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COMPARISON THEOREMS FOR RICCATI INEQUALITIES ARISING IN THE THEORY OF PDE'S WITH p-LAPLACIAN

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Abstract. In this article, we study a Riccati inequality that appears in the theory of partial differential equations with p-Laplacian. Our results allow to compare existence and nonexistence of positive solutions for Riccati type inequalities which are associated with equations with different powers p.

1. Introduction

In this article, we consider a problem which is closely related to the second-order elliptic half-linear differential operator; i. e., the operator with the p-Laplacian

$$\Delta_p u(x) = \operatorname{div}(\|\nabla u(x)\|^{p-2}\nabla u(x))$$

and signed power-type nonlinearity of degree $p-1$:

$$L[u](x) := \Delta_p u(x) + c(x)|u(x)|^{p-2}u(x), \tag{1.1}$$

where $p > 1$ and the norm $\|\cdot\|$ is the Euclidean norm. The corresponding differential equation

$$L[u] = 0 \tag{1.2}$$

attracted a considerable attention in the last years because of its applications in physics, glaceology, and biology, see the recent book [7] which summarizes results related to this equation up to 2005 and also the book [15] which deals with the application aspects of the problem.

Many problems related to the theory of equation (1.2) can be studied using the corresponding Riccati type operator

$$R[w](x) := \operatorname{div} w(x) + c(x) + (p-1)\|w(x)\|^q, \tag{1.3}$$

where $q = \frac{p}{p-1}$ is the conjugate number to the number p. In this paper we study the Riccati type partial differential inequality

$$R[w] \le 0. \tag{1.4}$$

Throughout the paper we suppose for simplicity that $c(x)$ is a Hölder continuous function on a domain with piecewise smooth boundary $\Omega \subseteq \mathbb{R}^n$ and the domain

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of the operators (1.1) and (1.3) is the set of $C^2(\Omega,\mathbb{R})$ and $C^1(\Omega,\mathbb{R}^n)$ functions, respectively.

The oscillation theory for (1.2) is similar to the classical oscillation theory of linear second order differential equations. Among others, it turns out that the Sturm type comparison theorems extend to (1.2) and equations can be classified as oscillatory or nonoscillatory. There are many results which guarantee that equation (1.2) is oscillatory; i. e., it possesses no positive solution on any exterior domain in $\mathbb{R}^n$. This is partly due to the fact that each known oscillation criterion for the half-linear ordinary differential equation

$$\big(r(t)|u'|^{p-2}u'\big)' + c(t)|u|^{p-2}u = 0, \quad ({}' = \frac{\mathrm{d}}{\mathrm{d}t}), \tag{1.5}$$

can be extended easily to equation (1.2) using results of [5] and [11]. Roughly speaking, oscillation of (1.2) can be deduced if (1.2) is a majorant (in the sense of integral average over spheres) of some radially symmetric equation, which can be reduced into an oscillatory ordinary differential equation. There is also an alternative approach (see [12]), which is based on the fact that the substitution $w = \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u}$ transforms a nonzero solution of (1.2) into a solution of the Riccati type equation

$$R[w] = 0. \tag{1.6}$$

Then one can employ integration over balls to convert (1.6) into an inequality in one variable and finally to follow known methods from the one-dimensional case, to finish the proof of the corresponding oscillation criteria. This approach allows (among others) to deal with more general unbounded domains than exterior ones. In the application of the Riccati technique in proofs of oscillation criteria, we prove in fact that the Riccati equation (1.6) has no solution on the domain under consideration (usually the complement of a ball centered at the origin with arbitrarily large radius). Criteria for the nonexistence of solutions of (1.4) and (1.6) have been derived in [13].

In contrast to a voluminous literature devoted to oscillation criteria, there are only a few nonoscillation criteria, even for the linear operator

$$L_2[u] := \Delta u + c(x)u \tag{1.7}$$

and the linear equation $L_2[u] = 0$, which is a special case $p = 2$ in (1.2). Neglecting some trivial results based on a comparison with radially symmetric nonoscillatory majorant, we have only a few results based on the investigation of positive solutions of the inequality $L_2[u] \le 0$ (see [2, 8]) or of the Riccati equation $R_2[w] = 0$ with the operator

$$R_2[w] = \operatorname{div} w + c(x) + \|w\|^2 \tag{1.8}$$

and the inequality $R_2[w] \le 0$ (see [8]).

