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INTERVAL CRITERIA FOR OSCILLATION OF SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATION WITH MIXED NONLINEARITIES

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Abstract. We establish sufficient conditions for the oscillation of all solutions to the second-order impulsive differential equation

$$\big(r(t)x'(t)\big)' + p(t)x'(t) + q(t)x(t) + \sum_{i=1}^{n} q_i(t)|x(t)|^{\alpha_i}\operatorname{sgn} x(t) = e(t), \quad t \neq \tau_k,$$
$$x(\tau_k{}^+) = a_k x(\tau_k), \quad x'(\tau_k{}^+) = b_k x'(\tau_k).$$

The results obtained in this paper extend some of the existing results and are illustrated by examples.

1. Introduction

Consider the second-order impulsive differential equation, with mixed nonlinearities,

$$\begin{gathered}\big(r(t)x'(t)\big)' + p(t)x'(t) + q(t)x(t) + \sum_{i=1}^{n} q_i(t)|x(t)|^{\alpha_i}\operatorname{sgn} x(t) = e(t), \quad t \neq \tau_k,\\ x(\tau_k{}^+) = a_k x(\tau_k), \quad x'(\tau_k{}^+) = b_k x'(\tau_k),\end{gathered} \tag{1.1}$$

where $t \geq t_0$, $k \in \mathbb{N}$, τ_k is the impulse moments sequence with

$$0 \leq t_0 = \tau_0 < \tau_1 < \dots < \tau_k < \dots, \lim_{k\to\infty} \tau_k = \infty,$$
$$x(\tau_k) = x(\tau_k^-) = \lim_{t\to\tau_k^{-0}} x(t), \quad x(\tau_k{}^+) = \lim_{t\to\tau_k^{+0}} x(t),$$
$$x'(\tau_k) = x'(\tau_k{}^-) = \lim_{h\to 0^-} \frac{x(\tau_k+h) - x(\tau_k)}{h}, \quad x'(\tau_k{}^+) = \lim_{h\to 0^+} \frac{x(\tau_k+h) - x(\tau_k{}^+)}{h}.$$

Throughout this paper, assume that the following conditions hold without further mention:

C1) $r \in C^1([t_0,\infty),(0,\infty))$, $p, q, q_i, e \in C([t_0,\infty),\mathbb{R})$, $i = 1, 2 \dots, n$;
(C2) $\alpha_1 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n > 0$ are constants;
(C3) $b_k \geq a_k > 0$, $k \in \mathbb{N}$ are constants.

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Let $J \subset \mathbb{R}$ be an interval and define

$$PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t) \text{ is piecewise-left-continuous and has discontinuity of first kind at } \tau_k' s\}.$$

By a *solution* of (1.1), we mean a function $x \in PC([t_0,\infty),\mathbb{R})$ with a property $(rx')' \in PC([t_0,\infty),\mathbb{R})$ such that (1.1) is satisfied for all $t \geq t_0$. A nontrivial solution is called *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*. An equation is called oscillatory if all its solutions are oscillatory.

In recent years the oscillation theory of impulsive differential equations emerging as an important area of research, since such equations have applications in control theory, physics, biology, population dynamics, economics, etc. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equation, see for example [3] and the references cited therein.

In [1, 5, 7], the authors established several oscillation criteria for second-order impulsive differential equations which are particular cases of (1.1). Compared to second order ordinary differential equations [2, 4, 6, 8, 9, 10, 11], the oscillatory behavior of impulsive second order differential equations received less attention even though such equations have many applications. Motivated by this observation, in this paper, we establish some new oscillation criteria for all solutions of (1.1). Our results extend those obtained in [10] for equation without impulses. Finally some examples are given to illustrate the results.

2. Main results

We begin with the following notation. Let $k(s) = \max\{i : t_0 < \tau_i < s\}$ and for $c_j < d_j$, let $r_j = \max\{r(t) : t \in [c_j, d_j]\}$, $j = 1, 2$. For two constants $c, d \notin \{\tau_k\}$ with $c < d$ and a function $\phi \in C([c,d],\mathbb{R})$, we define an operator $\Omega : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$\Omega_c^d[\phi] = \begin{cases} 0, & \text{for } k(c) = k(d), \\ \phi(\tau_{k(c)+1})\theta(c) + \sum_{i=k(c)+2}^{k(d)} \phi(\tau_i)\varepsilon(\tau_i), & \text{for } k(c) < k(d), \end{cases}$$

where

$$\theta(c) = \frac{b_{k(c)+1} - a_{k(c)+1}}{a_{k(c)+1}(\tau_{k(c)+1} - c)}, \quad \varepsilon(\tau_i) = \frac{b_i - a_i}{a_i(\tau_i - \tau_{i-1})}.$$

Following Kong [2] and Philos [5], we introduce a class of functions: Let $D = \{(t,s) : t_0 \leq s \leq t\}$, $H_1, H_2 \in C^1(D,\mathbb{R})$. A pair of functions (H_1, H_2) is said to belong to a function class $\mathcal{H}$, if $H_1(t,t) = H_2(t,t) = 0$, $H_1(t,s) > 0$, $H_2(t,s) > 0$ for $t > s$ and there exist $h_1, h_2 \in L_{\rm loc}(D,\mathbb{R})$ such that

$$\frac{\partial H_1(t,s)}{\partial t} = h_1(t,s)H_1(t,s), \quad \frac{\partial H_2(t,s)}{\partial s} = -h_2(t,s)H_2(t,s). \tag{2.1}$$

To prove our main results we need the following lemma due to Sun and Wong [9].

Lemma 2.1. *Let $\{\alpha_i\}$, $i = 1, 2, \dots, n$, be the n-tuple satisfying $\alpha_1 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n > 0$. Then there exist an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ satisfying*

$$\sum_{i=1}^n \alpha_i \eta_i = 1, \tag{2.2}$$

which also satisfies either

$$\sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1, \tag{2.3}$$

or

$$\sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{2.4}$$

Remark 2.2. For a given set of exponents $\{\alpha_i\}$ satisfying $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$, Lemma 1 ensures the existence of an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ such that either (2.2) and (2.3) hold or (2.2) and (2.4) hold. When $n = 2$ and $\alpha_1 > 1 > \alpha_2 > 0$, in the first case, we have

$$\eta_1 = \frac{1 - \alpha_2(1 - \eta_0)}{\alpha_1 - \alpha_2}, \quad \eta_2 = \frac{\alpha_1(1 - \eta_0) - 1}{\alpha_1 - \alpha_2},$$

where η_0 can be any positive number satisfying $0 < \eta_0 < (\alpha_1 - 1)/\alpha_1$. This will ensure that $0 < \eta_1, \eta_2 < 1$ and conditions (2.2) and (2.3) are satisfied. In the second case, we simply solve (2.2) and (2.4) and obtain

$$\eta_1 = \frac{1 - \alpha_2}{\alpha_1 - \alpha_2}, \quad \eta_2 = \frac{\alpha_1 - 1}{\alpha_1 - \alpha_2}.$$

