NONHOMOGENEOUS ELLIPTIC EQUATIONS WITH DECAYING CYLINDRICAL POTENTIAL AND CRITICAL EXPONENT

MOHAMMED BOUCHEKIF, MOHAMMED EL MOKHTAR OULD EL MOKHTAR

Abstract. We prove the existence and multiplicity of solutions for a nonhomogeneous elliptic equation involving decaying cylindrical potential and critical exponent.

1. Introduction

In this article, we consider the problem
\[-\text{div}(|y|^{-2a} \nabla u) - \mu |y|^{-2(a+1)} u = h|y|^{-2b}|u|^{2^*-2} u + \lambda g \quad \text{in } \mathbb{R}^N, \quad y \neq 0 \quad (1.1)\]
where each point in \( \mathbb{R}^N \) is written as a pair \((y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \); \( k \) and \( N \) are integers such that \( N \geq 3 \) and \( k \) belongs to \( \{1, \ldots, N\} \); \(-\infty < \mu < (k-2)/2 \); \( a \leq b < a + 1 \); \( 2_\ast = 2N/(N-2+2(b-a)) \); \(-\infty < \mu < \bar{\mu}_{a,k} := ((k-2(a+1))/2)^2 \); \( g \in H'_\mu \cap C(\mathbb{R}^N) \); \( h \) is a bounded positive function on \( \mathbb{R}^k \) and \( \lambda \) is real parameter. Here \( H'_\mu \) is the dual of \( H_\mu \), where \( H_\mu \) and \( D^{1,2}_0 \) will be defined later.

Some results are already available for (1.1) in the case \( k = N \); see for example [10, 11] and the references therein. Wang and Zhou [10] proved that there exist at least two solutions for (1.1) with \( a = 0, 0 < \mu \leq \bar{\mu}_{0,N} = ((N-2)/2)^2 \) and \( h \equiv 1 \), under certain conditions on \( g \). Bouchekif and Matallah [2] showed the existence of two solutions of (1.1) under certain conditions on functions \( g \) and \( h \), when \( 0 < \mu \leq \bar{\mu}_{0,N} \), \( \lambda \in (0, \Lambda^\ast) \), \(-\infty < \mu < (N-2)/2 \) and \( a \leq b < a + 1 \), with \( \Lambda^\ast \) a positive constant.

Concerning existence results in the case \( k < N \), we cite [6, 7] and the references therein. Musina [7] considered (1.1) with \(-a/2 \) instead of \( a \) and \( \lambda = 0 \), also (1.1) with \( a = 0, b = 0, \lambda = 0 \), with \( h \equiv 1 \) and \( a \neq 2 - k \). She established the existence of a ground state solution when \( 2 < k \leq N \) and \( 0 < \mu < \bar{\mu}_{a,k} = ((k-2 + a)/2)^2 \) for (1.1) with \(-a/2 \) instead of \( a \) and \( \lambda = 0 \). She also showed that (1.1) with \( a = 0, b = 0, \lambda = 0 \) does not admit ground state solutions. Badiale et al [11] studied (1.1) with \( a = 0, b = 0, \lambda = 0 \) and \( h \equiv 1 \). They proved the existence of at least a nonzero nonnegative weak solution \( u \), satisfying \( u(y,z) = u(|y|,z) \) when \( 2 \leq k < N \) and...
\( \mu < 0 \). Boucikif and El Mokhtar \cite{3} proved that \( (1.1) \) with \( a = 0 \), \( b = 0 \) admits two distinct solutions when \( 2 < k \leq N \), \( b = N - \frac{\mu(N - 2)}{2} \) with \( \mu \in [2, 2^*] \), \( \mu < \hat{\mu}_{a,k} \), and \( \lambda \in (0, \Lambda_\ast) \) where \( \Lambda_\ast \) is a positive constant. Terracini \cite{9} proved that there are no positive solutions of \( (1.1) \) with \( b = 0 \), \( \lambda = 0 \) when \( a \neq 0 \), \( h \equiv 1 \) and \( \mu < 0 \). The regular problem corresponding to \( a = b = \mu = 0 \) and \( h \equiv 1 \) has been considered on a regular bounded domain \( \Omega \) by Tarantello \cite{8}. She proved that for \( g \) in \( H^{-1}(\Omega) \), the dual of \( H_0^1(\Omega) \), not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notation. We denote by \( D_0^{1,2} = D_0^{1,2}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k}) \) and \( \mathcal{H}_\mu = \mathcal{H}_\mu(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k}) \), the closure of \( C_0^\infty(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k}) \) with respect to the norms
\[
\|u\|_{a,\mu} = \left( \int_{\mathbb{R}^N} |y|^{-2a} |\nabla u|^2 \, dx \right)^{1/2}
\]
and
\[
\|u\|_{a,\mu} = \left( \int_{\mathbb{R}^N} (|y|^{-2a} |\nabla u|^2 - \mu |y|^{-2(a+1)} |u|^2) \, dx \right)^{1/2},
\]
respectively, with \( \mu < \tilde{\mu}_{a,k} = ((k-2(a+1))/2)^2 \) for \( k \neq 2(a+1) \).