The approach based on the investigation of the inequality $L_2[u] \le 0$ can be also used for half-linear equations. Indeed, Allegretto and Huang [1] used Picone's identity and Harnack's inequality to prove the following theorem (g and g_1 are supposed to belong to $L^{n/p}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$).

Theorem 1.1. *Suppose that the inequality $-\Delta_p u \ge g_1|u|^{p-2}u$ has a positive solution in Ω. If $g \le g_1$ in Ω, then so does the inequality $-\Delta_p u = g|u|^{p-2}u$.*

Eliason and White [8] proved the following theorem for linear operator L_2 defined on $\mathbb{R}^2$. As mentioned in [8], this result extends to the more general operator

$$\operatorname{div}(r(x)\nabla u) + c(x)u$$

with an elliptic matrix $r(x)$ and $x \in \mathbb{R}^n$.

Theorem 1.2. *The inequality $R_2[w] \le 0$ has a conservative $C^1(G)$ vector field solution w on a subdomain $G \subseteq \Omega \subseteq \mathbb{R}^2$ if and only if the equation $R_2[w] = 0$ has one; and this holds if and only if the equation $L_2[u] = 0$ has a positive $C^2(G)$ solution on G.*

The aim of this article is to extend Theorem 1.2 to half-linear equations. This extension shows that the associated Riccati inequality (1.4) plays an important role not only in oscillation criteria, but also in problems related to the existence of (eventually) positive solutions, which are closely related to nonoscillation criteria.

Another aim of this article is to prove some comparison results for the existence of a solution of the Riccati type inequality. It is a well known fact from the theory of ordinary half-linear equations (1.5), that bigger p speeds up oscillation of the equation, see [14, 16]. Another approach which allows to compare oscillatory properties of half-linear differential equations with different power in nonlinearity appeared in works [4, 6, 9]. More precisely, the oscillation properties of half-linear equations are studied within the framework of the linear equations, as the following theorem shows.

Theorem 1.3 ([4, Theorem 1 and Theorem 2])**.** *Denote $\Phi(x) = |x|^{p-2}x$ and suppose that the equation*

$$\big(\widetilde{r}(t)\Phi(x')\big)' + \widetilde{c}(t)\Phi(x) = 0 \tag{1.9}$$

is nonoscillatory and possesses a positive solution $h(t)$ such that $h'(t) \neq 0$ for large t. Consider the equations

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0 \tag{1.10}$$

and

$$(R(t)y')' + \frac{p}{2}C(t)y = 0, \tag{1.11}$$

where

$$C(t) = h(t)\Big[\Big((r(t) - \widetilde{r}(t))\Phi(h'(t))\Big)' + (c(t) - \widetilde{c}(t))\Phi(h(t))\Big]$$

and $R(t) = r(t)h^2(t)|h'(t)|^{p-2}$.

1. *If $p \ge 2$ and* (1.11) *is nonoscillatory, then* (1.10) *is also nonoscillatory.*
2. *If $p \in (1, 2]$ and* (1.11) *is oscillatory, then* (1.10) *is also oscillatory.*

The second aim of this paper is to provide a version of Theorem 1.3 suitable for a differential inequality which appears in the theory of (1.2). In addition to the fact that we introduce a multidimensional version, we also provide more freedom in comparison. More precisely, we follow the idea suggested in [10] and the equation which is used as a replacement for (1.11) need not to be linear. However, we do not formulate the comparison theorems directly for the second order PDE's, but for the corresponding Riccati type inequalities. For an explanation and more details see Remarks 3.3 and 3.4 at the end of the paper.

2. Preliminary results

In this section we present some technical lemmas which allow us to formulate our main results in the last section.