Theorem 2.3. *Assume that for any $T > 0$, there exist $c_j, d_j, \delta_j \notin \{\tau_k\}$, $j = 1, 2$ such that $c_1 < \delta_1 < d_1 \le c_2 < \delta_2 < d_2$, and*

$$\begin{gathered} q(t), q_i(t) \ge 0, \quad t \in [c_1, d_1] \cup [c_2, d_2],\ i = 1, 2 \dots, n; \\ e(t) \le 0, \quad t \in [c_1, d_1]; \\ e(t) \ge 0, t \in [c_2, d_2] \end{gathered} \tag{2.5}$$

and if there exists $(H_1, H_2) \in \mathcal{H}$ such that

$$\begin{aligned} &\frac{1}{H_1(\delta_j, c_j)} \int_{c_j}^{\delta_j} H_1(t, c_j)\Big[Q(t) - \frac{1}{4} r(t)\Big(h_1(t, c_j) - \frac{p(t)}{r(t)}\Big)^2\Big] dt \\ &+ \frac{1}{H_2(d_j, \delta_j)} \int_{\delta_j}^{d_j} H_2(d_j, t)\Big[Q(t) - \frac{1}{4} r(t)\Big(h_2(d_j, t) + \frac{p(t)}{r(t)}\Big)^2\Big] dt \\ &> \Lambda(H_1, H_2; c_j, d_j), \end{aligned} \tag{2.6}$$

where

$$\Lambda(H_1, H_2; c_j, d_j) = \frac{r_j}{H_1(\delta_j, c_j)} \Omega_{c_j}^{\delta_j}[H_1(., c_j)] + \frac{r_j}{H_2(d_j, \delta_j)} \Omega_{\delta_j}^{d_j}[H_2(d_j, .)] \tag{2.7}$$

and

$$Q(t) = q(t) + k_0 |e(t)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(t), \quad k_0 = \prod_{i=0}^{n} \eta_i^{-\eta_i},\ \eta_0 = 1 - \sum_{i=1}^{n} \eta_i \tag{2.8}$$

and $\eta_1, \eta_2, \dots, \eta_n$ are positive constants satisfying (2.2) and (2.3) in Lemma 1, then (1.1) is oscillatory.

Proof. Let $x(t)$ be a solution of (1.1). Suppose $x(t)$ does not have any zero in $[c_1, d_1] \cup [c_2, d_2]$. Without loss of generality, we may assume that $x(t) > 0$ for

$t \in [c_1, d_1]$. When $x(t) < 0$ for $t \in [c_2, d_2]$, the proof follows the same argument using the interval $[c_2, d_2]$ instead of $[c_1, d_1]$. Define

$$w(t) = -\frac{r(t)x'(t)}{x(t)}, \quad t \in [c_1, d_1]. \tag{2.9}$$

Then for $t \in [c_1, d_1]$ and $t \neq \tau_k$, we have

$$w'(t) = q(t) + \sum_{i=1}^{n} q_i(t)x^{\alpha_i - 1}(t) - e(t)x^{-1}(t) - \frac{p(t)}{r(t)}w(t) + \frac{w^2(t)}{r(t)}. \tag{2.10}$$

Recall the arithmetic-geometric mean inequality,

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i}, \quad u_i \geq 0 \tag{2.11}$$

where $\eta_i > 0, i = 0, 1, 2, \dots, n$, are chosen according to given $\alpha_1, \alpha_2, \dots, \alpha_n$ as in Lemma 1 satisfying (2.2) and (2.3). Now identify $u_0 = \eta_0^{-1}|e(t)|x^{-1}(t)$ and $u_i = \eta_i^{-1}q_i(t)x^{\alpha_i - 1}(t)$, $i = 1, 2, \dots, n$, in (2.11). Then equation (2.10) becomes

$$\begin{aligned} w'(t) &\geq q(t) + \eta_0^{-\eta_0}|e(t)|^{\eta_0}x^{-\eta_0}(t)\prod_{i=1}^{n} \eta_i^{-\eta_i}q_i^{\eta_i}(t)x^{\eta_i(\alpha_i - 1)}(t) - \frac{p(t)}{r(t)}w(t) + \frac{w^2(t)}{r(t)} \\ &= Q(t) - \frac{p(t)}{r(t)}w(t) + \frac{w^2(t)}{r(t)}, \quad t \in (c_1, d_1),\ t \neq \tau_k. \end{aligned} \tag{2.12}$$

For $t = \tau_k$, $k = 1, 2, \dots$, from (2.9), we have

$$w({\tau_k}^+) = \frac{b_k}{a_k}w(\tau_k). \tag{2.13}$$

Notice that whether there are or not impulsive moments in $[c_1, \delta_1]$ and $[\delta_1, d_1]$, we must consider the following 4 cases, namely, $k(c_1) < k(\delta_1) < k(d_1)$; $k(c_1) = k(\delta_1) < k(d_1)$; $k(c_1) < k(\delta_1) = k(d_1)$ and $k(c_1) = k(\delta_1) = k(d_1)$.

Case 1. If $k(c_1) < k(\delta_1) < k(d_1)$, then there are impulsive moments $\tau_{k(c_1)+1}$, $\tau_{k(c_1)+2}, \dots, \tau_{k(\delta_1)}$ in $[c_1, \delta_1]$ and $\tau_{k(\delta_1)+1}, \tau_{k(\delta_1)+2}, \dots, \tau_{k(d_1)}$ in $[\delta_1, d_1]$ respectively. Multiplying both sides of inequality (2.12) by $H_1(t, c_1)$, then integrating it from c_1 to δ_1 and using (2.13), we have

$$\begin{aligned} &\int_{c_1}^{\delta_1} H_1(t, c_1)Q(t)dt \\ &\leq \int_{c_1}^{\delta_1} H_1(t, c_1)w'(t)dt - \int_{c_1}^{\delta_1} H_1(t, c_1)\frac{w^2(t)}{r(t)}dt + \int_{c_1}^{\delta_1} H_1(t, c_1)w(t)\frac{p(t)}{r(t)}dt \\ &= \Big(\int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \dots + \int_{k(\delta_1)}^{\delta_1}\Big)H_1(t, c_1)dw(t) \qquad (2.14) \\ &\quad - \int_{c_1}^{\delta_1} H_1(t, c_1)\Big[\frac{w^2(t)}{r(t)} - w(t)\frac{p(t)}{r(t)}\Big]dt \\ &= \sum_{i=k(c_1)+1}^{k(\delta_1)} H_1(\tau_i, c_1)\frac{a_i - b_i}{a_i}w(\tau_i) + H_1(\delta_1, c_1)w(\delta_1) \qquad (2.15) \end{aligned}$$

