EXPLICIT SOLUTIONS FOR A SYSTEM OF FIRST-ORDER
PARTIAL DIFFERENTIAL EQUATIONS-II

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Abstract. In this note we give an explicit formula for the solution of con-
servative form of a system studied in a previous article [6], in the domain
\( \{ (x,t) : x > 0, t > 0 \} \) with initial conditions at \( t = 0 \) and with Bardos Leroux
Nedelec boundary conditions at \( x = 0 \).

1. Introduction

In this note we consider the conservative form of a system considered in [6],
namely
\[
\begin{align*}
    u_t + f(u)_x &= 0, \\
    v_t + (f'(u)v)_x &= 0,
\end{align*}
\]
with \( f''(u) > 0 \), in the domain \( \Omega = \{ (x,t) : x > 0, t > 0 \} \). We give an explicit
formula for the solution of (1.1) with prescribed initial conditions
\[
\begin{pmatrix}
    u(x,0) \\
    v(x,0)
\end{pmatrix} = \begin{pmatrix}
    u_0(x) \\
    v_0(x)
\end{pmatrix},
\]
at \( t = 0 \), the Bardos Leroux and Nedelec boundary condition for \( u \)
either \( u(0+,t) = u^+_b(t) \)
or \( f'(u(0+,t)) \leq 0 \) and \( f(u(0+,t)) \geq f(u^+_b(t)) \),
and a weak form of Dirichlet boundary conditions for \( v \)
if \( f'(u(0+,t)) > 0 \), then \( v(0+,t) = v_b(t) \).

Here \( u^+_b(t) = \max\{u_b(t), \lambda\} \), where \( \lambda \) is the unique point where \( f'(u) \) changes sign.

In [6], explicit solution was constructed for the system where the second equation
in (1.1) was replaced by
\[
V_t + f'(u)V_x = 0
\]
with the weak form of Dirichlet boundary condition \( V(0,t) = V_b(t) \). Taking deriva-
tive of (1.5) with respect to \( x \) and setting \( v = V_x \) we obtain the conservative
equation for \( v \). In this note we give Dirichlet boundary condition for \( v \), which
is equivalent to giving Neumann Boundary condition for \( V \). Here we explain the

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required modification of the formula in [4] in the construction of solution to [1.1]-
[1.4].
LeFloch [8] was the first who studied the system [1.1] when \( f(u) \) is strictly
convex and constructed explicit formula for the pure initial value problem using
Lax formula. One important property of the system is the formation of \( \delta \) - wave
solutions for certain types of initial data which are of bounded variation. Such
systems come in applications, for example, the special case \( f(u) = u^2/2 \) in [1.1],
is the one-dimensional model in the large scale structure formation. Initial value
problem for this quadratic case was also studied by Joseph [2, 3] by different way,
using the vanishing viscosity method and Hopf-Cole transformation.

2. A FORMULA FOR THE SOLUTION IN THE QUARTER PLANE

We consider the system [1.1] with initial condition [1.2] and boundary condition
[1.3] and [1.4]. We assume \( u_0(x) \) is bounded measurable and \( v_0(x) \) is Lipschitz
continuous functions of \( x \geq 0 \), \( u_b(t) \) and \( v_b(t) \) are Lipschitz continuous functions of
t > 0.

We assume the flux \( f(u) \) satisfies the conditions
\[
f''(u) > 0, \quad \lim_{u \to -\infty} \frac{f(u)}{u} = \infty,
\]
and let \( f^*(u) \) be the convex dual of \( f(u) \) namely, \( f^*(u) = \max_{\theta \in \mathbb{R}^1} \{ \theta u - f(\theta) \} \).