First of all, we find (in Lemma 2.2 below) an upper and lower estimate for a function

$$P(u_1, u_2) = \frac{\|u_1\|^p}{p} - \langle u_1, u_2\rangle + \frac{\|u_2\|^q}{q} \tag{2.1}$$

which appears frequently in the qualitative theory of equations with p-Laplacian. In the proof of Lemma 2.2 we show that the problem can be reduced to an inequality for a function in one variable, which is studied in Lemma 2.1 below. Further, we derive an inequality between two Riccati type operators. This inequality is used to prove our main results.

Lemma 2.1. *Let $q = \frac{p}{p-1}$ and consider the function*

$$f(t) = \frac{|t|^q}{q} - t + \frac{1}{p} - \frac{4}{\alpha 2^\alpha}|t-1|^\alpha,$$

where $\alpha \in [2, q]$ in the case $1 < p \le 2$ and $\alpha \in [q, 2]$ for $p \ge 2$. Then $f(t) \ge 0$ for $1 < p \le 2$ and $f(t) \le 0$ for $p \ge 2$.

Proof. If $p = 2$, then $f \equiv 0$. Consider the case $p < 2$; i. e., $q > 2$, the case $p > 2$ can be treated analogically. We have

$$f'(t) = \Phi_q(t) - 1 - \frac{4}{2^\alpha}\Phi_\alpha(t-1), \quad f''(t) = (q-1)|t|^{q-2} - \frac{4(\alpha-1)}{2^\alpha}|t-1|^{\alpha-2},$$

where $\Phi_q(t) = |t|^{q-2}t$, Φ_α is defined analogically. Hence $f'(-1) = 0 = f'(1)$, $f(-1) = 2 - \frac{4}{\alpha} \ge 0$, and $f''(-1) = q - \alpha \ge 0$. Drawing the graphs of the functions $|t|$ and $\left(\frac{4(\alpha-1)}{2^\alpha(q-1)}\right)^{\frac{1}{q-2}} |t-1|^{\frac{\alpha-2}{q-2}}$ shows that the equation $f''(t) = 0$ has exactly 2 roots, one positive in the interval $(0, 1)$, and one negative in $[-1, 0)$. Hence f'' is positive outside of the interval determined by these roots and negative inside of it. This means that f has at both stationary points $t = \pm 1$ nonnegative local minima. This also implies that the equation $f'(t) = 0$ may have at most one zero in $(-1, 1)$, where the function f attains a positive local maximum. Consequently, summarizing these facts about the graph of the functions f we obtain that $f(t) \ge 0$ for $t \in \mathbb{R}$. $\square$

Observe also that substituting $t \to -t$ gives for $q \ge 2$ the inequality

$$\frac{|t|^q}{q} + t + \frac{1}{p} - \frac{4}{\alpha 2^\alpha}|t+1|^\alpha \ge 0, \quad t \in \mathbb{R} \tag{2.2}$$

and the opposite inequality for $q \in (1, 2]$.

The following lemma is an extension of [3, Lemma 2.4] which deals with the scalar case and $\alpha = 2$.

Lemma 2.2. (i) *Let $p \ge 2$ and $\|u_1\| \ne 0$. Then for every $\alpha \in [q, 2]$ there exists a number $\gamma(\alpha, p)$ such that*

$$P(u_1, u_2) \le \gamma(\alpha, p)\|u_1\|^{(p-1)(q-\alpha)}\|u_2 - \|u_1\|^{p-2}u_1\|^\alpha. \tag{2.3}$$

(ii) *Let $p \in (1, 2]$. Then for every $\alpha \in [2, q]$ there exists a number $\gamma(\alpha, p)$ such that*

$$P(u_1, u_2) \ge \gamma(\alpha, p)\|u_1\|^{(p-1)(q-\alpha)}\|u_2 - \|u_1\|^{p-2}u_1\|^\alpha. \tag{2.4}$$

Remark 2.3. In the proof we will show that we can take $\gamma(\alpha, p) = 4/\alpha 2^\alpha$. However, the numerical computations show that this constant is not optimal and can be improved. To find this optimal constant is a subject of the present investigation.