From the Hardy-Sobolev-Maz’ya inequality, it is easy to see that the norm \( \|u\|_{a,\mu} \) is equivalent to \( \|u\|_{a,0} \).

Since our approach is variational, we define the functional \( I_{a,b,\lambda,\mu} \) on \( \mathcal{H}_\mu \) by
\[
I(u) := I_{a,b,\lambda,\mu}(u) := (1/2)\|u\|_{a,\mu}^2 - (1/2) \int_{\mathbb{R}^N} h|y|^{-2b} |u|^2 \, dx - \lambda \int_{\mathbb{R}^N} gu \, dx.
\]
We say that \( u \in \mathcal{H}_\mu \) is a weak solution of \( (1.1) \) if it satisfies
\[
\langle I'(u), v \rangle = \int_{\mathbb{R}^N} (|y|^{-2a} \nabla u \nabla v - \mu |y|^{-2(a+1)} uv - h |y|^{-2b} |u|^2 - \lambda g v) \, dx = 0, \quad \text{for } v \in \mathcal{H}_\mu.
\]
Here \( \langle \cdot, \cdot \rangle \) denotes the product in the duality \( \mathcal{H}_\mu', \mathcal{H}_\mu \).

Throughout this work, we consider the following assumptions:

\( \text{(G)} \) There exist \( \rho_0 > 0 \) and \( \delta_0 > 0 \) such that \( g(x) \geq \rho_0 \), for all \( x \in B(0, 2\delta_0) \);
\( \text{(H)} \) \( \lim_{|y| \to 0} h(y) = \lim_{|y| \to \infty} h(y) = h_0 > 0 \), \( h(y) \geq h_0, y \in \mathbb{R}^k \).

Here, \( B(a, r) \) denotes the ball centered at \( a \) with radius \( r \).

Under some conditions on the coefficients of \( (1.1) \), we split \( \mathcal{N} \) in two disjoint subsets \( \mathcal{N}^+ \) and \( \mathcal{N}^- \), thus we consider the minimization problems on \( \mathcal{N}^+ \) and \( \mathcal{N}^- \).

**Remark 1.1.** Note that all solutions of \( (1.1) \) are nontrivial.

We shall state our main results.

**Theorem 1.2.** Assume that \( 3 \leq k \leq N, -1 < a < (k-2)/2, 0 \leq \mu < \hat{\mu}_{a,k} \), and \( \text{(G)} \) holds, then there exists \( \Lambda_1 > 0 \) such that the \( (1.1) \) has at least one nontrivial solution on \( \mathcal{H}_\mu \) for all \( \lambda \in (0, \Lambda_1) \).

**Theorem 1.3.** In addition to the assumptions of the Theorem 1.2, if \( \text{(H)} \) holds, then there exists \( \Lambda_2 > 0 \) such that \( (1.1) \) has at least two nontrivial solutions on \( \mathcal{H}_\mu \) for all \( \lambda \in (0, \Lambda_2) \).

This article is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1.2 and 1.3.
2. Preliminaries

We list here a few integral inequalities. The first one that we need is the Hardy inequality with cylindrical weights [7]. It states that

\[
\mu_{a,k} \int_{\mathbb{R}^N} |y|^{-2(a+1)}u^2 \, dx \leq \int_{\mathbb{R}^N} |y|^{-2a}|\nabla u|^2 \, dx, \quad \text{for all } v \in \mathcal{H}_\mu,
\]

The starting point for studying (1.1) is the Hardy-Sobolev-Maz’ya inequality that is particular to the cylindrical case \( k < N \) and that was proved by Maz’ya in [6]. It states that there exists positive constant \( C_{a,2} \), such that

\[
C_{a,2} \left( \int_{\mathbb{R}^N} |y|^{-2k}|v|^2 \, dx \right)^{2/2} \leq \int_{\mathbb{R}^N} (|y|^{-2a}|\nabla v|^2 - \mu |y|^{-2(a+1)}v^2) \, dx,
\]

for any \( v \in C_c^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}) \).

