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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR DISCRETE PROBLEMS WITH P-LAPLACIAN VIA VARIATIONAL METHODS

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Abstract. Using critical point theory, we prove the existence of multiple positive solutions for second-order discrete boundary-value problems with p-Laplacian.

1. Introduction

In recent years, a great deal of work has been done in the study of the existence of multiple positive solutions for discrete boundary value problems describing physical and biological phenomena. For the background and summary of results, we refer the reader to the monograph by Agarwal et al [2], and for some recent contributions to [1, 3]. Various fixed point theorems have been applied for obtaining solutions, among them, Krasnosel'skii fixed point theorem, Leggett-Williams fixed point theorem, fixed point theorem in cones; see [4, 5, 8, 10, 13] and the references therein.

There is also a trend to study difference equation using variational methods which lead to many interesting results; see for example [3, 6, 9, 14]. Li [9] studied the existence of solutions for the problem

$$\begin{gathered} \Delta(p(k)\Delta x(k-1)) + f(k,x(k)) = g(k) \\ x(0) = x(T+1) = 0, \end{gathered} \tag{1.1}$$

where $f \in C(\mathbb{R}^2,\mathbb{R})$, $p, g \in C(\mathbb{R},\mathbb{R})$. Using variational methods, the existence of at least one non-trivial solution was obtained. Agarwal et al [3] show the existence of multiple positive solutions for the discrete boundary-value problem

$$\begin{gathered} \Delta^2 y(k-1) + f(k,y(k)) = 0, \quad k \in [1,T], \\ y(0) = 0 = y(T+1), \end{gathered} \tag{1.2}$$

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where $[1,T]$ is the discrete interval $\{1,2,\dots,T\}$, $\Delta y(k)=y(k+1)-y(k)$, $f\in C([1,T]\times[0,\infty),\mathbb{R})$ satisfies $f(k,0)\geq 0$, for all $k\in[1,T]$. They applied critical point theory under the following conditions:

(a) $\min_{k\in[1,T]}\liminf_{u\to\infty}\frac{f(k,u)}{u}>\lambda_1$, where λ_1 is the smallest eigenvalue of $\Delta^2y(k-1)+\lambda y(k)=0$, $y\in H$;

(b) there is a positive constant M, independent of λ, such that $\|y\|\neq M$ for every solution $y\geq 0$ of the equation
$$\Delta^2y(k-1)+\lambda f(k,y(k))=0,\quad y\in H,\ \lambda\in(0,1].$$

We remark that is not easy to verify Condition (b) in applications.

To the best of our knowledge, very few authors have studied the existence of multiple positive solutions for discrete boundary value problem with a p-Laplacian by using variational methods. As a result the goal of this paper is to fill the gap in this area. It is well known that positive solutions are very important in applications. Motivated by the above results, in this paper, we study the existence of multiple positive solutions for the second-order discrete boundary-value problem (BVP)
$$\begin{gathered}\Delta(\Phi_p(\Delta y(k-1)))+f(k,y(k))=0,\quad k\in[1,T],\\ y(0)=0=y(T+1),\end{gathered}\tag{1.3}$$
where T is a positive integer, $[1,T]$ is the discrete interval $\{1,\dots,T\}$ and $\Delta y(k)=y(k+1)-y(k)$ is the forward difference operator, $p>1$, $\Phi_p(y):=|y|^{p-2}y$, $f\in C([1,T]\times[0,+\infty),[0,+\infty))$, $f(k,0)\not\equiv 0$ for $k\in[1,T]$, $F(k,x)=\int_0^x f(k,s)ds$. For a review of variational methods, we refer the reader to [11, 12].

Our aim of this paper is to apply critical point theory to (1.3) and prove the existence of two positive solutions. We impose some conditions on the nonlinearity f that are different from those in [2] for $p=2$, and are easy to verify.