As in [6], we introduce some notation and describe the construction of \((u, v)\) and
then verify it is a solution. For each fixed \((x, y, t)\), \( x \geq 0, y \geq 0, t > 0 \), \( C(x, y, t) \)
denotes the following class of paths \( \beta \) in the quarter plane \( \Omega = \{(z, s) : z \geq 0, s \geq 0 \} \).
Each path is connected from the initial point \((y, 0)\) to \((x, t)\) and is of the form
\( z = \beta(s) \), where \( \beta \) is a piecewise linear function of maximum three lines and always
linear in the interior of \( \Omega \). Thus for \( x > 0 \) and \( y > 0 \), the curves are either a straight
line or have exactly three straight lines with one lying on the boundary \( x = 0 \). For
\( y = 0 \) the curves are made up of one straight line or two straight lines with one
piece lying on the boundary \( x = 0 \). Associated with the flux \( f(u) \) and boundary
data \( u_b(t) \), we define the functional \( J(\beta) \) on \( C(x, y, t) \)
\[
J(\beta) = -\int_{\{s: \beta(s) = 0\}} f(u_B(s)^+)ds + \int_{\{s: \beta(s) \neq 0\}} f^*(\frac{d\beta(s)}{ds})ds.
\]
We call \( \beta_0 \) straight line path connecting \((y, 0)\) and \((x, t)\) which does not touch the boundary \( x = 0 \), \{\((0, t), t > 0\}\), then let
\[
A(x, y, t) = J(\beta_0) = tf^*(\frac{x-y}{t}).
\]
For any \( \beta \in C^*(x, y, t) = C(x, y, t) - \{\beta_0\} \), that is made up of three straight lines
connecting \((y, 0)\) to \((0, t_2)\) in the interior and \((0, t_2)\) to \((0, t_1)\) on the boundary and
\((0, t_1)\) to \((x, t)\) in the interior, \( t_2 < t_1 < t \), it can be easily seen that
\[
J(\beta) = J(x, y, t_1, t_2) = -\int_{t_2}^{t_1} f(u_B(s)^+)ds + t_2 f^*(\frac{y}{t_2}) + (t_1 - t_2) f^*(\frac{x}{t_1}).
\]
For the curves made up of two straight lines with one piece lying on the boundary
\( x = 0 \) which connects \((0, 0)\) and \((0, t_1)\) and the other connecting \((0, t_1)\) to \((x, t)\).
\[
J(\beta) = J(x, y, t_1, t_2 = 0) = -\int_0^{t_1} f(u_B(s)^+)ds + (t_1 - t_2) f^*(\frac{x}{t_1}).
\]
In the following, we list some facts which was proved in [3], that are used later in the construction of solution, which follow from some basic convex analysis and arguments of Lax [4].

There exists a $\beta^{**} \in C^*(x, y, t)$ or correspondingly $t_1(x, y, t), t_2(x, y, t)$ so that

$$B(x, y, t) = J(\beta^{**}) = J(x, y, t, t_1(x, y, t), t_2(x, y, t)) = \min\{J(\beta) : \beta \in C^*(x, y, t)\}$$

is a locally Lipschitz continuous function of $(x, y, t), x \geq 0, y \geq 0, t \geq 0$.

Secondly, the functions

$$Q(x, y, t) = \min\{J(\beta) : \beta \in C(x, y, t)\} = \min\{A(x, y, t), B(x, y, t)\},$$

and

$$U(x, t) = \min\{Q(x, y, t) + U_0(y), 0 \leq y < \infty\}$$

are locally Lipschitz continuous functions in their variables, where we have taken $U_0(y) = \int_0^y u_0(z)dz$.

Thirdly minimum in (2.1) is attained at some value of $y \geq 0$ which depends on $(x, t)$, we call it $y(x, t)$. For each fixed $t > 0$, this minimizer is unique except for a countable number of points of $x > 0$.

Finally, for each fixed $t > 0$, except for one point of $x$, either $A(x, y(x, t), t) < B(x, y(x, t), t)$ or $A(x, y(x, t), t) > B(x, y(x, t), t)$. If $A(x, y(x, t), t) < B(x, y(x, t), t),

$$U(x, t) = tf^*(\frac{x - y(x, t)}{t}) + U_0(y),$$

and if $A(x, y(x, t), t) > B(x, y(x, t), t),

$$U(x, t) = J(x, y(x, t), t, t_1(x, y(x, t), t), t_2(x, y(x, t), t)) + U_0(y).$$

Here and hence forth $y(x, t)$ is a minimizer in (2.1) and we denote $A(x, t) = A(x, y(x, t), t), B(x, t) = B(x, y(x, t), t), t_2(x, t) = t_2(x, y(x, t), t)$ and $t_1(x, t) = t_1(x, y(x, t), t)$.