Proposition 2.1 ([6]). The value

\[
S_{\mu,2} = S_{\mu,2}(k,2_*) := \inf_{v \in \mathcal{H}_\mu \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|y|^{-2a}|\nabla v|^2 - \mu |y|^{-2(a+1)}v^2) \, dx}{\left( \int_{\mathbb{R}^N} |y|^{-2,b}|v|^2 \, dx \right)^{2/2}}
\]

is achieved on \( \mathcal{H}_\mu \), for \( 2 \leq k < N \) and \( \mu \leq \mu_{a,k} \).

Definition 2.2. Let \( c \in \mathbb{R} \), \( E \) be a Banach space and \( I \in C^1(E, \mathbb{R}) \).

(i) \( (u_n) \) is a Palais-Smale sequence at level \( c \) (in short \( (PS)_c \)) in \( E \) for \( I \) if \( I(u_n) = c + o_n(1) \) and \( I'(u_n) = o_n(1) \), where \( o_n(1) \to 0 \) as \( n \to \infty \).

(ii) We say that \( I \) satisfies the \( (PS)_c \) condition if any \( (PS)_c \) sequence in \( E \) for \( I \) has a convergent subsequence.

2.1. Nehari manifold. It is well known that \( I \) is of class \( C^1 \) in \( \mathcal{H}_\mu \) and the solutions of (1.1) are the critical points of \( I \) which is not bounded below on \( \mathcal{H}_\mu \).

Consider the Nehari manifold

\[
\mathcal{N} = \{ u \in \mathcal{H}_\mu \setminus \{0\} : \langle I'(u), u \rangle = 0 \},
\]

Thus, \( u \in \mathcal{N} \) if and only if

\[
\|u\|_{a,\mu}^2 - \int_{\mathbb{R}^N} h|y|^{-2,b}|u|^2 \, dx - \lambda \int_{\mathbb{R}^N} gu \, dx = 0. \tag{2.2}
\]

Note that \( \mathcal{N} \) contains every nontrivial solution of (1.1). Moreover, we have the following results.

Lemma 2.3. The functional \( I \) is coercive and bounded from below on \( \mathcal{N} \).

Proof. If \( u \in \mathcal{N} \), then by (2.2) and the H"older inequality, we deduce that

\[
I(u) = ((2_* - 2)/2,2)\|u\|_{a,\mu}^2 - \lambda(1 - (1/2_*)) \int_{\mathbb{R}^N} gu \, dx
\geq ((2_* - 2)/2,2)\|u\|_{a,\mu}^2 - \lambda(1 - (1/2_*))\|u\|_{a,\mu}\|g\|_{\mathcal{H}'}^a
\geq -\lambda^2 C_0,
\]

where

\[
C_0 := C_0(\|g\|_{\mathcal{H}'}^a) = [(2_* - 1)^2/2,2(2_* - 2)]\|g\|_{\mathcal{H}'}^2 > 0.
\]

Thus, \( I \) is coercive and bounded from below on \( \mathcal{N} \). \( \square \)
Lemma 2.4. Suppose that there exists a local minimizer $u_0$ for $I$ on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$. Then, $I'(u_0) = 0$ in $\mathcal{H}^\mu_{\mu}$.

Proof. If $u_0$ is a local minimizer for $I$ on $\mathcal{N}$, then there exists $\theta \in \mathbb{R}$ such that

$$\langle I'(u_0), \varphi \rangle = \theta \langle \Psi'_\lambda(u_0), \varphi \rangle$$

for any $\varphi \in \mathcal{H}^\mu_{\mu}$.

If $\theta = 0$, then the lemma is proved. If not, taking $\varphi \equiv u_0$ and using the assumption $u_0 \in \mathcal{N}$, we deduce

$$0 = \langle I'(u_0), u_0 \rangle = \theta \langle \Psi'_\lambda(u_0), u_0 \rangle.$$

Thus

$$\langle \Psi'_\lambda(u_0), u_0 \rangle = 0,$$

which contradicts that $u_0 \notin \mathcal{N}^0$.