$$
\begin{aligned}
&- \int_{c_1}^{\delta_1} \frac{H_1(t,c_1)}{r(t)} \Big[w^2(t) + r(t)w(t)\Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)\Big] dt \\
&\le \sum_{i=k(c_1)+1}^{k(\delta_1)} H_1(\tau_i, c_1) \frac{a_i - b_i}{a_i} w(\tau_i) + H_1(\delta_1, c_1) w(\delta_1) \\
&- \int_{c_1}^{\delta_1} \frac{H_1(t,c_1)}{r(t)} \Big[w(t) + \frac{r(t)}{2}\Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)\Big]^2 dt \\
&+ \int_{c_1}^{\delta_1} \frac{H_1(t,c_1) r(t)}{4} \Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)^2 dt \\
&= \sum_{i=k(c_1)+1}^{k(\delta_1)} H_1(\tau_i, c_1) \frac{a_i - b_i}{a_i} w(\tau_i) + H_1(\delta_1, c_1) w(\delta_1) \\
&+ \frac{1}{4} \int_{c_1}^{\delta_1} H_1(t,c_1) r(t) \Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)^2 dt.
\end{aligned}
$$

On the other hand, multiplying both sides of inequality (2.12) by $H_2(d_1, t)$, then integrating it from δ_1 to d_1, we have

$$
\begin{aligned}
\int_{\delta_1}^{d_1} H_2(d_1,t) Q(t) dt &\le \sum_{i=k(\delta_1)+1}^{k(d_1)} H_2(d_1, \tau_i) \frac{a_i - b_i}{a_i} w(\tau_i) - H_2(d_1, \delta_1) w(\delta_1) \\
&+ \frac{1}{4} \int_{\delta_1}^{d_1} H_2(d_1,t) r(t) \Big(h_2(d_1,t) + \frac{p(t)}{r(t)}\Big)^2 dt.
\end{aligned} \tag{2.16}
$$

Dividing (2.15) and (2.16) by $H_1(\delta_1, c_1)$ and $H_2(d_1, \delta_1)$ respectively, then adding them, we obtain

$$
\begin{aligned}
&\frac{1}{H_1(\delta_1, c_1)} \int_{c_1}^{\delta_1} H_1(t,c_1) \Big[Q(t) - \frac{1}{4} r(t) \Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)^2 \Big] dt \\
&+ \frac{1}{H_2(d_1, \delta_1)} \int_{\delta_1}^{d_1} H_2(d_1,t) \Big[Q(t) - \frac{1}{4} r(t) \Big(h_2(d_1,t) + \frac{p(t)}{r(t)}\Big)^2 \Big] dt \\
&\le \frac{1}{H_1(\delta_1, c_1)} \sum_{i=k(c_1)+1}^{k(\delta_1)} H_1(\tau_i, c_1) \frac{a_i - b_i}{a_i} w(\tau_i) \\
&\quad + \frac{1}{H_2(d_1, \delta_1)} \sum_{i=k(\delta_1)+1}^{k(d_1)} H_2(d_1, \tau_i) \frac{a_i - b_i}{a_i} w(\tau_i).
\end{aligned} \tag{2.17}
$$

Now for $t \in (c_1, \tau_{k(c_1)+1}]$,

$$
(r(t)x'(t))' + p(t)x'(t) = e(t) - q(t)x(t) - \sum_{i=1}^{n} q_i(t) x^{\alpha_i}(t) \le 0
$$

which implies that $x'(t) \exp\big(\int_{c_1}^{t} \frac{r'(s)+p(s)}{r(s)} ds\big)$ is non-increasing on $(c_1, \tau_{k(c_1)+1}]$. So for any $t \in (c_1, \tau_{k(c_1)+1}]$, we have

$$
x(t) - x(c_1) = x'(\xi)(t - c_1)
$$

$$
\begin{aligned}
&\geq \frac{x'(t)\exp\big(\int_{c_1}^{t}\frac{r'(s)+p(s)}{r(s)}ds\big)}{\exp\big(\int_{c_1}^{\xi}\frac{r'(s)+p(s)}{r(s)}ds\big)}(t-c_1)\\
&\geq x'(t)(t-c_1).
\end{aligned}
$$

for some $\xi\in(c_1,t)$. Since $x(c_1)>0$, we have

$$
-\frac{x'(t)}{x(t)}\geq-\frac{1}{t-c_1}.
$$

Letting $t\to\tau_{k(c_1)+1}^{-}$, it follows that

$$
w(\tau_{k(c_1)+1})\geq-\frac{r(\tau_{k(c_1)+1})}{\tau_{k(c_1)+1}-c_1}\geq-\frac{r_1}{\tau_{k(c_1)+1}-c_1}. \tag{2.18}
$$

Similarly we can prove that on (τ_{i-1},τ_i),

$$
w(\tau_i)\geq-\frac{r_1}{\tau_i-\tau_{i-1}}\quad\text{for } i=k(c_1)+2,\dots,k(\delta_1). \tag{2.19}
$$

Using (2.18), (2.19) and (C3), we obtain

$$
\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(\delta_1)}\frac{b_i-a_i}{a_i}w(\tau_i)H_1(\tau_i,c_1)\\
&=\frac{b_{k(c_1)+1}-a_{k(c_1)+1}}{a_{k(c_1)+1}}w(\tau_{k(c_1)+1})H_1(\tau_{k(c_1)+1},c_1)+\sum_{i=k(c_1)+2}^{k(\delta_1)}\frac{b_i-a_i}{a_i}w(\tau_i)H_1(\tau_i,c_1)\\
&\geq-r_1\Big[H_1(\tau_{k(c_1)+1},c_1)\theta(c_1)+\sum_{i=k(c_1)+2}^{k(\delta_1)}H_1(\tau_i,c_1)\varepsilon(\tau_i)\Big]\\
&=-r_1\Omega_{c_1}^{\delta_1}[H_1(.,c_1)].
\end{aligned}
$$

Thus, we have

$$
\sum_{i=k(c_1)+1}^{k(\delta_1)}\frac{a_i-b_i}{a_i}w(\tau_i)H_1(\tau_i,c_1)\leq r_1\Omega_{c_1}^{\delta_1}[H_1(.,c_1)],
$$

and

$$
\sum_{i=k(\delta_1)+1}^{k(d_1)}\frac{a_i-b_i}{a_i}w(\tau_i)H_2(d_1,\tau_i)\leq r_1\Omega_{\delta_1}^{d_1}[H_2(d_1,.)].
$$

Therefore, (2.17) becomes

$$
\begin{aligned}
&\frac{1}{H_1(\delta_1,c_1)}\int_{c_1}^{\delta_1}H_1(t,c_1)\Big[Q(t)-\frac14 r(t)\Big(h_1(t,c_1)-\frac{p(t)}{r(t)}\Big)^2\Big]dt\\
&+\frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1}H_2(d_1,t)\Big[Q(t)-\frac14 r(t)\Big(h_2(d_1,t)+\frac{p(t)}{r(t)}\Big)^2\Big]dt\\
&\leq\frac{r_1}{H_1(\delta_1,c_1)}\Omega_{c_1}^{\delta_1}[H_1(.,c_1)]+\frac{r_1}{H_2(d_1,\delta_1)}\Omega_{\delta_1}^{d_1}[H_2(d_1,.)]\\
&=\Lambda(H_1,H_2;c_1,d_1)
\end{aligned} \tag{2.20}
$$

which contradicts (2.6).