In this article, we assume the following conditions:

(C1) there exist $\mu>p$, $h\in C([1,T]\times[0,+\infty),[0,+\infty))$, $l:[1,T]\to(0,+\infty)$, $\min_{k\in[1,T]}l(k)>0$ such that
$$f(k,y)=l(k)\Phi_\mu(y)+h(k,y);$$

(C2) there exist functions $c,d:[1,T]\to[0,+\infty)$ such that
$$h(k,y)\leq c(k)+d(k)\Phi_p(y).$$

2. Related Lemmas

Here, and in the sequel, we denote
$$Y=W_0^{1,p}[0,T+1]=\{y:[0,T+1]\to R: y(0)=y(T+1)=0\}$$
whihc is a T-dimensional Banach space with the norm
$$\|y\|=\Big(\sum_{k=1}^{T+1}|\Delta y(k-1)|^p\Big)^{1/p}.$$

Lemma 2.1. *Let $y^{\pm}=\max\{\pm y,0\}$, then the following five properties hold:*

(i) $y=y^+-y^-$;

(ii) $\|y^+\|\leq\|y\|$;

(iii) $y^+(t)y^-(t)=0,(y^+)'(t)(y^-)'(t)=0$ *for* $t\in[0,T+1]$;

(iv) $\Phi_p(y)y^+=|y^+|^p$, $\Phi_p(y)y^-=-|y^-|^p$.

Lemma 2.2. *If y is a solution of the equation*
$$\Delta(\Phi_p(\Delta y(k-1))) + f(k, y^+(k)) = 0, \quad y \in Y, \tag{2.1}$$
then $y \geq 0$, $y(k) \not\equiv 0$, $k \in [0, T+1]$ and hence it is a solution of (1.3).

Proof. If y is a solution of (2.1), then
$$\begin{aligned}
0 &= \sum_{k=1}^{T} \big[\Delta(\Phi_p(\Delta y(k-1))) + f(k, y^+(k))\big]\, y^-(k) \\
&= \Phi_p(\Delta y(k-1))y^-(k)|_{k=1}^{T+1} - \sum_{i=1}^{T} \Phi_p(\Delta y(k))\Delta y^-(k) + \sum_{k=1}^{T} f(k, y^+(k))y^-(k) \\
&\geq -\Phi_p(y(1))y^-(1) + \sum_{k=1}^{T} |\Delta y^-(k)|^p \\
&= |y^-(1)|^p + \sum_{k=2}^{T+1} |\Delta y^-(k-1)|^p,
\end{aligned} \tag{2.2}$$
so $\Delta y^-(k) = 0$, $k \in [1, T]$ and $y^-(1) = 0$, which yield that $y^-(k) = 0, k \in [1, T+1]$; that is, $y \geq 0$. If $y(k) = 0$ for every $k \in [0, T+1]$, the fact $f(k, 0) \not\equiv 0$ for every $k \in [1, T]$ gives a contradiction. □

Remark 2.3. By Lemma 2.2, to find positive solutions of (1.3) it suffices to obtain solutions of (2.1).

For $y \in Y$, put
$$\varphi(y) := \sum_{k=1}^{T+1} \big[\frac{1}{p}|\Delta y(k-1)|^p - F(k, y^+(k)) + f(k, 0)y^-(k)\big]. \tag{2.3}$$
Clearly, the functional φ is C^1 with
$$\langle \varphi'(y), z\rangle = \sum_{k=1}^{T+1} \big[\Phi_p(\Delta y(k-1))\Delta z(k-1) - f(k, y^+(k))z(k)\big] \tag{2.4}$$
for every $z \in Y$. So the solutions of (2.1) are precisely the critical points of the functional φ.

