ENTIRE SOLUTIONS FOR A NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this article, we study the existence of solutions to the differential equation
\[ f^n(z) + P(f) = P_1e^{h_1} + P_2e^{h_2}, \]
where \( n \geq 2 \) is a positive integer, \( f \) is a transcendental entire function, \( P(f) \) is a differential polynomial in \( f \) of degree less than or equal \( n - 1 \), \( P_1, P_2 \) are small functions of \( e^z \), \( h_1, h_2 \) are polynomials, and \( z \) is in the open complex plane \( C \). Our results extend those obtained by Li [6, 7, 8, 9].

1. Introduction and main results

Nevanlinna value distribution theory of meromorphic functions has been extensively applied to resolve growth (see [6]), value distribution [6], and solvability of meromorphic solutions of linear and nonlinear differential equations [4, 6, 10, 11]. Considering meromorphic functions \( f \) in the complex plane, we assume that the reader is familiar with the standard notations and results such as the proximity function \( m(r, f) \), counting function \( N(r, f) \), characteristic function \( T(r, f) \), the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevanlinna theory; see e.g. [3, 6]. Given a meromorphic function \( f \), we shall call a meromorphic function \( a(z) \) a small function of \( f(z) \) if \( T(r, a) = S(r, f) \), where \( S(r, f) \) is used to denote any quantity that satisfies \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \), possibly outside a set of \( r \) of finite logarithmic measure. A differential polynomial \( P(f) \) in \( f \) is a polynomial in \( f \) and its derivatives with small functions of \( f \) as the coefficients. The notation \( \mathcal{F} \) is defined to the family of all meromorphic functions which satisfy \( N(r, \frac{1}{r}) + N(r, h) = S(r, h) \). Note that all functions in family \( \mathcal{F} \) are transcendental, and all functions of the form \( be^{\lambda z} \) are functions in family \( \mathcal{F} \), where \( \lambda \) is any nonzero constant and \( b \) is a rational function.

In 2006, Li and Yang [7] obtain the following results.

Theorem 1.1. Let \( n \geq 4 \) be an integer, and \( P(f) \) denote an algebraic differential polynomial in \( f \) of degree \( \leq n - 3 \). Let \( P_1, P_2 \) be two nonzero polynomials, \( \alpha_1 \) and
\( \alpha_2 \) be two nonzero constants with \( \frac{\alpha_1}{\alpha_2} \neq \text{rational}. \) Then the differential equation

\[
f^n(z) + P(f) = P_1 e^{\alpha_1 z} + P_2 e^{\alpha_2 z}
\]

has no transcendental entire solutions.

**Theorem 1.2.** Let \( n \geq 3 \) be an integer, and \( P(f) \) be an algebraic differential polynomial in \( f \) of degree \( \leq n - 3 \), \( b(z) \) be a meromorphic function, and \( \lambda, c_1, c_2 \) and three nonzero constants. Then the differential equation

\[
f^n(z) + P(f) = b(z)(c_1 e^{\lambda z} + c_2 e^{-\lambda z})
\]

has no transcendental entire solutions \( f(z) \), satisfying \( T(r, b) = S(r, f) \).

Recently, Considering the degree of the differential polynomial \( P(f) \) of \( n - 2 \) or \( n - 1 \), P. Li [9] proved the following results which are improvements or complementarity of Theorems 1.1 and 1.2.

**Theorem 1.3.** Let \( n \geq 2 \) be an integer. Let \( f \) be a transcendental entire function, \( P(f) \) be a differential polynomial in \( f \) of degree \( \leq n - 1 \). If

\[
f^n(z) + P(f) = P_1 e^{\alpha_1 z} + P_2 e^{\alpha_2 z}, \quad (1.1)
\]

where \( P_i(i = 1, 2) \) are nonvanishing small functions of \( e^z \), \( \alpha_i(i = 1, 2) \) are positive numbers satisfying \( (n - 1)\alpha_2 \geq n\alpha_1 > 0 \), then there exists a small function \( \gamma \) of \( f \) such that

\[
(f - \gamma)^n = P_2 e^{\alpha_2 z}. \quad (1.2)
\]