**Theorem 2.1.** Assume $u_0$ is bounded measurable and locally Lipschitz continuous, $v_0$ is Lipschitz continuous in $x \geq 0$ and $u_0(t)$ ans $v_0(t)$ are Lipschitz continuous functions. Then for every $\{(x, t), x \geq 0, t > 0, U(x, t)\}$ defined by the minimization problem (2.1) is a locally Lipschitz continuous function. For almost every $(x, t)$ there is only one minimizer $y(x, t)$ and let $t_1(x, t)$ and $t_2(x, t)$ as described before. Define

$$u(x, t) = \begin{cases} (f^*)'(\frac{x - y(x, t)}{t}) & \text{if } A(x, t) < B(x, t), \\ (f^*)'(\frac{t - t_1(x, t)}{t_1}) & \text{if } A(x, t) > B(x, t), \end{cases}$$

(2.2)

and

$$V(x, t) = \begin{cases} f^0(x, t) v_0(z)dz, & \text{if } A(x, t) < B(x, t), \\ -f^t(x, t) f'(v_0^+(s))v_0(s)ds, & \text{if } A(x, t) > B(x, t), \end{cases}$$

(2.3)

and set

$$v(x, t) = \partial_x(V(x, t)).$$

(2.4)

Then the function $(u(x, t), v(x, t))$ is a weak solution of (1.1), satisfying the initial condition (1.2) and boundary conditions (1.3) and (1.4). Further $u$ satisfies the entropy condition $u(x-, t) \geq u(x+, t)$ for $x > 0, t > 0$. 
Proof. The proof is by direct verification and most part is identical to [4] and so that part is omitted. We give here only the verification of the boundary condition [1.4].

Suppose $f'(u(0+, t)) > 0$ then $f'(u(x, t)) > 0$ for $0 < x \leq \epsilon$ for some sufficiently small $\epsilon$ and $u$ and $v$ are given by (2.2)-(2.4). Then $t_2(x, t)$ is constant for $x \in [0, \epsilon)$ and

$$u(x, t) = (f')'(\frac{x}{t - t_1(x, t)}),$$

so that $t - t_1(x, t) = \frac{x}{f'(u(x, t))}$. It follows that $\lim_{x \to 0} t_1(x, t) = t$, since we assumed that

$$\lim_{x \to 0} f'(u(x, t)) = f'(u(0+, t)) = f'(u_b(t)) > 0.$$ \hspace{1cm} (2.5)

Now

$$v(x, t) = -\partial_t \int_{t_2(x, t)}^{t_1(x, t)} f'(u_b^+)(s)v_b(s) ds = -f'(u_b(t_1(x, t)))v_b(t_1(x, t))\partial_x t_1(x, t)$$

Again differentiating the relation $t - t_1(x, t) = \frac{x}{f'(u(x, t))}$ with respect to $x$, we have

$$\partial_x t_1(x, t) = \frac{xf''(u(x, t))u_x - f'(u(x, t))}{(f'(u(x, t)))^2}.$$ \hspace{1cm} (2.7)

By (2.5)-(2.7) and using the fact $\lim_{x \to 0} t_1(x, t) = t$, we get the weak boundary condition [1.4].

**Explicit formula for Riemann initial boundary value problem.** It is illustrative to compute the solution constructed in the above theorem for the Riemann type initial boundary data, namely $u_0$, $v_0$, $u_b$ and $v_b$ are all constants.

**Theorem 2.2.** For Riemann initial boundary value problems, the formulae [2.2] - [2.4] takes the form

Case 1: $f'(u_0) = f'(u_b) > 0,$

$$(u(x, t), v(x, t)) = \begin{cases} (u_0, v_0), & \text{if } x < f'(u_0)t, \\ (u_0, v_0), & \text{if } x > f'(u_0)t. \end{cases}$$

Case 2: $f'(u_0) = f'(u_b) < 0,$

$$(u(x, t), v(x, t)) = (u_0, v_0).$$

Case 3: $0 < f'(u_b) < f'(u_0),$  

$$(u(x, t), v(x, t)) = \begin{cases} (u_b, v_b), & \text{if } x < f'(u_b)t, \\ (x/t, 0), & \text{if } f'(u_b)t < x < f'(u_0)t \\ (u_0, v_0), & \text{if } x > f'(u_0)t \end{cases}$$

Case 4: $f'(u_b) < 0 < f'(u_0),$

$$(u(x, t), v(x, t)) = \begin{cases} (x/t, 0), & \text{if } 0 < x < f'(u_0)t \\ (u_0, v_0), & \text{if } x > f'(u_0)t \end{cases}$$

