. □

Theorem 3.2. *Suppose* (H1)-(H2), (H4)-(H6) *hold and the following two conditions hold:*

(H8) *there is an constant $\sigma \geq 0$ such that*

$$\begin{gathered} |f(t,x)| \leq \sigma|x|, \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \\ xf(t,x) \leq -\beta|x|^2, \forall (t,x) \in [0,T] \times \mathbb{R}, \end{gathered}$$

(H9) $yJ_k(x,y) \geq 0$ *and there are constants $b_k \geq 0$ such that $|J_k(x,y)| \leq b_k$.*

Then (1.1) *has at least one T-periodic solution provided* (3.5) *and* (3.6) *hold.*

The proof of the above theorem is similar to that of Theorem 3.1, so we omit it.

Example. Consider the equation

$$\begin{gathered} x''(t) + \frac{1}{3}x'(t) + \frac{1}{15}x(t - \frac{1}{10}\cos t) = \sin t, \quad t \neq k, \\ \Delta x(k) = \frac{\sin(k\pi/3)}{120}x(k) + \frac{x'(t_k)}{1 + x'^2(t_k)}, \\ \Delta x'(k) = -\frac{2x^2(t_k)x'(t_k)}{1 + x^4(t_k)x'^2(t_k)}, \end{gathered} \tag{3.21}$$

where $t_k = k$, $f(t,x) = \frac{1}{3}x$, $g(y) = \frac{1}{15}y$, $p(t) = \sin t$, $\tau(t) = \frac{1}{10}\cos t$, $I_k(x,y) = \frac{\sin\frac{k\pi}{3}}{120}x + \frac{y}{1+y^2}$, $J_k(x,y) = -\frac{2x^2y}{1+x^4y^2}$, it is easy to see that $|\tau(t)|_\infty = \frac{1}{10}$, $T = 2\pi$, $\{k\} \cap [0,2\pi] = \{1,2,3,4,5,6\}$, $\sigma = \beta = \frac{1}{3}$, $\beta_1 = 0$, $\beta_2 = \beta_3 = \frac{1}{15}$. Since $|I_k(x,y)| \leq \frac{1}{120}|x| + \frac{1}{2}$, $|J_k(x,y)| \leq 1$, $|\int_x^{x+I_k(x,y)} g(s)ds| \leq |I_k(x,y)|(\frac{1}{15}|x| + \frac{1}{30}|I_k(x,y)|)$, then we take $a_k = \frac{1}{120}$, $a'_k = \frac{1}{2}$, $b'_k = 1$ $(k = 1,2,3,4,5,6)$, $\gamma_1 = 0$, $\gamma_2 = 1/15$, $\gamma_3 = 1/30$. Thus assumption (H1)–(H7) hold and

$$\sum_{k=1}^6 a_k = \frac{1}{20} < 1,$$

$$M = \frac{1}{1 - \sum_{k=1}^n a_k}\Big(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2}\Big) = \frac{1}{1 - \frac{1}{20}}\Big(\frac{\frac{1}{3}}{\frac{1}{15}(2\pi)^{1/2}} + (2\pi)^{1/2}\Big) < 6.$$

Thus

$$\Big[\gamma_2(\sum_{k=1}^n a_k) + \gamma_3(\sum_{k=1}^n a_k^2)\Big]M^2 + \beta_3[2|\tau(t)|_\infty^2$$

$$+ 2|\tau(t)|_\infty A(|\tau(t)|_\infty)M + C(|\tau(t)|_\infty)M^2]^{1/2} < \beta.$$

By Theorem 3.1, Equation (3.21) has at least one 2π-periodic solution.

References

[1] D. Bainov, M. Dimitrova; *Oscillations of sub- and superlinear impulsive differential equations with constant delay*, Appl. Anal. 64 (1997) 57-67.

[2] Y. S. Chen, W. Z. Feng; *Oscillations of second order nonlinear ODE with impulses*, J. Math. Anal. Appl. 210 (1997) 150-169.

[3] C. Fabry, J. Mawhin, M. Nkashama; *A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations*, Bull London Math soc. 10 (1986) 173-180.

[4] R. Gaines, J. Mawhin; *Concidence degree and nonlinear differential equations*, Lecture Notes in Math, 568, berlin, New York: Springer-Verlag, 1977.

[5] K. Gopalsamy, B. G. Zhang; *On delay differential equations with impulses*, J. Math. Anal. Appl. 139 (1989) 110-122.

[6] I. T. Kiguradze, B. Puza; *On periodic solutions of system of differential equations with deviating arguments*, Nonlinear Anal.42 (2000) 229-242.

[7] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov; *Theory of impulsive differential equations*, World Scientific Singapore, 1989.

[8] J. L. Li, J. H. Shen; *Periodic solutions of duffing equation with delays and impulses*, Acta Mathematicae Applicatae Sinica. 28 (2005) 124-133 (in Chinese).

[9] S. Lu, W. Ge; *On the existence of periodic solutions for Linenard equation with a deviating argument*, J. Math. Anal. Appl. 289 (2004) 241-243.

[10] S. Lu, W. Ge; *Sufficient conditions for the existence of periodic solutions to some second order differential equation with a deviating argument*, J. Math. Anal. Appl. 308 (2005) 393-419.

[11] J. Mawhin; *Degré topologique et solutions périodiques des systémes différentiels nonlineares*, Bull Soc Roy Sci Liége. 38 (1969) 308-398.

[12] J. J. Neito, R. R. Lopez; *Green′s functions for second order periodic boundary value problems with piesewise constant arguments*, J. Math. Anal. Appl. 304 (2005) 33-57.

[13] P. Omari, G. villari, F. zanolin; *Periodic solutions of lienard equation with one-side growth restrictions*, J. Diffential Equations. 67 (1987) 278-293.

[14] D. B. Qian, X. Y. Li; *Periodic solutions for ordinary differential equations with sublinear impulsive effects*, J. Math. Anal. Appl. 303 (2005) 288-303.

[15] J. H. Shen; *The nonoscillatory solutions of delay differential equations with impulses*, Appl. Math. comput. 77 (1996) 153-165.

[16] G. Q. Wang; *A priori bounds for periodic solutions of a delay rayleigh equation*, Appl. Math. Lett. 12 (1999) 41-44.

[17] J. R. Yan, A. M. Zhao, W. P. Yan; *Existence and global attractivity of periodic solution for an impulsive delay differential equation with allee effect*, J. Math. Anal. Appl. 309 (2005) 489-504.

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