**Theorem 1.4.** Let \( n \geq 2 \) be an integer, \( \alpha_1, \alpha_2 \) be real numbers and \( \alpha_1 < 0 < \alpha_2 \). Let \( P_1, P_2 \) be small functions of \( e^z \). If there exists a transcendental entire function \( f \) satisfying the differential equation \( (1.1) \), where \( P(f) \) is a differential polynomial in \( f \) of degree not exceeding \( n - 2 \), then \( \alpha_1 + \alpha_2 = 0 \), and there exist constants \( c_1, c_2 \) and small functions \( \beta_1, \beta_2 \) with respect to \( f \) such that

\[
f = c_1 \beta_1 e^{\alpha_1 z/n} + c_2 \beta_2 e^{\alpha_2 z/n}, \quad (1.3)
\]

moreover, \( \beta_1^n = P_1, i = 1, 2 \).

**Theorem 1.5.** Let \( n \geq 2 \) be an integer, \( \alpha_1, \alpha_2 \) be positive numbers satisfying \( (n - 1)\alpha_2 \geq n\alpha_1 > 0 \). Let \( P_1, P_2 \) be small functions of \( e^z \). If \( \frac{\alpha_1}{\alpha_2} \) is irrational, then the differential equation \( (1.1) \) has no entire solutions, where \( P(f) \) is a differential polynomial in \( f \) of degree \( \leq n - 1 \).

**Remark 1.6.** By an example, Li [9] pointed if the degree of \( P(f) \) is \( n - 1 \), then the solutions of \( (1.1) \) may not be the form in \( (1.3) \).

It is natural to ask whether \( \alpha_1 z \) and \( \alpha_2 z \) in \( (1.1) \) can be replaced by two polynomials. In this article, by the same method as in [9], we obtain the following results.

**Theorem 1.7.** Let \( n \geq 2 \) be an integer. Let \( f \) be a transcendental entire function, \( P(f) \) be a differential polynomial in \( f \) of degree \( \leq n - 1 \). If

\[
f^n(z) + P(f) = P_1 e^{Q_1(z)} + P_2 e^{Q_2(z)}, \quad (1.4)
\]

where \( P_i(i = 1, 2) \) are nonvanishing small meromorphic functions of \( e^z \), \( Q_1(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \cdots + \alpha_1 z + \alpha_0, Q_2(z) = \beta_k z^k + \beta_{k-1} z^{k-1} + \cdots + \beta_1 z + \beta_0 \) are two polynomials satisfying \( (n - 1)\beta_k \geq n\alpha_k > 0 \) (where \( \alpha_{k-1}, \alpha_0, \beta_{k-1}, \ldots, \beta_0 \)
are finite constants and \( k \geq 1 \) is a positive integer, then there exists a small meromorphic function \( \gamma \) of \( f \) such that

\[
(f - \gamma)^n = P_2e^{Q_2}.
\]

**Theorem 1.8.** Let \( n \geq 2 \) be an integer and \( P_1, P_2 \) be small functions of \( e^z \). If there exists a transcendental entire function \( f \) satisfying the differential equation \([1.4]\), where \( P(f) \) is a differential polynomial in \( f \) of degree not exceeding \( n - 2 \) and \( \alpha_k < 0 < \beta_k \), then \( \alpha_k + \beta_k = 0 \), and there exist constants \( c_1, c_2 \) and small functions \( \beta_1, \beta_2 \) with respect to \( f \) such that

\[
f = c_1\beta_1e^{\frac{Q_1}{n}} + c_2\beta_2e^{\frac{Q_2}{n}},
\]

moreover, \( \beta_i^n = P_i, i = 1, 2 \).