Let

$$\Lambda_1 := (2s - 2)(2s - 1)^{-(2s - 1)/(2s - 2)} \bigl( (h_0)^{-1} S_{\mu, 2s} \bigr)^{2s/(2s - 2)} \| g \|_{H^s_{\mu}}^{-1}. \quad (2.5)$$

Lemma 2.5. We have $\mathcal{N}^0 = \emptyset$ for all $\lambda \in (0, \Lambda_1)$.

Proof. Let us reason by contradiction. Suppose $\mathcal{N}^0 \neq \emptyset$ for some $\lambda \in (0, \Lambda_1)$. Then, by (2.4) and for $u \in \mathcal{N}^0$, we have

$$\| u \|^2_{a, \mu} = (2s - 1) \int_{\mathbb{R}^N} h|y|^{-2s} |u|^{2s} \, dx$$

$$= \lambda((2s - 1)/(2s - 2)) \int_{\mathbb{R}^N} gu \, dx. \quad (2.6)$$

Moreover, by (G), the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\left[ (h_0)^{-1} S_{\mu, 2s} \right]^{2s/(2s - 2)} \| u \|_{a, \mu} \leq \lambda((2s - 1)] \| g \|_{H^s_{\mu}}/(2s - 2)) \). \quad (2.7)$$

This implies that $\lambda \geq \Lambda_1$, which is a contradiction to $\lambda \in (0, \Lambda_1)$. \qed
Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ for $\lambda \in (0, \Lambda_1)$. Define
\[ c := \inf_{u \in \mathcal{N}} I(u), \quad c^+ := \inf_{u \in \mathcal{N}^+} I(u), \quad c^- := \inf_{u \in \mathcal{N}^-} I(u). \]

We need also the following Lemma.

**Lemma 2.6.** (i) If $\lambda \in (0, \Lambda_1)$, then $c \leq c^+ < 0$.
(ii) If $\lambda \in (0, (1/2)\Lambda_1)$, then $c^- > C_1$, where
\[ C_1 = C_1(\lambda, S_{\mu,2}, \|g\|_{\mathcal{H}_\mu'}) = ((2_* - 2)/2, 2)(2_* - 1)^{2/(2_* - 2)}(S_{\mu,2})^{2/(2_* - 2)} - \lambda(1 - (1/2_*))(2_* - 1)^{2/(2_* - 2)}\|g\|_{\mathcal{H}_\mu'}.

**Proof.** (i) Let $u \in \mathcal{N}^+$. By (2.4),
\[ [1/(2_* - 1)]\|u\|_{a,\mu}^2 > \int_{\mathbb{R}^N} h|y|^{-2,b}|u|^{2_*} \, dx \]
and so
\[ I(u) = (-1/2)\|u\|_{a,\mu}^2 + (1 - (1/2_*)) \int_{\mathbb{R}^N} h|y|^{-2,b}|u|^{2_*} \, dx \]
\[ < [((-1/2) + (1 - (1/2_*))(1/(2_* - 1))]\|u\|_{a,\mu}^2 \]
\[ = -((2_* - 2)/2, 2)|u|_{a,\mu}^2; \]
we conclude that $c \leq c^+ < 0$.
(ii) Let $u \in \mathcal{N}^-$. By (2.4),
\[ [1/(2_* - 1)]\|u\|_{a,\mu}^2 < \int_{\mathbb{R}^N} h|y|^{-2,b}|u|^{2_*} \, dx. \]
Moreover, by Sobolev embedding theorem, we have
\[ \int_{\mathbb{R}^N} h|y|^{-2,b}|u|^{2_*} \, dx \leq (S_{\mu,2})^{-2/2}\|u\|_{a,\mu}^{2_*}. \]
This implies
\[ \|u\|_{a,\mu} > [(2_* - 1)^{-1/(2_* - 2)}(S_{\mu,2})^{2/2_*}]^{2/(2_* - 2)}, \quad \text{for all } u \in \mathcal{N}^-.
\]
By (2.3),
\[ I(u) \geq ((2_* - 2)/2, 2)|u|_{a,\mu}^2 - \lambda(1 - (1/2_*))\|u\|_{a,\mu}\|g\|_{\mathcal{H}_\mu'}.
\]
Thus, for all $\lambda \in (0, (1/2)\Lambda_1)$, we have $I(u) \geq C_1$. \hfill \Box