Lemma 2.4. *For $y \in Y$, we have $\|y\|_\infty \leq (T+1)^{1/q}\|y\|$, where*
$$\|y\|_\infty = \max_{i \in [0, T+1]} |y(i)|.$$

Proof. For $y \in Y$, it follows from Hölder's inequality, that
$$\begin{aligned}
|y(k)| &= \big|y(0) + \sum_{i=0}^{k-1} \Delta y(i)\big| \leq \sum_{i=0}^{T} |\Delta y(i)| \\
&\leq (T+1)^{1/q}\Big(\sum_{i=0}^{T} |\Delta y(i)|^p\Big)^{1/p} = (T+1)^{1/q}\|y\|,
\end{aligned}$$
which completes the proof. □

Lemma 2.5 ([15, Theorem 38.A]). *For the functional $F : M \subseteq X \to [-\infty, +\infty]$ with $M \neq \emptyset$, $\min_{u \in M} F(u) = \alpha$ has a solution when the following conditions hold:*

(i) *X is a real reflexive Banach space;*
(ii) *M is bounded and weak sequentially closed; i.e., by definition, for each sequence (u_n) in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we always have $u \in M$;*
(iii) *F is weak sequentially lower semi-continuous on M.*

Lemma 2.6 ([6]). *Let E be a Banach space and $\varphi \in C^1(E,R)$ satisfy Palais-Smale condition. Assume there exist $x_0, x_1 \in E$, and a bounded open neighborhood Ω of x_0 such that $x_1 \notin \overline{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Let $\Gamma = \{h : h : [0,1] \to E$ is continuous, $h(0) = x_0, h(1) = x_1\}$ and

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \varphi(h(s)).$$

Then c is a critical value of φ; that is, there exists $x^* \in E$ such that $\varphi'(x^*) = \Theta$ and $\varphi(x^*) = c$, where $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

Lemma 2.7. *Suppose that* (C1), (C2) *hold. Furthermore, we assume*

(C3) $(T+1)^{p/q} \sum_{k=1}^{T+1} d(k) < \frac{\mu}{p} - 1$.

Then the functional φ satisfies Palais-Smale condition; i.e., every sequence $\{y_n\}$ in Y satisfying $\varphi(y_n)$ is bounded and $\varphi'(y_n) \to 0$ has a convergent subsequence.

Proof. Since Y is a finite dimensional Banach space, we only need to show that (y_n) is a bounded sequence in Y.

For this, by Lemma 2.1 (iv) and (2.4) we have

$$\begin{aligned} \langle \varphi'(y_n), y_n^- \rangle &= \sum_{k=1}^{T+1} \left[\Phi_p(\Delta y_n(k-1))\Delta y_n^-(k-1) - f(k, y_n^+(k))y_n^-(k)\right] \\ &\le -\sum_{k=1}^{T+1} |\Delta y_n^-(k-1)|^p = -\|y_n^-\|^p. \end{aligned} \tag{2.5}$$

Set $w_n^- = \frac{y_n^-}{\|y_n^-\|}$. Dividing by $\|y_n^-\|$ on the both sides of the above inequality, we have

$$\|y_n^-\|^{p-1} \le -\langle \varphi'(y_n), w_n^- \rangle \to 0 \quad \text{as } n \to \infty.$$

So $y_n^- \to 0$ in Y.

Now we show that (y_n^+) is bounded. By (2.3) (2.4) we have

$$\begin{aligned} \frac{\mu}{p}\|y_n\|^p - \|y_n^+\|^p &= \mu\varphi(y_n) - \langle \varphi'(y_n), y_n^+ \rangle - \sum_{k=1}^{T+1} \mu f(k,0) y_n^+(k) \\ &\quad + \sum_{k=1}^{T+1} \left[\mu F(k, y_n^+(k)) - f(k, y_n^+(k))y_n^+(k)\right]. \end{aligned} \tag{2.6}$$