**Theorem 1.9.** Let \( n \geq 2 \) be an integer, \( P_1, P_2 \) be small functions of \( e^z \). If \( \alpha_k \) is irrational, then the differential equation \([1.4]\) has no entire solutions, where \( P(f) \) is a differential polynomial in \( f \) of degree \( \leq n - 1 \) and \( (n - 1)\beta_k \geq n\alpha_k > 0 \).

Obviously, our results generalize the results in \([6, 7, 8, 9]\).

## 2. Preliminary Lemmas

In order to prove our theorems, we need the following lemmas. First, we need the following well-known Clunie’s lemma, which has been extensively applied in studying the value distribution of a differential polynomial \( P(z, f) \), as well as the growth estimates of solutions and meromorphic solvability of differential equations in the complex plane.

**Lemma 2.1** (*) \([1, 2]\). Let \( f \) be a transcendental meromorphic solution of

\[
f^n A(z, f) = B(z, f),
\]

where \( A(z, f) \), \( B(z, f) \) are differential polynomials in \( f \) and its derivatives with small meromorphic coefficients \( a_\lambda \), in the sense of \( T(r, a_\lambda) = S(r, f) \) for all \( \lambda \in I \), where \( I \) is an index set. If the total degree of \( B(z, f) \) as a polynomial in \( f \) and its derivatives is less than or equal \( n \), then \( m(r, A(z, f)) = S(r, f) \).

**Lemma 2.2** (*) \([3]\). Suppose that \( f \) is a nonconstant meromorphic function and \( F = f^n + Q(f) \), where \( Q(f) \) is a differential polynomial in \( f \) with degree \( \leq n - 1 \). If \( N(r, f) + N(r, \frac{1}{f}) = S(r, f) \), then

\[
F = (f + \gamma)^n,
\]

whereby \( \gamma \) is meromorphic and \( T(r, \gamma) = S(r, f) \).

**Lemma 2.3** (*) \([8]\). Suppose that \( h \) is a function in family \( \mathcal{F} \). Let \( f = a_0 h^p + a_1 h^{p-1} + \cdots + a_p \) and \( g = b_0 h^q + b_1 h^{q-1} + \cdots + b_q \) be polynomials in \( h \) with all coefficients being small functions of \( h \) and \( a_0b_0a_p \neq 0 \) if \( q \leq p \), then \( m(r, \frac{f}{g}) = S(r, h) \).
3. Proofs of main theorems

Proof of Theorem 1.2: First of all, we write \( P(f) \) as

\[
P(f) = \sum_{j=0}^{n-1} b_j M_j(f),
\]

where \( b_j \) are small functions of \( f \), \( M_0(f) = 1 \), \( M_j(f)(j = 1, 2, \ldots, n - 1) \) are homogeneous differential monomials in \( f \) of degree \( j \). Without loss of generality, we assume that \( b_0 \neq 0 \), otherwise, we do the transformation \( f = f_1 + c \) for a suitable constant \( c \). From (1.4), we have

\[
\frac{1}{P_1 e^{Q_1} + P_2 e^{Q_2} - b_0} + \sum_{j=1}^{n-1} \frac{b_j}{P_1 e^{Q_1} + P_2 e^{Q_2} - b_0} M_j(f) \left( \frac{1}{f} \right)^{n-j} = \left( \frac{1}{f} \right)^n. \tag{3.2}
\]

Note that \( m(r, -\frac{M(f)}{f}) = S(r, f) \),

\[
m(r, -\frac{M(f)}{P_1 e^{Q_1}(z) + P_2 e^{Q_2}(z) - b_0}) = m(r, -\frac{1}{P_1 e^{Q_1}(z) + P_2 e^{Q_2}(z) - b_0}) = S(r, e^k).
\]

We take \( h = e^k \), \( q = 0 \), \( p = \beta_k \), by Lemma 2.3, we obtain

\[
m(r, \frac{1}{P_1 e^{Q_1}(z) + P_2 e^{Q_2}(z) - b_0}) = S(r, e^k) = S(r, P_1 e^{Q_1}(z) + P_2 e^{Q_2}(z) - b_0) = S(r, f(z)).
\]