Case 2. If $k(c_1) = k(\delta_1) < k(d_1)$, there is no impulsive moment in $[c_1, \delta_1]$.Then we have

$$
\begin{aligned}
&\int_{c_1}^{\delta_1} H_1(t,c_1)Q(t)dt \\
&\le \int_{c_1}^{\delta_1} H_1(t,c_1)w'(t)dt - \int_{c_1}^{\delta_1} H_1(t,c_1)\Big[\frac{w^2(t)}{r(t)} - w(t)\frac{p(t)}{r(t)}\Big]dt \\
&= H_1(\delta_1,c_1)w(\delta_1) - \int_{c_1}^{\delta_1} H_1(t,c_1)\Big[\frac{w^2(t)}{r(t)} + \Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)w(t)\Big]dt \\
&\le H_1(\delta_1,c_1)w(\delta_1) + \frac{1}{4}\int_{c_1}^{\delta_1} H_1(t,c_1)r(t)\Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)^2 dt.
\end{aligned}
\tag{2.21}
$$

Thus using $\Omega_{c_1}^{\delta_1}[H_1(.,c_1)] = 0$, we obtain

$$
\begin{aligned}
&\frac{1}{H_1(\delta_1,c_1)}\int_{c_1}^{\delta_1} H_1(t,c_1)\Big[Q(t) - \frac{1}{4}r(t)\Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)^2\Big]dt \\
&\quad + \frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1} H_2(d_1,t)\Big[Q(t) - \frac{1}{4}r(t)\Big(h_2(d_1,t) + \frac{p(t)}{r(t)}\Big)^2\Big]dt \\
&\le \frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)} H_2(d_1,\tau_i)\frac{a_i - b_i}{a_i}w(\tau_i) \\
&\le \frac{r_1}{H_2(d_1,\delta_1)}\Omega_{\delta_1}^{d_1}[H_2(d_1,.)] \\
&\le \Lambda(H_1,H_2;c_1,d_1),
\end{aligned}
$$

which is a contradiction. By a similar argument, we can prove the other two cases. Hence the proof is complete. □

Remark 2.4. When $p(t) = 0$, Theorem 2.3 reduces to [5, Theorem 2.2] with $\rho(t) = 1$.

The following theorem gives an interval oscillation criteria for equation (1.1) with $e(t) \equiv 0$.

Theorem 2.5. *Assume that for any $T > 0$, there exist $c_1, d_1, \delta_1 \notin \{\tau_k\}$ such that $c_1 < \delta_1 < d_1$, and $q(t), q_i(t) \ge 0$ for $t \in [c_1, d_1]$ and if there exists $(H_1, H_2) \in \mathcal{H}$ such that*

$$
\begin{aligned}
&\frac{1}{H_1(\delta_1,c_1)}\int_{c_1}^{\delta_1} H_1(t,c_1)\Big[\overline{Q}(t) - \frac{1}{4}r(t)\Big(h_1(t,c_1) - \frac{p(t)}{r(t)}\Big)^2\Big]dt \\
&\quad + \frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1} H_2(d_1,t)\Big[\overline{Q}(t) - \frac{1}{4}r(t)\Big(h_2(d_1,t) + \frac{p(t)}{r(t)}\Big)^2\Big]dt \\
&\ge \Lambda(H_1,H_2;c_1,d_1),
\end{aligned}
\tag{2.22}
$$

where

$$
\overline{Q}(t) = q(t) + k_1\prod_{i=1}^{n} q_i^{\eta_i}(t), \quad k_1 = \prod_{i=1}^{n}\eta_i^{-\eta_i},
\tag{2.23}
$$

Λ is defined as in Theorem 2.3 and $\eta_1, \eta_2, \dots, \eta_n$ are positive constants satisfying (2.2) and (2.4) in Lemma 1, then (1.1) is oscillatory.

Proof. The proof is immediate from Theorem 2.3, if we put $e(t) \equiv 0, \eta_0 = 0$ and applying conditions (2.2) and (2.4) of Lemma 1. □

Next we introduce another function class: a function u belongs to the class $E_{c,d}$ if $u \in C^1[c,d]$, $u(t) \not\equiv 0$ and $u(c) = u(d) = 0$.

Theorem 2.6. *Assume that for any $T > 0$, there exist $c_j, d_j \notin \{\tau_k\}$, $j = 1, 2$ such that $c_1 < d_1 \le c_2 < d_2$, and* (2.5) *holds. Moreover if there exists $u_j \in E_{c_j,d_j}$ such that*

$$\int_{c_j}^{d_j} \Big[Q(t){u_j}^2(t) - \frac{1}{4}r(t)\Big(2{u_j}'(t) - \frac{p(t)}{r(t)}u_j(t)\Big)^2\Big]dt > r_j\Omega_{c_j}^{d_j}[u_j^2], \quad j = 1, 2 \tag{2.24}$$

where $Q(t)$ is the same as in Theorem 2.3, then (1.1) *is oscillatory.*

Proof. Proceed as in the proof of Theorem 2.3 to get (2.12) and (2.13).