By (C1) (C2) Lemma 2.4 one has

$$\begin{aligned} &\sum_{k=1}^{T+1}\left[\mu F(k,y_n^+(k))-f(k,y_n^+(k))y_n^-(k)\right]\\ &\le \sum_{k=1}^{T+1}\left[c(k)y_n^+(k)+d(k)|y_n^+(k)|^p\right] \\ &\le (T+1)^{1/q}\|y_n^+\|\sum_{k=1}^{T+1}c(k)+(T+1)^{p/q}\|y_n^+\|^p\sum_{k=1}^{T+1}d(k). \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6), in view of Lemma 2.1 (ii), one has

$$\begin{aligned} \big(\frac{\mu}{p}-1\big)\|y_n^+\|^p \le \mu\varphi(y_n)-\langle\varphi'(y_n),y_n^+\rangle+(T+1)^{1/q}\|y_n^+\|\sum_{k=1}^{T+1}c(k)\\ +(T+1)^{p/q}\|y_n^+\|^p\sum_{k=1}^{T+1}d(k). \end{aligned} \tag{2.8}$$

Suppose that (y_n^+) is unbounded. Passing to a subsequence, we may assume if necessary, that $\|y_n^+\|\to\infty$ as $n\to\infty$. Dividing the both sides of (2.8) by $\|y_n^+\|^p$, denoting $w_n^+=\frac{y_n^+}{\|y_n^+\|}$, we have

$$\begin{aligned} \frac{\mu}{p}-1 \le \frac{\mu\varphi(y_n)}{\|y_n^+\|^p}-\frac{\langle\varphi'(y_n),w_n^+\rangle}{\|y_n^+\|^{p-1}}+(T+1)^{1/q}\|y_n^+\|^{1-p}\sum_{k=1}^{T+1}c(k)\\ +(T+1)^{p/q}\sum_{k=1}^{T+1}d(k). \end{aligned} \tag{2.9}$$

Since $\varphi(y_n)$ is bounded and $\varphi'(y_n)\to 0, y_n^-\to 0$ in Y, let $n\to\infty$, we have

$$\frac{\mu}{p}-1\le (T+1)^{p/q}\sum_{k=1}^{T+1}d(k),$$

which contradicts to (C3). Therefore, (y_n) is bounded in Y. □

3. Main Results

Theorem 3.1. *Suppose that* (C1)–(C3) *hold. Furthermore, we assume*

(C4) $(T+1)^{\frac{\mu}{q}}\sum_{k=1}^{T}b(k)+(T+1)^{1/q}\sum_{k=1}^{T}c(k)+(T+1)^{p/q}\sum_{k=1}^{T}d(k)<1.$

Then (1.3) *has two positive solutions* x_0, x^*.

Proof. By Lemma 2.7 the functional φ satisfies Palais-Smale condition. Now we shall show that there exists $R>0$ such that the functional φ has a local minimum $x_0\in B_R:=\{x\in X:\|x\|<R\}$.

Let $R=1$. First we claim that the functional φ has a minimum on $\overline{B}_R$. Clearly $\overline{B}_R$ is a bounded and weak sequentially closed. Now we claim that φ has a minimum $x_0\in\overline{B}_R$. We will show that φ is weak sequentially lower semi-continuous on $\overline{B}_R$. For this, let

$$\varphi^1(y)=\frac{1}{p}\sum_{k=1}^{T+1}|\Delta y(k-1)|^p,\quad \varphi^2(y)=\sum_{k=1}^{T+1}\left[-F(k,y^+(k))+f(k,0)y^-(k)\right],$$

then $\varphi(y) = \varphi^1(y) + \varphi^2(y)$. By $y_n \rightharpoonup y$ on Y we have (y_n) uniformly converges to y in $C([0, T+1])$. So φ^2 is weak sequentially continuous. Clearly φ^1 is continuous, which together with the convexity of φ^1 we have φ^1 is weak sequentially lower semi-continuous. Therefore, φ is weak sequentially lower semi-continuous on B_R. Besides, Y is a reflexive Banach space, $\overline{B}_R$ is a bounded and weak sequentially closed, so our claim follows from Lemma 2.5.