Therefore, the left-hand side of (3.2) is a polynomial in \( 1/f \) of degree at most \( n - 1 \) with coefficients being small proximate functions of \( 1/f \). Hence

\[
m(r, \frac{1}{f}) = S(r, f). \tag{3.3}
\]

Taking the derivatives in both sides of (1.4) gives

\[
nf^{n-1} f' + (P(f))' = (P_1' + Q_1' P_1) e^{Q_1} + (P_2' + Q_2' P_2) e^{Q_2}. \tag{3.4}
\]

By eliminating \( e^{Q_1} \) and \( e^{Q_2} \), respectively from (1.4) and the above equation, we obtain

\[
(P_2' + Q_2' P_2) f^n - P_2 n f^{n-1} f' + (P_2' + Q_2' P_2) P(f) - P_2 (P(f))' = \beta e^{Q_1}, \tag{3.5}
\]

\[
(P_1' + Q_1' P_1) f^n - P_1 n f^{n-1} f' + (P_1' + Q_1' P_1) P(f) - P_1 (P(f))' = -\beta e^{Q_2}, \tag{3.6}
\]

where \( \beta = P_1' P_2 - P_2 P_1 + (Q_2' - Q_1') P_1 P_2 \) which is a small function of \( f \). We note that \( \beta \) cannot vanish identically, otherwise, by integration we obtain \( e^{Q_2} - Q_1 = C \frac{P_1}{P_2} \) for a constant, which is impossible. From (3.5) and (3.6), we obtain

\[
m(r, e^{Q_1}) \leq nT(r, f) + S(r, f), \quad j = 1, 2. \tag{3.7}
\]

On the other hand, from (1.4), we have

\[
nT(r, f) = nT(r, f^n) = m(r, f^n + P(f)) \leq T(r, P_1 e^{Q_1} + P_2 e^{Q_2}) + S(r, f). \tag{3.8}
\]
For a fixed $r > 0$, let $z = \text{e}^{i\theta}$. The interval $[0, 2\pi)$ can be expressed as the union of the following three disjoint sets:

$$E_1 = \{ \theta \in [0, 2\pi) \} \bigg\{ \frac{|f(z)|}{|e^{Q_2(z)} - Q_1(z)|} \leq 1 \},$$

$$E_2 = \{ \theta \in [0, 2\pi) \} \bigg\{ \frac{|f(z)|}{|e^{Q_2(z)} - Q_1(z)|} > 1, |\text{e}^{z^k}| \leq 1 \},$$

$$E_3 = \{ \theta \in [0, 2\pi) \} \bigg\{ \frac{|f(z)|}{|e^{Q_2(z)} - Q_1(z)|} > 1, |\text{e}^{z^k}| > 1 \}.$$

By the definition of the proximate function, we have

$$m(r, \frac{e^{Q_1(z)}}{f^{n-1}(z)}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \bigg| \frac{e^{Q_1(z)}}{f^{n-1}(z)} \bigg| d\theta = I_1 + I_2 + I_3,$$

where

$$I_j = \frac{1}{2\pi} \int_{E_j} \log^+ \bigg| \frac{e^{Q_1(z)}}{f^{n-1}(z)} \bigg| d\theta, \quad (j = 1, 2, 3).$$

For $\theta \in E_1$, we have $|f(z)| \leq |e^{Q_2(z)} - Q_1(z)|$. Since $\frac{e^{Q_2(z)}}{f^{n-1}(z)} = \frac{e^{Q_2(z)}}{e^{Q_2(z)} - Q_1(z)}$, we obtain

$$I_1 = m(r, \frac{e^{Q_2}}{f^{n-1}}) = S(r).$$

For $\theta \in E_2$, we have $|e^{Q_1(z)}| = |\text{e}^{\alpha z^k (1+o(1))}| \leq 1$, and thus $|\frac{e^{Q_1(z)}}{f^{n-1}(z)}| \leq \frac{1}{|f^{n-1}(z)|}$. It follows from (3.3) that