Case 5: $f'(u_b) < 0$ and $f'(u_0) \leq 0,$

$$(u(x, t), v(x, t)) = (u_0, v_0)$$
3. Solution in a strip

The solution we have obtained for the quarter plane problem can be easily generalized to the strip \( \Omega = \{(x, t) : 0 < x < 1, t > 0\} \). Here we prescribe

\[
(u(x, 0+), v(x, 0+)) = (u_0(x), v_0(x)), \quad 0 \leq x \leq 1.
\]

(3.1)

As before for \( u \) component we prescribe a weak form of Dirichlet boundary conditions at \( x = 0 \) and at \( x = 1 \):

\[
\text{either} \quad u(0+, t) = u_1^+(t) \\
\text{or} \quad f'(u(0+, t)) \leq 0 \quad \text{and} \quad f(u(0+, t)) \geq f(u_1^+(t)),
\]

(3.2)

\[
\text{either} \quad u(1-, t) = u_r^+(t) \\
\text{or} \quad f'(u(1-, t)) \geq 0 \quad \text{and} \quad f(u(1-, t)) \geq f(u_r^+(t)).
\]

(3.3)

Here \( u_1^+(t) = \max\{u(t), \lambda\} \), \( u_r^-(t) = \min\{u_r(t), \lambda\} \) where as before \( \lambda \) is the point of minimum of \( f \). We get explicit formula for the entropy weak solution of the first component \( u \) of \( \frac{\partial}{\partial t} u + f(u) \frac{\partial}{\partial x} u = 0 \) with initial condition \( u(x, 0) = u_0(x) \) and the boundary conditions (3.2) and (3.3) by Joseph and Gowda [4]. Once \( u \) is obtained, the boundary conditions for \( v(0+, t) = v_l(t) \) is prescribed only if the characteristics at \( (0, t) \) has positive speed, ie \( f'(u(0+, t)) > 0 \). So the weak form of boundary conditions for \( v \) component at \( x = 0 \) is

\[
\text{if} \quad f'(u(0+, t)) > 0, \quad \text{then} \quad v(0+, t) = v_l(t).
\]

(3.4)

Similarly the weak form of the boundary condition at \( x = 1 \) is

\[
\text{if} \quad f'(u(1-, t)) < 0, \quad \text{then} \quad v(1-, t) = v_r(t).
\]

(3.5)

We assume the initial conditions \( u_0(x) \) is bounded measurable, and locally Lipschitz, and \( v_0(x) \) is Lipschitz continuous on \( 0 \leq x \leq 1 \) and boundary datas \( u_l(t), v_l(t) \) are Lipschitz continuous \([0, T]\), for each \( T > 0 \).

For the statement of the theorem, we introduce some notations. For each fixed \((x, y, t), \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ t > 0, \ |i - j| \leq 1, i, j = 0, 1, 2, 3, \ldots, C_{ij}(x, y, t)\) denotes the following class of paths \( \beta \) in the strip

\[
\Omega = \{(z, s) : 0 \leq z \leq 1, s \geq 0\}
\]

Each path connects \((y, 0)\) to \((x, t)\) and is of the form \( z = \beta(s) \) where \( \beta(s) \) is piecewise linear function which are straight lines in the interior of \( D \), and having \( i \) straight line pieces lie on \( x = 0 \) and \( j \) of them lie on \( x = 1 \). The points of intersection of the straight line pieces of the curve lying in \( \Omega \) with the boundaries \( x = 0 \) and \( x = 1 \) are called corners of the curve \( \beta \).

Denote

\[
C(x, y, t) = \cup_{i \geq 0, j \geq 0, |i - j| \leq 1} C_{i,j}(x, y, t)
\]
For fixed \((x, y, t)\), we define

\[
J(\beta) = -\int_{\{s; \beta(s) = 0\}} f(u^+_x(s))ds - \int_{\{s; \beta(s) = 1\}} f(u^-_x(s))ds + \int_{\{s; 0 < \beta(s) < 1\}} f^*\left(\frac{d\beta}{ds}\right)ds.
\]

(3.6)

Denote \(C^*(x, y, t) = C(x, y, t) - \{\beta_0\}\), where \(\beta_0\) is the straight line path joining \((x, t)\) to \((y, 0)\).

Let us define \(A(x, y, t)\) and \(B(x, y, t)\) by

\[
A(x, y, t) = J(\beta_0), \quad B(x, y, t) = \min_{\beta \in C^*(x, y, t)} J(\beta)
\]

(3.7)

where \(J(\beta)\) be defined by \((3.6)\).