Proof of Lemma 2.2. Observe that (2.4) trivially holds for $\|u_1\| = 0$. Therefore, in the remaining part of the proof we suppose $\|u_1\| \neq 0$. We will prove the first statement of lemma ($p \geq 2$), the proof of the second part is analogical. By dividing both sides of (2.3) with the factor $\|u_1\|^p$ we get the inequality

$$\frac{1}{p} - \Big\langle \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_1\|^{p-1}} \Big\rangle + \frac{\|\frac{u_2}{\|u_1\|^{p-1}}\|^q}{q} \leq \gamma(\alpha, p) \| \frac{u_2}{\|u_1\|^{p-1}} - \frac{u_1}{\|u_1\|} \|^\alpha. \tag{2.5}$$

Define $x = \frac{u_2}{\|u_1\|^{p-1}}$ and $a = \frac{u_1}{\|u_1\|}$. Then $\|a\| = 1$ and (2.5) can be written in the form

$$\frac{\|x\|^q}{q} - \langle a, x\rangle + \frac{1}{p} \leq \gamma(\alpha, p)\|x - a\|^\alpha. \tag{2.6}$$

As mentioned above, we show that this inequality holds with $\gamma(\alpha, p) = \frac{4}{\alpha 2^\alpha}$.

Let $g(x) = \langle x, a\rangle + \frac{4}{\alpha 2^\alpha}\|x - a\|^\alpha$. We will examine the minimal value of this function over the sphere $\|x\| = t$, $t \geq 0$. Any $x \in \mathbb{R}^n$ can be written in the form $x = \mu a + \nu a^\perp$ for some unit vector $a^\perp$ with $\langle a, a^\perp\rangle = 0$. Then

$$t^2 = \|x\|^2 = \langle \mu a + \nu a^\perp, \mu a + \nu a^\perp\rangle = \mu^2 + \nu^2.$$

We have

$$\begin{aligned} g(x) =& \langle \mu a + \nu a^\perp, a\rangle + \frac{4}{\alpha 2^\alpha}\langle \mu a + \nu a^\perp - a, \mu a + \nu a^\perp - a\rangle^{\alpha/2} \\ =& \mu + \frac{4}{\alpha 2^\alpha}(t^2 - 2\mu + 1)^{\alpha/2}. \end{aligned}$$

Now we solve the extremal problem $g(x) \to \min$, $\|x\| = t$ which can be written in the form

$$\mu + \frac{4}{\alpha 2^\alpha}(t^2 - 2\mu + 1)^{\alpha/2} \to \min, \quad \mu \in [-t, t].$$

Since $\alpha/2 \leq 1$, the minimized function is concave and hence it attains its minimum over $[-t, t]$ at the boundary point of this interval; i. e.,

$$g(x)\big|_{\|x\|=t} \geq \min\big\{ -t + \frac{4}{\alpha 2^\alpha}|t + 1|^\alpha, t + \frac{4}{\alpha 2^\alpha}|t - 1|^\alpha \big\}.$$

Consequently, inequality (2.6) holds if

$$\frac{|t|^q}{q} + \frac{1}{p} \mp t - \frac{4}{\alpha 2^\alpha}|t \mp 1|^2 \leq 0.$$

But this is just the inequality from Lemma 2.1 for the sign "−" or its equivalent reformulation after the substitution $t \to -t$ (see (2.2) in case $q \in (1, 2]$) . The Lemma is proved. $\square$

The next lemma presents a link between two Riccati type operators, namely the operator which corresponds to half-linear equation (1.2) (the power at the dependent variable is q) and the Riccati operator with α-degree nonlinearity, where $\alpha \in [\min\{q, 2\}, \max\{q, 2\}]$.

Lemma 2.4. *Let $h \in C^2(\Omega, \mathbb{R}^+)$. Define $G = h\|\nabla h\|^{p-2}\nabla h$ and $v = h^p w - G$. Further, let $\alpha \in [\min\{q, 2\}, \max\{q, 2\}]$ and $\gamma(\alpha, p)$ be the number from Lemma 2.2.*

(i) *If* $1 < p \le 2$*, then*

$$h^p R[w] \ge \operatorname{div} v + hL[h] + \gamma(\alpha, p) p h^{-\alpha} \|\nabla h\|^{(p-1)(q-\alpha)} \|v\|^\alpha \tag{2.7}$$

holds on Ω.