If $k(c_1) < k(d_1)$, there are all impulsive moments in $[c_1, d_1]$; $\tau_{k(c_1)+1}$, $\tau_{k(c_1)+2}$, $\dots$, $\tau_{k(d_1)}$. Multiplying both sides of (2.12) by $u_1^2(t)$ and integrating over $[c_1, d_1]$, then using integration by parts, we obtain

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} u_1^2(\tau_i)[w(\tau_i) - w(\tau_i^+)]\\
&\ge \int_{c_1}^{d_1} \Big[Q(t)u_1^2(t) - \frac{1}{4}r(t)\Big(2u_1'(t) - \frac{p(t)}{r(t)}u_1(t)\Big)^2\Big]dt + \Big(\int_{c_1}^{\tau_{k(c_1)+1}}\\
&\quad + \sum_{i=k(c_1)+1}^{k(d_1)} \int_{\tau_{i-1}}^{\tau_i} + \int_{\tau_{k(d_1)}}^{d_1}\Big)\frac{1}{r(t)}\Big[u_1(t)w(t) + \frac{1}{2}r(t)\Big(2u_1'(t) - \frac{p(t)}{r(t)}u_1(t)\Big)\Big]^2 dt\\
&\ge \int_{c_1}^{d_1} \Big[Q(t)u_1^2(t) - \frac{1}{4}r(t)\Big(2u_1'(t) - \frac{p(t)}{r(t)}u_1(t)\Big)^2\Big]dt.
\end{aligned}$$

Thus, we have

$$\sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i - b_i}{a_i} w(\tau_i)u_1^2(\tau_i) \ge \int_{c_1}^{d_1} \Big[Q(t)u_1^2(t) - \frac{1}{4}r(t)\Big(2u_1'(t) - \frac{p(t)}{r(t)}u_1(t)\Big)^2\Big]dt. \tag{2.25}$$

Proceeding as in the proof of Theorem 2.3 and using (2.18) and (2.19), we obtain

$$\int_{c_1}^{d_1} \Big[Q(t)u_1^2(t) - \frac{1}{4}r(t)\Big(2u_1'(t) - \frac{p(t)}{r(t)}u_1(t)\Big)^2\Big]dt \le r_1\Omega_{c_1}^{d_1}[u_1^2]$$

which contradicts our assumption (2.24).

If $k(c_1) = k(d_1)$ then $\Omega_{c_1}^{d_1}[u_1^2] = 0$ and there is no impulsive moments in $[c_1, d_1]$. Similar to the proof of (2.25), we obtain

$$\int_{c_1}^{d_1} \Big[Q(t)u_1^2(t) - \frac{1}{4}r(t)\Big(2u_1'(t) - \frac{p(t)}{r(t)}u_1(t)\Big)^2\Big]dt \le 0.$$

This is a contradiction which completes the proof. □

Remark 2.7. When $p(t) = 0$, Theorem 2.6 reduces to [5, Theorem 2.1] with $\rho(t) = 1$.

Theorem 2.8. *Assume that for any $T > 0$, there exist $c_1, d_1 \notin \{\tau_k\}$ such that $c_1 < d_1$, and $q(t), q_i(t) \geq 0$ for $t \in [c_1, d_1]$ and if there exists $u \in E_{c_1,d_1}$ such that*

$$\int_{c_1}^{d_1} \Big[\overline{Q}(t)u^2(t) - \frac{1}{4}r(t)\Big(2u'(t) - \frac{p(t)}{r(t)}u(t)\Big)^2\Big]dt > r_1\Omega_{c_1}^{d_1}[u^2] \tag{2.26}$$

where $\overline{Q}(t)$ is the same as in Theorem 2.5, then (1.1) *with $e(t) = 0$ is oscillatory.*

The proof of the above theorem is immediate by putting $e(t) = 0$ and $\eta_0 = 0$ in the proof of Theorem 2.6. Next we discuss the oscillatory behavior of the equation

$$\begin{gathered} (r(t)x'(t))' + p(t)x'(t) + q(t)x(t) + q_1(t)x^{\alpha_1}(t) = 0, \quad t \neq \tau_k, \\ x(\tau_k{}^+) = a_k x(\tau_k), \quad x'(\tau_k{}^+) = b_k x'(\tau_k), \end{gathered} \tag{2.27}$$

where α_1 is a ratio of odd positive integers. Before stating our result, we prove another lemma.

Lemma 2.9. *Let u, B and C be positive real numbers and l, m be ratio of odd positive integers. Then*

(i) $l > m + 1$, $0 < m \leq 1$, $u^{l-1} + M_1 B^{\frac{l-1}{l-m-1}} \geq Bu^m$,
(ii) $0 < l + m < 1$, $u^{l+m-1} + M_2 C^{\frac{1-l}{1-l-m}} u^m \geq C$,

where

$$M_1 = \Big(\frac{m}{l-1}\Big)^{\frac{m}{l-m-1}} \Big(\frac{l-m-1}{l-1}\Big), \quad M_2 = \Big(\frac{1-l-m}{1-l}\Big)\Big(\frac{m}{1-l}\Big)^{\frac{m}{1-l-m}}.$$

The proof of the above lemma follows by using elementary differential calculus, and hence it is omitted.

Theorem 2.10. *Assume that for any $T > 0$, there exist $c, d \notin \{\tau_k\}$ such that $c < d$, and $q(t) > 0, q_1(t) \geq 0$ for $t \in [c, d]$ and if there exists $u \in E_{c,d}$ such that*

$$\int_c^d \Big[Q_1(t)u^2(t) - \frac{1}{4}r(t)\Big(2u'(t) - \frac{p(t)}{r(t)}u(t)\Big)^2\Big]dt > r_1\Omega_c^d[u^2] \tag{2.28}$$

where $\alpha_1 > \beta + 1$, $0 < \beta \leq 1$, $Q_1(t) = q(t) - M_1 q_1(t)(\rho(t))^{\frac{\alpha_1 - 1}{\alpha_1 - \beta - 1}}$ where $M_1 = \big(\frac{\beta}{\alpha_1 - 1}\big)^{\frac{\beta}{\alpha_1-\beta-1}} \big(\frac{\alpha_1-\beta-1}{\alpha_1-1}\big)$ and $\rho(t)$ is a positive continuous function, then (2.27) *is oscillatory.*

Proof. Let $x(t)$ be a solution of (2.27). Suppose $x(t)$ does not have any zero in $[c, d]$. Without loss of generality, we may assume that $x(t) > 0$ for $t \in [c, d]$. Define

$$w(t) = -\frac{r(t)x'(t)}{x(t)}, \quad t \in [c, d].$$

Then for $t \in [c, d]$ and $t \neq \tau_k$, we have

$$w'(t) = q(t) + q_1(t)x^{\alpha_1 - 1}(t) - \frac{p(t)}{r(t)}w(t) + \frac{w^2(t)}{r(t)}, \tag{2.29}$$

or

$$\begin{aligned} w'(t) &\geq q(t) + q_1(t)x^{\alpha_1-1}(t) - q_1(t)\rho(t)x^\beta(t) - \frac{p(t)}{r(t)}w(t) + \frac{w^2(t)}{r(t)} \\ &= q(t) + q_1(t)\Big(x^{\alpha_1-1}(t) - \rho(t)x^\beta(t)\Big) - \frac{p(t)}{r(t)}w(t) + \frac{w^2(t)}{r(t)}. \end{aligned}$$