For each $u \in \mathcal{H}_\mu$, we write
\[ t_m := t_{\max}(u) = \left[ \frac{\|u\|_{a,\mu}}{(2_* - 1) \int_{\mathbb{R}^N} h|y|^{-2,b}|u|^{2_*} \, dx} \right]^{1/(2_* - 2)} > 0. \]

**Lemma 2.7.** Let $\lambda \in (0, \Lambda_1)$. For each $u \in \mathcal{H}_\mu$, one has the following:
(i) If $\int_{\mathbb{R}^N} g(x)u \, dx \leq 0$, then there exists a unique $t^+ > t_m$ such that $t^- u \in \mathcal{N}^-$ and
\[ I(t^- u) = \sup_{t \geq 0} I(tu). \]
(ii) If $\int_{\mathbb{R}^N} g(x)u \, dx > 0$, then there exist unique $t^+$ and $t^-$ such that $0 < t^+ < t_m < t^-$, $t^+ u \in \mathcal{N}^+$, $t^- u \in \mathcal{N}^-$,
\[ I(t^+ u) = \inf_{0 \leq t \leq t_m} I(tu) \text{ and } I(t^- u) = \sup_{t \geq 0} I(tu). \]
The proof of the above lemma follows from a proof in [5], with minor modifications.

3. Proof of Theorem 1.2

For the proof we need the following results.

Proposition 3.1 ([5]). (i) If \( \lambda \in (0, \Lambda_1) \), then there exists a minimizing sequence \((u_n)_n\) in \( \mathcal{N} \) such that

\[
I(u_n) = c + o_n(1), \quad I'(u_n) = o_n(1) \quad \text{in} \quad \mathcal{H}',
\]

where \( o_n(1) \) tends to 0 as \( n \) tends to \( \infty \).

(ii) if \( \lambda \in (0, (1/2)\Lambda_1) \), then there exists a minimizing sequence \((u_n)_n\) in \( \mathcal{N}^- \) such that

\[
I(u_n) = c^- + o_n(1), \quad I'(u_n) = o_n(1) \quad \text{in} \quad \mathcal{H}'.
\]

Now, taking as a starting point the work of Tarantello [8], we establish the existence of a local minimum for \( I \) on \( \mathcal{N}^+ \).

Proposition 3.2. If \( \lambda \in (0, \Lambda_1) \), then \( I \) has a minimizer \( u_1 \in \mathcal{N}^+ \) and it satisfies

(i) \( I(u_1) = c = c^+ < 0 \),

(ii) \( u_1 \) is a solution of (1.1).

Proof. (i) By Lemma 2.3, \( I \) is coercive and bounded below on \( \mathcal{N} \). We can assume that there exists \( u_1 \in \mathcal{H}_\mu \) such that

\[
u_n \rightharpoonup u_1 \quad \text{weakly in} \quad \mathcal{H}_\mu, \\
u_n \rightharpoonup u_1 \quad \text{weakly in} \quad L^2(\mathbb{R}^N, |y|^{-2}b), \\
u_n \rightharpoonup u_1 \quad \text{a.e in} \quad \mathbb{R}^N.
\]

Thus, by (3.1) and (3.2), \( u_1 \) is a weak solution of (1.1) since \( c < 0 \) and \( I(0) = 0 \).

Now, we show that \( u_n \) converges to \( u_1 \) strongly in \( \mathcal{H}_\mu \). Suppose otherwise. Then

\[
\|u_1\|_{a,\mu} < \liminf_{n \to \infty} \|u_n\|_{a,\mu}
\]

and we obtain

\[
c \leq I(u_1) = ((2^* - 2)/2, 2)||u_1||^2_{a,\mu} - \lambda(1 -(1/2^*)) \int_{\mathbb{R}^N} gu_1 \, dx
\]

\[
< \liminf_{n \to \infty} I(u_n) = c.
\]