(ii) *If* $p \ge 2$ *and* $\|\nabla h\| \ne 0$ *on* Ω*, then*

$$h^p R[w] \le \operatorname{div} v + hL[h] + \gamma(\alpha, p) p h^{-\alpha} \|\nabla h\|^{(p-1)(q-\alpha)} \|v\|^\alpha \tag{2.8}$$

holds on Ω.

Proof. We start with the following obvious identities

$$\operatorname{div} G = \|\nabla h\|^p + h\Delta_p h \tag{2.9}$$

and

$$\begin{aligned} h^p \operatorname{div} w &= h^p \operatorname{div}(h^{-p}(v + G)) \\ &= \operatorname{div} v + \operatorname{div} G - p h^{-1}\langle v + G, \nabla h\rangle \\ &= \operatorname{div} v + \|\nabla h\|^p + h\Delta_p h - p h^{-1}\langle v + G, \nabla h\rangle. \end{aligned} \tag{2.10}$$

Now a direct computation shows

$$\begin{aligned} h^p R[w] &= h^p \operatorname{div} w + h^p c(x) + (p-1) h^p \|w\|^q \\ &= \operatorname{div} v + \|\nabla h\|^p + h\Delta_p h - p h^{-1}\langle v + G, \nabla h\rangle + h^p c(x) + (p-1) h^{-q} \|h^p w\|^q \\ &= \operatorname{div} v + hL[h] + p h^{-q}\Big(\frac{\|h^{q-1}\nabla h\|^p}{p} - \langle v + G, h^{q-1}\nabla h\rangle + \frac{\|v + G\|^q}{q}\Big) \\ &= \operatorname{div} v + hL[h] + p h^{-q} P(h^{q-1}\nabla h, v + G). \end{aligned}$$

For $u_1 = h^{q-1}\nabla h$ and $u_2 = v + G$ we have

$$\begin{aligned} &\|u_1\|^{(p-1)(q-\alpha)} \|u_2 - \|u_1\|^{p-2} u_1\|^\alpha \\ &= h^{(q-1)(p-1)(q-\alpha)} \|\nabla h\|^{(p-1)(q-\alpha)} \|v + G - h\|\nabla h\|^{p-2}\nabla h\|^\alpha \\ &= h^{q-\alpha} \|\nabla h\|^{(p-1)(q-\alpha)} \|v\|^\alpha. \end{aligned}$$

Now the lemma follows from the estimates in Lemma 2.2. □

3. Main results

In this section we introduce the main results of the paper. Since most of the work has been already done in the previous section, the proofs are short and straightforward. Our first theorem certifies the importance of Riccati type inequality (1.4) in the theory of half-linear differential equations (1.2).

Theorem 3.1. *The following statements are equivalent:*

(i) *The equation* $L[u] = 0$ *has a positive* C^2 *solution on* Ω.
(ii) *The inequality* $L[u] \le 0$ *has a positive* C^2 *solution on* Ω.
(iii) *The equation* $R[w] = 0$ *has a* C^1 *solution* w *on* Ω *such that the vector field* $\|w\|^{q-2} w$ *is conservative.*
(iv) *The inequality* $R[w] \le 0$ *has a* C^1 *solution* w *on* Ω *such that the vector field* $\|w\|^{q-2} w$ *is conservative.*

Proof. Define

$$w = \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2} u}. \tag{3.1}$$

By a direct calculation, the i-th component of the vector w satisfies

$$\frac{\partial w_i}{\partial x_i} = \frac{\frac{\partial}{\partial x_i}\left(\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_i}\right)}{|u|^{p-2}u} - (p-1)\frac{\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_i}}{|u|^p}\frac{\partial u}{\partial x_i}$$

and summing up over all independent variables we get

$$\begin{aligned}\operatorname{div} w &= \frac{\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right)}{|u|^{p-2}u} - (p-1)\frac{\|\nabla u\|^{p-2}}{|u|^p}\|\nabla u\|^2 \\ &= \frac{\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right)}{|u|^{p-2}u} - (p-1)\|w\|^q.\end{aligned}$$

Using this computation we easily observe that

$$R[w] = \frac{L[u]}{|u|^{p-2}u} \tag{3.2}$$

holds.