Thus by Lemma 2(i), we have

$$w'(t) \geq Q_1(t) - \frac{p(t)}{r(t)} w(t) + \frac{w^2(t)}{r(t)}. \tag{2.30}$$

Then following the proof of Theorem 2.6, we obtain a contradiction to (2.28). Hence the proof is complete. □

Theorem 2.11. *Assume that for any $T > 0$, there exist $c, d \notin \{\tau_k\}$ such that $c < d$, and $q(t) > 0, q_1(t) \geq 0$ for $t \in [c, d]$ and if there exists $u \in E_{c,d}$ such that*

$$\int_c^d \Big[Q_2(t)u^2(t) - \frac{1}{4} r(t)\Big(2u'(t) - \frac{p(t)}{r(t)} u(t)\Big)^2\Big] dt > r_1 \Omega_c^d [u^2] \tag{2.31}$$

where $0 < \alpha_1 + \alpha_2 < 1$, $Q_2(t) = q(t) - M_2 q_1(t)(\rho(t))^{\frac{1-\alpha_1}{1-\alpha_1-\alpha_2}}$ where $M_2 = \left(\frac{1-\alpha_1-\alpha_2}{1-\alpha_1}\right)\left(\frac{\alpha_2}{1-\alpha_1}\right)^{\frac{\alpha_2}{1-\alpha_1-\alpha_2}}$ and $\rho(t)$ is a positive continuous function, then (2.27) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.10, we obtain (2.29) or

$$w'(t) \geq q(t) + q_1(t)\Big(x^{\alpha_1 - 1}(t) - \rho(t) x^{-\alpha_2}(t)\Big) - \frac{p(t)}{r(t)} w(t) + \frac{w^2(t)}{r(t)}.$$

Now use Lemma 2(ii) and then proceed as in the proof of Theorem 2.6. Thus we obtain a contradiction to condition (2.31). This completes the proof. □

Remark 2.12. When $q_1(t) \equiv 0$, then the results of Theorems 2.10 and 2.11 are the same and it seems to be new. However, Theorem 2.10 and Theorem 2.11 are not applicable when $q(t) \equiv 0$. Therefore, it would be interesting to obtain results similar to Theorems 2.10 and 2.11 which are applicable to the case $q(t) \equiv 0$ and $q_1(t) \not\equiv 0$.

3. Examples

In this section, we give some examples to illustrate our results.

Example 3.1. Consider the impulsive differential equation

$$\begin{aligned} &x''(t) + \sin t x'(t) + (l \cos t) x(t) + (l_1 \sin t)|x(t)|^{\frac{3}{2}} \operatorname{sgn} x(t) \\ &+ (l_2 \cos t)|x(t)|^{1/2} \operatorname{sgn} x(t) = -\cos 2t, \quad t \neq \tau_k, \\ &x(\tau_k{}^+) = \frac{1}{2} x(\tau_k), \quad x'(\tau_k{}^+) = \frac{3}{4} x'(\tau_k), \quad \tau_k = 2k\pi + \frac{\pi}{6}, \end{aligned} \tag{3.1}$$

where $k \in \mathbb{N}$, $t \geq t_0 > 0, \tau_{2n} = 2n\pi + \frac{\pi}{6}$, $\tau_{2n+1} = 2n\pi + \frac{\pi}{3}$, $n = 0, 1, 2, \dots, l, l_1, l_2$ are positive constants. Also note that $r_1 = r_2 = 1$. Now choose $\eta_0 = \frac{1}{4}$, $\eta_1 = \frac{5}{8}$, $\eta_2 = \frac{1}{8}$ to get $k_0 = 4\frac{2^{3/4}}{5^{5/8}}$ and $Q(t) = l \cos t + 4\frac{2^{3/4}}{5^{5/8}} |-\cos 2t|^{1/4} (l_1 \sin t)^{5/8} (l_2 \cos t)^{1/8}$.

For any $T \geq 0$, we can choose n large enough such that $T < c_1 = 2n\pi < \delta_1 = 2n\pi + \frac{\pi}{8} < d_1 = 2n\pi + \frac{\pi}{4} = c_2 < \delta_2 = 2n\pi + \frac{3\pi}{8} < d_2 = 2n\pi + \frac{\pi}{2}$, $n = 0, 1, 2, \dots$.

If we choose $H_1(t, s) = H_2(t, s) = (t - s)^2$ then $h_1(t, s) = -h_2(t, s) = \frac{2}{t-s}$. Then by using the mathematical software Mathematica 5.2, the left hand side of the inequality (2.6) with $j = 1$ is

$$\frac{1}{H_1(\delta_1, c_1)} \int_{c_1}^{\delta_1} H_1(t, c_1)\Big[Q(t) - \frac{1}{4} r(t)\Big(h_1(t, c_1) - \frac{p(t)}{r(t)}\Big)^2\Big] dt$$

$$
\begin{aligned}
&+ \frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1} H_2(d_1,t)\Big[Q(t)-\frac{1}{4}r(t)\Big(h_2(d_1,t)+\frac{p(t)}{r(t)}\Big)^2\Big]dt\\
&= \frac{64}{\pi^2}\Big[l\int_{2n\pi}^{2n\pi+\frac{\pi}{8}}(\cos t)(t-2n\pi)^2dt\\
&\quad -\frac{1}{4}\int_{2n\pi}^{2n\pi+\frac{\pi}{8}}(t-2n\pi)^2\Big(\frac{2}{t-2n\pi}-\sin t\Big)^2dt + l\int_{2n\pi+\frac{\pi}{8}}^{2n\pi+\frac{\pi}{4}}(\cos t)(2n\pi+\frac{\pi}{4}-t)^2dt\\
&\quad + 4\frac{2^{3/4}}{5^{5/8}}{l_1}^{5/8}{l_2}^{1/8}\int_{2n\pi+\frac{\pi}{8}}^{2n\pi+\frac{\pi}{4}}|-\cos 2t|^{1/4}(\sin t)^{5/8}(\cos t)^{1/8}(2n\pi+\frac{\pi}{4}-t)^2dt\\
&\quad -\frac{1}{4}\int_{2n\pi+\frac{\pi}{8}}^{2n\pi+\frac{\pi}{4}}(2n\pi+\frac{\pi}{4}-t)^2\Big(\frac{-2}{2n\pi+\frac{\pi}{4}-t}+\sin t\Big)^2dt\Big]\\
&\approx 0.240013l+0.306144{l_1}^{5/8}{l_2}^{1/8}-4.72546.
\end{aligned}
$$