We recall a few facts from [4]. For each \((x, t) \in \Omega\) and \(0 \leq y \leq 1\), the minimum in \((3.3)\) is attained for a path \(\beta^*\) over \(C^*(x, y, t)\). Let the corner points of the minimizer \(\beta\) be

\[
(\beta^*(t_1(x, y, t)), t_1(x, y, t)), \quad (\beta^*(t_2(x, y, t)), t_2(x, y, t)),
\]

\[
\ldots, \quad (\beta^*(t_k(x, y, t)), t_k(x, y, t)),
\]

\(t > t_1(x, y, t) > t_2(x, y, t) \cdots > t_k(x, y, t) > 0\). For a given \(T > 0\), there exits positive integer \(N(T)\) such that for any \(t \leq T\), and \(k < N(T)\). This is due to the bound of \(u_0(t)\) on \([0, T]\) and due to the conditions on \(f(u)\). Then the function \(B\) is expressed in terms of \(x, y, t, t_1(x, y, t), \ldots t_k(x, y, t)\) which we denote by \(B(x, y, t) = J(x, y, t, t_1(x, y, t), \ldots t_k(x, y, t) = J(\beta^*).\) Similarly define \(A(x, y, t) = t_j(t^*(\frac{x-\gamma}{t})) = J(\beta_0), \beta_0\) is the straight line path connecting \((x, t)\) and \((y, 0)\). Define the function

\[
Q(x, y, t) = \min\{A(x, y, t), B(x, y, t)\}.
\]

(3.8)

The function

\[
U(x, t) = \min_{0 \leq y \leq 1} \left[ \int_0^y u_0(z)dz + Q(x, y, t) \right]
\]

(3.9)

is Lipschitz continuous function of \((x, t)\) in \(\Omega\). For almost every \((x, t)\) in \(\Omega\), there exists a unique minimizer \(y(x, t)\) and either

\[
A(x, y(x, t), t) < B(x, y(x, t), t) \quad \text{and} \quad U(x, t) = \int_0^y (x, t)u_0(z)dz + A(x, y, t)
\]

or

\[
A(x, y(x, t), t) > B(x, y(x, t), t) \quad \text{in which case} \quad U(x, t) = \int_0^y u_0(z)dz + B(x, y, t).
\]

In the second case, let \(t_j(x, y, t), j = 1, 2, \ldots, k\) corresponds to the corner points of the curve \(\beta^*\) in the evaluation of \(B(x, y, t)\). Denote \(t_j(x, y, t) = t_j(x, y(x, t), t),\)

\[
A(x, t) = A(x, y(x, t), t) \quad \text{and} \quad B(x, t) = B(x, y(x, t), t).
\]

With these notations we have the following theorem.

**Theorem 3.1.** Let \(U\) be defined by the minimization problem \((3.9)\) and \(y(x, t)\) be a minimizer (which is unique for a.e points of \(\Omega\)). Let \(u = U_z(x, t)\) exists for a.e. points of \(\Omega\) and has the form

\[
u(x, t) = \begin{cases} f^*(\frac{z-y(x, t)}{t}), & \text{if } A(x, t) < B(x, t), \\ f^*(\frac{x-t(x, t)}{t}), & \text{if } A(x, t) > B(x, t), \end{cases}
\]

and

\[
V(x, t) = \begin{cases} \int_0^{y(x, t)} v_0(z)dz, & \text{if } A(x, t) < B(x, t), \\ -\int_{t_1(x, t)}^{t_2(x, t)} f'(u^+_x(s))v_1(s)ds, & \text{if } A(x, t) > B(x, t) \quad \text{and} \quad \beta^*(t_1(x, t)) = 0, \\ -\int_{t_2(x, t)}^{t_1(x, t)} f'(u^-_x(s))v_2(s)ds, & \text{if } A(x, t) > B(x, t) \quad \text{and} \quad \beta^*(t_1(x, t)) = 1, \end{cases}
\]
and set
\[ v(x, t) = \partial_x(V(x, t)). \]
Then \((u, v)\) is a solution to (1.1) with initial conditions (3.1) and boundary conditions (3.2)-(3.5). Further \(u\) satisfies the entropy condition \(u(x-, t) \geq u(x+, t)\) for \(0 < x < 1\), \(t > 0\).

Proof. The assertions on \(u\) is proved in [4]. Once we have that, the verification that \(v\) solves the equation and the initial and boundary conditions follows exactly as in section 2 and is omitted. \(\square\)

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