(i) $\Longrightarrow$ (iii). Follows from (3.2) and from the fact that if w is defined by (3.1), then $\|w\|^{q-2}w = \frac{\nabla u}{u} = \nabla(\ln u)$ and $\ln u$ is a scalar potential to $\|w\|^{q-2}w$.

(iii) $\Longrightarrow$ (iv). Clearly holds.

(iv) $\Longrightarrow$ (ii). Since $\|w\|^{q-2}w$ has a scalar potential, there exists a scalar function φ, such that $\nabla\varphi = \|w\|^{q-2}w$. Define function $u = e^{\varphi}$. The function u satisfies (3.1) and in view of (3.2) the implication holds.

(ii) $\Longrightarrow$ (i). Follows from Theorem 1.1. $\square$

Our second theorem relates two Riccati type inequalities. One of them is inequality (1.4) which is associated to the half-linear equation with p-Laplacian (1.2) (the dependent variable appears in the inequality in the power q), while the second one contains the dependent variable in the power α; i. e., the equation is associated with a half-linear PDE with β-degree Laplacian, where β is the conjugate number to the number α (see also Remark 3.3 below).

Theorem 3.2. *Let $h \in C^2(\Omega, \mathbb{R}^+)$.*

(i) *Let $p \in (1, 2]$, (1.4) has a C^1 solution on Ω, $\alpha \in [2, q]$ be arbitrary number and $\gamma(\alpha, p)$ be the number from Lemma 2.2. Then*

$$\operatorname{div} v + h(x)L[h(x)] + p\gamma(\alpha,p)h^{-\alpha}(x)\|\nabla h(x)\|^{(p-1)(q-\alpha)}\|v\|^{\alpha} \le 0 \tag{3.3}$$

has also a C^1 solution on Ω.

(ii) *Let $p \ge 2$, $\alpha \in [q, 2]$ be arbitrary number, $\gamma(\alpha, p)$ be the number from Lemma 2.2 and let h satisfy $\|\nabla h\| \ne 0$ on Ω. If (3.3) has a C^1 solution on Ω, then (1.4) has also a C^1 solution on Ω.*

The proof of the above theorem is a direct consequence of the inequalities from Lemma 2.4.

Remark 3.3. Suppose that both h and $\|\nabla h\|$ do not vanish in Ω. The equation

$$\operatorname{div} v + h(x)L[h](x) + p\gamma(\alpha,p)h^{-\alpha}(x)\|\nabla h(x)\|^{(p-1)(q-\alpha)}\|v\|^{\alpha} = 0$$

is the Riccati equation for the second order partial differential equation

$$\operatorname{div}\Big(A(x)\|\nabla u\|^{\beta-2}\nabla u\Big) + h(x)L[h](x)|u|^{\beta-2}u = 0, \tag{3.4}$$

where $\beta = \frac{\alpha}{\alpha-1}$ and $A(x) = \left[\frac{p\gamma(\alpha,p)}{\beta-1}\right]^{1-\beta} h^\beta(x)\|\nabla h(x)\|^{p-\beta}$. Thus if (3.4) has a positive solution on Ω, then (3.3) has a solution. Conversely, if (3.3) has a C^1 solution v on Ω and $\|v\|^{\alpha-2}v$ is conservative, then (3.4) has a positive C^2 solution on Ω. If $\alpha = 2$, then (3.4) becomes the linear partial differential equation

$$\operatorname{div}\Big(h^2(x)\|\nabla h(x)\|^{p-2}\nabla u\Big) + \frac{p}{2}h(x)L[h](x)u = 0.$$

In this case, Theorem 3.2 allows us to transfer results from the linear theory to half-linear equations.

Remark 3.4. Note that we are not able to guarantee that the condition on the existence of scalar potential from (iv) part of Theorem 3.1 holds. For this reason we are not able yet to formulate the results from Theorem 3.2 in terms of second order half-linear differential equations, like in Theorem 1.3.

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