Note that there is no impulsive moment in (c_1,δ_1) and $\tau_{2n}\in(\delta_1,d_1)$. Also $k(\delta_1)=2n-1, k(d_1)=2n$. Hence the right side of the inequality (2.6) with $j=1$ is

$$
\begin{aligned}
\Lambda(H_1,H_2;c_1,d_1) &= \frac{r_1}{H_2(d_1,\delta_1)}\Omega_{\delta_1}^{d_1}[H_2(d_1,.)]\\
&= \frac{64}{\pi^2}H_2(d_1,\tau_{2n})\theta(\delta_1)\\
&= \frac{32}{3\pi}\Big(\frac{b_{2n}-a_{2n}}{a_{2n}}\Big)=\frac{16}{3\pi}.
\end{aligned}
$$

Thus (2.6) is satisfied with $j=1$ if

$$
0.240013l+0.306144{l_1}^{5/8}{l_2}^{1/8} > 4.72546+\frac{16}{3\pi}=6.42311.
$$

In a similar way, the left hand side of the inequality (2.6) with $j=2$ is

$$
\begin{aligned}
&\frac{1}{H_1(\delta_2,c_2)}\int_{c_2}^{\delta_2} H_1(t,c_2)\Big[Q(t)-\frac{1}{4}r(t)\Big(h_1(t,c_2)-\frac{p(t)}{r(t)}\Big)^2\Big]dt\\
&+\frac{1}{H_2(d_2,\delta_2)}\int_{\delta_2}^{d_2} H_2(d_2,t)\Big[Q(t)-\frac{1}{4}r(t)\Big(h_2(d_2,t)+\frac{p(t)}{r(t)}\Big)^2\Big]dt\\
&\approx 0.0994167l+0.48437{l_1}^{5/8}{l_2}^{1/8}-4.23605.
\end{aligned}
$$

Note that $\tau_{2n+1}\in(c_2,\delta_2)$ and there is no impulsive moment in (δ_2,d_2). Also $k(c_2)=2n$, $k(\delta_2)=2n+1$. Hence the right side of the inequality (2.6) with $j=2$ is

$$
\begin{aligned}
\Lambda(H_1,H_2;c_2,d_2) &= \frac{r_2}{H_1(\delta_2,c_2)}\Omega_{c_2}^{\delta_2}[H_1(.,c_2)]\\
&= \frac{64}{\pi^2}H_1(\tau_{2n+1},c_2)\theta(c_2)\\
&= \frac{16}{3\pi}\Big(\frac{b_{2n+1}-a_{2n+1}}{a_{2n+1}}\Big)=\frac{8}{3\pi}.
\end{aligned}
$$

Thus (2.6) is satisfied with $j=2$ if

$$
0.0994167l+0.48437{l_1}^{5/8}{l_2}^{1/8} > 4.23605+\frac{8}{3\pi}=5.08488.
$$

So, if we choose the constants l, l_1, l_2 large enough such that

$0.240013l + 0.306144{l_1}^{5/8}{l_2}^{1/8} > 6.42311, 0.0994167l + 0.48437{l_1}^{5/8}{l_2}^{1/8} > 5.08488,$

then by Theorem 2.3, equation (3.1) is oscillatory. In fact, for $l = 20$, $l_1 = 30$, $l_2 = 40$, equation (3.1) is oscillatory.

Example 3.2. Consider the impulsive differential equation

$$\begin{gathered}\Big(\frac{1}{2+\sin 2t}x'(t)\Big)' + (2\cos 4t)x'(t) + (\gamma_0\cos t)x(t) + (\gamma_1\cos 2t)|x(t)|^{5/2}\operatorname{sgn} x(t) \\ + (\gamma_2\cos 2t)|x(t)|^{1/2}\operatorname{sgn} x(t) = \sin 2t, \quad t \neq 2k\pi - \frac{\pi}{8}, \\ x({\tau_k}^+) = \frac{1}{3}x(\tau_k), \quad x'({\tau_k}^+) = \frac{2}{3}x'(\tau_k), \quad \tau_k = 2k\pi - \frac{\pi}{8},\end{gathered} \tag{3.2}$$

where $k \in \mathbb{N}$, $t \geq t_0 > 0$, γ_i, $i = 0, 1, 2$ are positive constants.

Now choose $\eta_0 = \frac{1}{2}$, $\eta_1 = 3/8$, $\eta_2 = 1/8$ to get $k_0 = \frac{4}{3^{3/8}}$ and $Q(t) = \gamma_0\cos t + \frac{4}{3^{3/8}}|\sin 2t|^{1/2}(\gamma_1\cos 2t)^{3/8}(\gamma_2\cos 2t)^{1/8}$. For any $T \geq 0$, we can choose n large enough such that $T < c_1 = 2n\pi - \frac{\pi}{4}$, $d_1 = c_2 = 2n\pi$, $d_2 = 2n\pi + \frac{\pi}{4}$, $n = 1, 2, \dots$. If we take $u_1(t) = \sin 4t$, $u_2(t) = \sin 8t$ then by using the mathematical software Mathematica 5.2, we obtain

$$\begin{aligned}&\int_{c_1}^{d_1}\Big[Q(t)u_1^2(t) - \frac{1}{4}r(t)\Big(2u_1'(t) - \frac{p(t)}{r(t)}u_1(t)\Big)^2\Big]dt \\ &= \gamma_0\int_{2n\pi-\frac{\pi}{4}}^{2n\pi}\cos t\sin^2 4t\,dt \\ &\quad + \frac{4}{3^{3/8}}{\gamma_1}^{3/8}{\gamma_2}^{1/8}\int_{2n\pi-\frac{\pi}{4}}^{2n\pi}|\sin 2t|^{1/2}(\cos 2t)^{3/8}(\cos 2t)^{1/8}\sin^2 4t\,dt \\ &\quad - \frac{1}{4}\int_{2n\pi-\frac{\pi}{4}}^{2n\pi}\frac{1}{2+\sin 2t}\Big(8\cos 4t - 2\cos 4t(2+\sin 2t)\sin 4t\Big)^2 dt \\ &\approx 0.359165\gamma_0 + 0.673369{\gamma_1}^{3/8}{\gamma_2}^{1/8} - 6.28071,\end{aligned}$$

and

$$\begin{aligned}&\int_{c_2}^{d_2}\Big[Q(t)u_2^2(t) - \frac{1}{4}r(t)\Big(2u_2'(t) - \frac{p(t)}{r(t)}u_2(t)\Big)^2\Big]dt \\ &= \gamma_0\int_{2n\pi}^{2n\pi+\frac{\pi}{4}}\cos t\sin^2 8t\,dt \\ &\quad + \frac{4}{3^{3/8}}{\gamma_1}^{3/8}{\gamma_2}^{1/8}\int_{2n\pi}^{2n\pi+\frac{\pi}{4}}|\sin 2t|^{1/2}(\cos 2t)^{3/8}(\cos 2t)^{1/8}\sin^2 8t\,dt \\ &\quad - \frac{1}{4}\int_{2n\pi}^{2n\pi+\frac{\pi}{4}}\frac{1}{2+\sin 2t}\Big(16\cos 8t - 2\cos 4t(2+\sin 2t)\sin 8t\Big)^2 dt \\ &\approx 0.35494\gamma_0 + 0.598551{\gamma_1}^{3/8}{\gamma_2}^{1/8} - 9.18481.\end{aligned}$$