We have a contradiction. Therefore, \( u_n \) converges to \( u_1 \) strongly in \( \mathcal{H}_\mu \). Moreover, we have \( u_1 \in \mathcal{N}^+ \). If not, then by Lemma 2.7, there are two numbers \( t_0^+ \) and \( t_0^- \), uniquely defined so that \( t_0^+ u_1 \in \mathcal{N}^+ \) and \( t_0^- u_1 \in \mathcal{N}^- \). In particular, we have

\[
t_0^+ < t_0^- = 1.
\]

Since

\[
\frac{d}{dt}I(tu_1)|_{t=t_0^+} = 0, \quad \frac{d^2}{dt^2}I(tu_1)|_{t=t_0^+} > 0,
\]

there exists \( t_0^+ < t^+ \leq t_0^- \) such that \( I(t_0^+ u_1) < I(t^+ u_1) < I(t^- u_1) \). By Lemma 2.7

\[
I(t_0^+ u_1) < I(t^+ u_1) < I(t^- u_1) = I(u_1),
\]

which is a contradiction.  \( \square \)
4. Proof of Theorem 1.3

In this section, we establish the existence of a second solution of (1.1). For this, we require the following Lemmas, with $C_0$ is given in (2.3).

**Lemma 4.1.** Assume that (G) holds and let $(u_n)_n \subset \mathcal{H}_\mu$ be a $(PS)_c$ sequence for $I$ for some $c \in \mathbb{R}$ with $u_n \rightharpoonup u$ in $\mathcal{H}_\mu$. Then, $I'(u) = 0$ and

$$I(u) \geq -C_0\lambda^2.$$ 

**Proof.** It is easy to prove that $I'(u) = 0$, which implies that $(I'(u), v) = 0$, and

$$\int_{\mathbb{R}^N} h|y|^{-2,b}|u|^2 \, dx = \|u\|_{a,\mu}^2 - \lambda \int_{\mathbb{R}^N} gu \, dx.$$ 

Therefore,

$$I(u) = ((2^*_s - 2)/2, 2\|u\|_{a,\mu}^2 - \lambda(1 - (1/2_s)) \int_{\mathbb{R}^N} gu \, dx.$$ 

Using (2.3), we obtain

$$I(u) \geq -C_0\lambda^2.$$ 

**Lemma 4.2.** Assume that (G) holds and for any $(PS)_c$ sequence with $c$ is a real number such that $c < c^*_\lambda$. Then, there exists a subsequence which converges strongly. Here $c^*_\lambda := ((2^*_s - 2)/2_s)(h_0)^{-2/(2_s - 2)}(S_{\mu, 2})^{-2/(2_s - 2)} - C_0\lambda^2$.

**Proof.** Using standard arguments, we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Thus, there exist a subsequence of $(u_n)_n$ which we still denote by $(u_n)_n$ and $u \in \mathcal{H}_\mu$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathcal{H}_\mu,$$

$$u_n \rightharpoonup u \quad \text{weakly in } L^{2_s}(\mathbb{R}^N, |y|^{-2,b}).$$

$$u_n \rightarrow u \quad \text{a.e in } \mathbb{R}^N.$$ 

Then, $u$ is a weak solution of (1.1). Let $v_n = u_n - u$, then by Brézis-Lieb [4], we obtain

$$\|v_n\|_{a,\mu}^2 = \|u_n\|_{a,\mu}^2 - \|u\|_{a,\mu}^2 + o_n(1) \quad (4.1)$$

and

$$\int_{\mathbb{R}^N} h|y|^{-2,b}|v_n|^{2_s} \, dx = \int_{\mathbb{R}^N} h|y|^{-2,b}|u_n|^{2_s} \, dx - \int_{\mathbb{R}^N} h|y|^{-2,b}|u|^{2_s} \, dx + o_n(1). \quad (4.2)$$

On the other hand, by using the assumption (H), we obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} h(x)|y|^{-2,b}|v_n|^{2_s} \, dx = h_0 \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |y|^{-2,b}|v_n|^{2_s} \, dx. \quad (4.3)$$

Since $I(u_n) = c + o_n(1)$, $I'(u_n) = o_n(1)$ and by (4.1), (4.2), and (4.3) we deduce that

$$\left(1/2\right)\|v_n\|_{a,\mu}^2 - (1/2_s) \int_{\mathbb{R}^N} h|y|^{-2,b}|v_n|^{2_s} \, dx = c - I(u) + o_n(1),$$

$$\|v_n\|_{a,\mu}^2 - \int_{\mathbb{R}^N} h|y|^{-2,b}|v_n|^{2_s} \, dx = o_n(1). \quad (4.4)$$