If $y_0 \in \partial B_R$ and y_0 is a local minimum of the functional φ, then it is also a minimizer of $\varphi|_{\partial B_R}$, so the gradient of φ at y_0 point is in the direction of the inward normal to ∂B_R. Since $y_0 \in \partial B_R = \partial B_1$ is a local minimum of the functional φ, $\varphi(y)$ have a conditional minimum at the point y_0 about the condition $\varphi(y) = \frac{1}{p}(\|y\|^p - 1)$. By [6], there exists $\gamma \in [0, \infty)$ such that

$$\langle \varphi'(y_0), v\rangle = -\gamma\langle \psi'(y_0), v\rangle \quad \text{for all } v \in Y.$$

That is,

$$\Delta(\Phi_p(\Delta y_0(k-1))) + \lambda f(k, y_0^+(k)) = 0, \quad y_0 \in Y \tag{3.1}$$

with $\lambda = \frac{1}{1+\gamma} \in (0, 1]$, $\|y_0\| = R = 1$ holds.

Multiplying $y_0(t)$ on the both sides of equation in (3.1), then summing on $[1, T]$, we have

$$\begin{aligned}
0 &= \sum_{k=1}^{T} \big[\Delta(\Phi_p(\Delta y_0(k-1))) + \lambda f(k, y_0^+(k))\big] \times y_0(k) \\
&= \Phi_p(\Delta y_0(k-1))y_0(k)|_{k=1}^{T+1} - \sum_{k=1}^{T} \Phi_p(\Delta y_0(k))\Delta y_0(k) + \sum_{k=1}^{T} \lambda f(k, y_0^+(k))y_0(k) \\
&= -\Phi_p(y_0(1))y_0(1) - \sum_{k=1}^{T} |\Delta y_0(k)|^p + \sum_{k=1}^{T} \lambda f(k, y_0^+(k))y_0(k) \\
&\le -\|y_0\|^p + \sum_{k=1}^{T} \lambda f(k, y_0^+(k))y_0(k).
\end{aligned}$$

Then

$$\begin{aligned}
\|y_0\|^p &\le \sum_{k=1}^{T} \lambda f(k, y_0(k))y_0(k) \\
&\le \sum_{k=1}^{T} b(k)|y_0(k)|^\mu + c(k)y_0(k) + d(k)|y_0(k)|^p \\
&\le (T+1)^{\frac{\mu}{q}}\|y_0\|^\mu \sum_{k=1}^{T} b(k) + (T+1)^{1/q}\|y_0\|^{1/p} \sum_{k=1}^{T} c(k) \\
&\quad + (T+1)^{p/q}\|y_0\|^p \sum_{k=1}^{T} d(k).
\end{aligned}$$

Since $\|y_0\| = 1$, we have

$$1 \le (T+1)^{\mu/q} \sum_{k=1}^{T} b(k) + (T+1)^{1/q} \sum_{k=1}^{T} c(k) + (T+1)^{p/q} \sum_{k=1}^{T} d(k),$$

which contradicts (C4). Therefore, for any $\lambda \in (0,1]$, the solution of (3.1) is not on ∂B_R. Therefore, $y_0 \in B_R$ and hence it is a local minimizer of φ, and $\varphi(y_0) < \min_{y\in\partial B_R} \varphi(y)$.

Next we show that there exists y_1 with $\|y_1\| > R = 1$ such that $\varphi(y_1) < \min_{y\in\partial B_R} \varphi(y)$. Let $\widetilde{e}(k) = 1 \in Y$. Then

$$\begin{aligned}
\varphi(\overline{\lambda}\widetilde{e}) &\le -\sum_{k=1}^{T}[F(k,\overline{\lambda}) - f(k,0)\overline{\lambda}] \\
&= -\sum_{k=1}^{T}\big[\frac{l(k)\overline{\lambda}^{\mu}}{\mu} + H(k,\overline{\lambda}) - f(k,0)\overline{\lambda}\big] \\
&\le -\sum_{k=1}^{T}\frac{l(k)\overline{\lambda}^{\mu}}{\mu} + \sum_{k=1}^{T}\Big[c(k)\overline{\lambda} + d(k)\overline{\lambda}^{p} + f(k,0)\overline{\lambda}\Big].
\end{aligned} \tag{3.2}$$