Since $k(c_1) = n - 1$, $k(d_1) = n$, $r_1 = 1$ and $k(c_2) = k(d_2)$, we obtain

$$r_1\Omega_{c_1}^{d_1}[u_1^2] = \frac{8(b_n - a_n)}{\pi a_n} = \frac{8}{\pi}, \quad r_2\Omega_{c_2}^{d_2}[u_2^2] = 0.$$

So, if we choose the constants γ_0 or γ_1, γ_2 large enough such that

$$0.359165\gamma_0 + 0.673369{\gamma_1}^{3/8}{\gamma_2}^{1/8} > 6.28071 + \frac{8}{\pi} = 8.82719,$$
$$0.35494\gamma_0 + 0.598551{\gamma_1}^{3/8}{\gamma_2}^{1/8} > 9.1848,$$

then by Theorem 2.6, equation (3.2) is oscillatory. In fact, for $\gamma_0 = 40, \gamma_1 = 20, \gamma_2 = 30$, equation (3.2) is oscillatory.

Example 3.3. Consider the impulsive differential equation

$$\begin{gathered} x''(t) + (2\sin t)x'(t) + (l\sin t)x(t) + (l_1\cos t)x^5(t) = 0, \quad t \neq 2k\pi + \frac{\pi}{4}, \\ x({\tau_k}^+) = 4x(\tau_k), \quad x'({\tau_k}^+) = 5x'(\tau_k), \quad \tau_k = 2k\pi + \frac{\pi}{4}, \end{gathered} \tag{3.3}$$

where $k \in \mathbb{N}$, $t \geq t_0 > 0$, l, l_1 are positive constants. For any $T \geq 0$, we can choose n large enough such that $T < c = 2n\pi + \frac{\pi}{6}$, $d = 2n\pi + \frac{\pi}{2}$, $n = 1, 2, \dots$. Then $q(t) = l\sin t > 0$, $q_1(t) = l_1\cos t > 0$ on $[c, d]$. If we take $u(t) = \sin 6t$, $\beta = 1, \rho(t) = 4$, we have $M_1 = (\frac{1}{4})^{1/3}(\frac{3}{4})$ and $Q_1(t) = l\sin t - 4^{4/3}M_1(l_1\cos t)$. Thus by using the mathematical software Mathematica 5.2, we have

$$\begin{aligned} &\int_c^d \Big[Q_1(t)u^2(t) - \frac{1}{4}r(t)\Big(2u'(t) - \frac{p(t)}{r(t)}u(t)\Big)^2\Big]dt \\ &= l\int_{2n\pi+\frac{\pi}{6}}^{2n\pi+\frac{\pi}{2}} \sin t\sin^2 6t\,dt - 4^{4/3}\Big(\frac{1}{4}\Big)^{1/3}\Big(\frac{3}{4}\Big)l_1\int_{2n\pi+\frac{\pi}{6}}^{2n\pi+\frac{\pi}{2}} \cos t\sin^2 6t\,dt \\ &\quad - \frac{1}{4}\int_{2n\pi+\frac{\pi}{6}}^{2n\pi+\frac{\pi}{2}} \Big(12\cos 6t - 2\sin t\sin 6t\Big)^2 dt \\ &\approx 0.436041l - 0.755245l_1 - 19.4744. \end{aligned}$$

Since $k(c) = n-1, k(d) = n, r_1 = 1$ and $k(c) < k(d)$, we obtain

$$r_1\Omega_c^d[u^2] = \frac{12}{\pi}\Big(\frac{b_n - a_n}{a_n}\Big) = \frac{3}{\pi}.$$

So, if we choose the constants l, l_1 such that

$$0.436041l - 0.755245l_1 > 19.4744 + \frac{3}{\pi} = 20.4293$$

then by Theorem 2.10, equation (3.3) is oscillatory. In fact, for $l = 50, l_1 = 0.01$, equation (3.3) is oscillatory.

Example 3.4. Consider the impulsive differential equation

$$\begin{gathered} x''(t) - (\sin 2t)x'(t) + ke^{t/2}x(t) + k_1e^{t/4}x^3(t) = 0, \quad t \neq 2k\pi, \\ x({\tau_k}^+) = \frac{1}{4}x(\tau_k), \quad x'({\tau_k}^+) = \frac{1}{2}x'(\tau_k), \quad \tau_k = 2k\pi, \end{gathered} \tag{3.4}$$

where $k \in \mathbb{N}$, $t \geq t_0 > 0$, k, k_1 are positive constants. Note that $\alpha_1 = \frac{1}{3}$, $p(t) = -\sin 2t$. For any $T \geq 0$, we can choose n large enough such that $T < c = 2n\pi - \frac{\pi}{4}$, $d = 2n\pi + \frac{\pi}{4}$, $n = 1, 2, \dots$. Then $q(t) = ke^{t/2} > 0$, $q_1(t) = k_1e^{t/4} > 0$ on $[c, d]$. If we take $u(t) = \cos 2t$, $\alpha_2 = \frac{1}{3}$, $\rho(t) = 3$, we have $M_2 = \frac{1}{4}$ and $Q_2(t) = ke^{t/2} - \frac{9}{4}k_1e^{t/4}$. Thus by using the mathematical software Mathematica 5.2, we have

$$\int_c^d \Big[Q_2(t)u^2(t) - \frac{1}{4}r(t)\Big(2u'(t) - \frac{p(t)}{r(t)}u(t)\Big)^2\Big]dt$$

$$
\begin{aligned}
&= k\int_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}} e^{t/2}\cos^2 2t\,dt - \frac{9}{4}k_1\int_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}} e^{t/4}\cos^2 2t\,dt\\
&\quad - \frac{1}{4}\int_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}} \Big(-4\sin 2t + (\sin 2t)(\cos 2t)\Big)^2 dt\\
&\approx 18.3585k - 8.52225k_1 - 2.52401.
\end{aligned}
$$

Since $k(c) = n-1$, $k(d) = n$, $r_1 = 1$ and $k(c) < k(d)$, we obtain

$$
r_1\Omega_c^d[u^2] = \frac{4}{\pi}.
$$

So, if we choose the constants k, k_1 such that

$$
18.3585k - 8.52225k_1 > 2.52401 + \frac{4}{\pi} = 3.79725
$$

then by Theorem 2.11, equation (3.4) is oscillatory. In fact, for $k = k_1 = 1$, equation (3.4) is oscillatory.

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