Hence, we may assume that

$$\|v_n\|_{a,\mu}^2 \rightarrow l, \quad \int_{\mathbb{R}^N} h|y|^{-2,b}|v_n|^{2_s} \, dx \rightarrow l. \quad (4.5)$$
In fact, if where Sobolev inequality gives \( \|v_n\|_{a,\mu}^2 \geq (S_{\mu,2}) \int_{\mathbb{R}^N} h|y|^{-2,b}|v_n|^{2^*} \, dx \). Combining this inequality with (4.5), we obtain

\[
\|v_n\|_{a,\mu}^2 \geq (1-h_0)^{-2/\gamma}. 
\]

Either \( l = 0 \) or \( l \geq (h_0)^{-2/(2^*-2)}(S_{\mu,2})^{2^*/(2^*-2)} \). Suppose that

\[
l \geq (h_0)^{-2/(2^*-2)}(S_{\mu,2})^{2^*/(2^*-2)}. 
\]

Then, from (4.4), (4.5) and Lemma 4.1, we obtain

\[
c \geq ((2^*-2)/2,2)l + I(u) \geq c^*_\lambda,
\]

which is a contradiction. Therefore, \( l = 0 \) and we conclude that \( u_n \) converges to \( u \) strongly in \( \mathcal{H}_\mu \).

**Lemma 4.3.** Assume that \( (G) \) and \( (H) \) hold. Then, there exist \( v \in \mathcal{H}_\mu \) and \( \Lambda_* > 0 \) such that for \( \lambda \in (0,\Lambda_*), \) one has

\[
\sup_{t \geq 0} I(tv) < c^*_\lambda.
\]

In particular, \( c^- < c^*_\lambda \) for all \( \lambda \in (0,\Lambda_*) \).

**Proof.** Let \( \varphi_\varepsilon \) be such that

\[
\varphi_\varepsilon(x) = \begin{cases}
\omega_\varepsilon(x) & \text{if } g(x) \geq 0 \text{ for all } x \in \mathbb{R}^N \\
\omega_\varepsilon(x-x_0) & \text{if } g(x_0) > 0 \text{ for } x_0 \in \mathbb{R}^N \\
-\omega_\varepsilon(x) & \text{if } g(x) \leq 0 \text{ for all } x \in \mathbb{R}^N
\end{cases}
\]

where \( \omega_\varepsilon \) satisfies (2.1). Then, we claim that there exists \( \varepsilon_0 > 0 \) such that

\[
\lambda \int_{\mathbb{R}^N} g(x)\varphi_\varepsilon(x) \, dx > 0 \quad \text{for any } \varepsilon \in (0,\varepsilon_0).
\]

In fact, if \( g(x) \geq 0 \) or \( g(x) \leq 0 \) for all \( x \in \mathbb{R}^N \), (4.6) obviously holds. If there exists \( x_0 \in \mathbb{R}^N \) such that \( g(x_0) > 0 \), then by the continuity of \( g(x) \), there exists \( \eta > 0 \) such that \( g(x) > 0 \) for all \( x \in B(x_0,\eta) \). Then by the definition of \( \omega_\varepsilon(x-x_0) \), it is easy to see that there exists an \( \varepsilon_0 \) small enough such that

\[
\lambda \int_{\mathbb{R}^N} g(x)\omega_\varepsilon(x-x_0) \, dx > 0, \quad \text{for any } \varepsilon \in (0,\varepsilon_0).
\]

Now, we consider the functions

\[
f(t) = I(t\varphi_\varepsilon), \quad \tilde{f}(t) = (t^2/2)||\varphi_\varepsilon||_{a,\mu}^2 - (t^2/2) \int_{\mathbb{R}^N} h|y|^{-2,b}|\varphi_\varepsilon|^{2^*} \, dx.
\]

Then, for all \( \lambda \in (0,\Lambda_1) \),

\[
f(0) = 0 < c^*_\lambda.
\]

By the continuity of \( f \), there exists \( t_0 > 0 \) small enough such that

\[
f(t) < c^*_\lambda, \quad \text{for all } t \in (0,t_0).
\]