Since $\mu > p$, we have $\lim_{\overline{\lambda}\to+\infty} \varphi(\overline{\lambda}\widetilde{e}) = -\infty$. So there exists sufficiently large $\overline{\lambda}_0$ with $\|\overline{\lambda}_0\widetilde{e}\| > R$ such that $\varphi(\overline{\lambda}_0\widetilde{e}) < \min_{y\in\partial B_R} \varphi(y)$.

Lemma 2.6 now gives the critical value

$$c = \inf_{h\in\Gamma}\max_{t\in[0,1]} \varphi(h(t)),$$

where $\Gamma = \{h : h : [0,1] \to E \text{ is continuous}, h(0) = y_0, h(1) = y_1\}$; that is, there exists $y^* \in Y$ such that $\varphi'(y^*) = 0$. Therefore, y_0, y^* are two critical points of φ, and hence they are classical solutions of (2.1). Lemma 2.2 means y_0, y^* are positive solutions of problem (1.3). □

Corollary 3.2. *Suppose that* (C1) (C4) *hold. Moreover we assume*

(C2') *there exists* $0 \le s < p$, $c \in L^1([a,b],[0,+\infty))$, $d \in C([a,b],[0,+\infty))$ *such that*

$$h(t,x) \le c(t) + d(t)\Phi_s(x).$$

Then (1.3) *has two positive solutions* x_0, x^*.

References

[1] R. P. Agarwal; *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] R. P. Agarwal, O'Regan, P. J. Y. Wong; *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, 1999.

[3] Ravi P. Agarwal, Kanishka Perera, Donal O'Regan; *Multiple positive solutions of singular and nonsingular discrete problems via variational methods*, Nonlinear Anal. 58(2004), 69-73.

[4] R. I. Avery, A. C. Peterson; *Multiple positive solutions of a discrete second order conjugate problem*, Panamer. Math. J., 1-12, (1998).

[5] R. I. Avery, A. C. Peterson; *Three positive fixed points of nonlinear operators on ordered Banach space*, Comput. Math. Appl., 42(2001)313-322.

[6] Guo Dajun; *Nonlinear Functional Analysis*, Shandong science and technology Press, 1985.

[7] Zhiming Guo, Jianshe Yu; *Existence of periodic and subharmonic solutions for second-order superlinear difference equations*, Science in China, Vol. 46, No. 4(2003), 506-515.

[8] J. Henderson, H. B. Thompson; *Existence of multiple solutions for second-order discrete boundary value problems*, Comput. Math. Applic., 43(2002)1239-1248.

[9] Yongjin Li; *The existence of solutions for second-order difference equations*, J. Difference Equ. Appl., Vol. 12, No. 2 (2006), 209-212.

[10] R. Y. Ma and Y. N. Raffoul; *Positive solutions of three-point nonlinear discrete second order boundary value problem*, J. Difference Equations Applications, 10 (2): 129-138, 2004.

[11] J. Mawhin, M. Willem; *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, Berlin, 1989.

[12] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applicatins to Differential Equations*, in CBMS Regional Conf. Ser. in Math., vol. 65, American Mathematical Society, Providence, RI, 1986.
[13] Yu Tian, Dexiang Ma, Weigao Ge; *Multiple positive solutions of four point boundary value problems for finite difference equations*, J. Difference Equ. Appl., Vol. 12, No. 1 (2006) 57-68.
[14] Jianshe Yu, Zhiming Guo, Xingfu Zou; *Periodic solutions of second order self-adjoint difference equations*, J. London Math. Soc. (2) 71 (2005), 146-160.
[15] E. Zeidler; *Nonlinear Functional Analysis and its Applications*, Vol. III, Springer, 1985.

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