On the other hand,

\[
\max_{t \geq 0} \tilde{f}(t) = ((2^*-2)/2,2)(h_0)^{-2/(2^*-2)}(S_{\mu,2})^{2^*/(2^*-2)}.
\]

Then, we obtain

\[
\sup_{t \geq 0} I(t\varphi_\varepsilon) < ((2^*-2)/2,2)(h_0)^{-2/(2^*-2)}(S_{\mu,2})^{2^*/(2^*-2)} - \lambda t_0 \int_{\mathbb{R}^N} g\varphi_\varepsilon \, dx.
\]
Now, taking $\lambda > 0$ such that
\[-\lambda t_0 \int_{\mathbb{R}^N} g\varphi_0 \, dx < -C_0 \lambda^2,\]
and by (4.6), we obtain
\[0 < \lambda < (t_0/C_0)\left(\int_{\mathbb{R}^N} g\varphi_0\right), \quad \text{for } \varepsilon << \varepsilon_0.\]
Set
\[\Lambda_* = \min\{\Lambda_1, (t_0/C_0)(\int_{\mathbb{R}^N} g\varphi_0)\}.\]
We deduce that
\[\sup_{t \geq 0} I(t\varphi_\varepsilon) < c_\lambda, \quad \text{for all } \lambda \in (0, \Lambda_*). \quad (4.7)\]
Now, we prove that
\[c^- < c^*_\lambda, \quad \text{for all } \lambda \in (0, \Lambda_*).\]
By (G) and the existence of $w_n$ satisfying (2.1), we have
\[\lambda \int_{\mathbb{R}^N} gw_n \, dx > 0.\]
Combining this with Lemma 2.7 and from the definition of $c^-$ and (4.7), we obtain that there exists $t_n > 0$ such that $t_n w_n \in \mathcal{N}^-$ and for all $\lambda \in (0, \Lambda_*),$
\[c^- \leq I(t_n w_n) \leq \sup_{t \geq 0} I(tw_n) < c^*_\lambda.\]

Now we establish the existence of a local minimum of $I$ on $\mathcal{N}^-$. 

**Proposition 4.4.** There exists $\Lambda_2 > 0$ such that for $\lambda \in (0, \Lambda_2)$, the functional $I$ has a minimizer $u_2$ in $\mathcal{N}^-$ and satisfies
\[(i) \ I(u_2) = c^-, \quad (ii) \ u_2 \text{ is a solution of (1.1) in } \mathcal{H}_\mu,\]
where $\Lambda_2 = \min\{(1/2)\Lambda_1, \Lambda_*\}$ with $\Lambda_1$ defined as in (2.5) and $\Lambda_*$ defined as in the proof of Lemma 4.3.

**Proof.** By Proposition 3.1(ii), there exists a $(PS)_{c^-}$ sequence for $I, (u_n)\_n$ in $\mathcal{N}^-$ for all $\lambda \in (0, (1/2)\Lambda_1)$. From Lemmas 4.2, 4.3 and 2.6(ii), for $\lambda \in (0, \Lambda_*), I$ satisfies $(PS)_{c^-}$ condition and $c^- > 0$. Then, we get that $(u_n)\_n$ is bounded in $\mathcal{H}_\mu$. Therefore, there exist a subsequence of $(u_n)\_n$ still denoted by $(u_n)\_n$ and $u_2 \in \mathcal{N}^-$ such that $u_n$ converges to $u_2$ strongly in $\mathcal{H}_\mu$ and $I(u_2) = c^-$ for all $\lambda \in (0, \Lambda_2)$. Finally, by using the same arguments as in the proof of the Proposition 3.2, for all $\lambda \in (0, \Lambda_1)$, we have that $u_2$ is a solution of (1.1).

Now, we complete the proof of Theorem 1.3. By Propositions 3.2 and 4.4, we obtain that (1.1) has two solutions $u_1$ and $u_2$ such that $u_1 \in \mathcal{N}^+$ and $u_2 \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that $u_1$ and $u_2$ are distinct.
References


Mohammed Boucheikif
University of Tlemcen, Departement of Mathematics, BO 119, 13 000 Tlemcen, Algeria
E-mail address: m.boucheikif@yahoo.fr

Mohammed El Mokhtar Ould El Mokhtar
University of Tlemcen, Departement of Mathematics, BO 119, 13 000 Tlemcen, Algeria
E-mail address: med.mokhtar66@yahoo.fr