$$I_2 \leq m(r, \frac{1}{f^{n-1}}) = S(r).$$

For $\theta \in E_3$, we have $|f(z)| > |e^{Q_2(z)} - Q_1(z)|$. Therefore,

$$\bigg| \frac{e^{Q_1(z)}}{f^{n-1}(z)} \bigg| \leq \frac{|e^{Q_1(z)}|}{|e^{(n-1)Q_2(z) - Q_1(z)}|} = \frac{1}{|e^{(n-1)Q_2(z) - Q_1(z)}|} = \frac{1}{|e^{((n-1)\beta_k - \alpha_k)z^k (1+o(1))}|}.$$
where \( \varphi = (P'_2 + P_2Q'_2)f - nP_2f' \), and

\[
R(f) = (P'_2 + P_2Q'_2)P(f) - P_2P'(f)
\]

which is a differential polynomial in \( f \) of degree at most \( n-1 \). By Lemma 2.1, we obtain \( m(r, \varphi) = S(r, f) \). Note that since \( \varphi \) is entire, we have \( N(r, \varphi) = S(r, f) \). Hence \( T(r, \varphi) = S(r, f) \), i.e., \( \varphi \) is a small function of \( f \). By the definition of \( \varphi \), we obtain

\[
f' = \frac{P'_2 + Q'_2P_2}{nP_2} f - \frac{\varphi}{nP_2}.
\]

Substituting the above equation into (3.6) gives

\[
f^n - \frac{nP_1\varphi}{\beta} f^{n-1} - \frac{P_2(P'_1 + Q'_1P_1)}{\beta} P(f) + \frac{P_1P_2}{\beta} (P'(f))' = P_2e^{Q_2}.
\]

By Lemma 2.2, we see that there exists a small function \( \gamma \) of \( f \) such that \((f - \gamma)^n = P_2e^{Q_2}\). This also completes the proof of Theorem 1.7. \( \square \)

**Proof of Theorem 1.8.** We discuss only the case \( \alpha_k + \beta_k \geq 0 \). The case \( \alpha_k + \beta_k \leq 0 \) can be discussed similarly. Suppose that \( f \) is a transcendental entire solution of (1.4). Similar to the proof of Theorem 1.7, we can still get (3.3)-(3.9). For a fixed \( r > 0 \), let \( z = re^{i\theta} \). We can express the interval \([0, 2\pi]\) as the union of the following three disjoint sets:

\[
E_1 = \{ \theta \in [0, 2\pi) \mid \frac{|f^2(z)|}{|e^{Q_2(z)} - Q_1(z)|} \leq 1 \},
\]

\[
E_2 = \{ \theta \in [0, 2\pi) \mid \frac{|f^2(z)|}{|e^{Q_2(z)} - Q_1(z)|} > 1, |e^{z^k}| \leq 1 \},
\]

\[
E_3 = \{ \theta \in [0, 2\pi) \mid \frac{|f^2(z)|}{|e^{Q_2(z)} - Q_1(z)|} > 1, |e^{z^k}| > 1 \}.
\]

By the definition of the proximate function, we have

\[
m(r, \frac{e^{Q_1(z) + Q_2(z)}}{f^{2n-2}(z)}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{e^{Q_1(z) + Q_2(z)}}{f^{2n-2}(z)} \right| d\theta = I_1 + I_2 + I_3,
\]

where

\[
I_j = \frac{1}{2\pi} \int_{E_j} \log^+ \left| \frac{e^{Q_1(z) + Q_2(z)}}{f^{2n-2}(z)} \right| d\theta, \quad j = 1, 2, 3.
\]

For \( \theta \in E_1 \), we have

\[
\left| \frac{e^{Q_1(z) + Q_2(z)}}{f^{2n-2}(z)} \right| = \left| \frac{e^{2Q_2(z)}}{f^{2n}(z)} \frac{f^2(z)}{e^{Q_2(z)} - Q_1(z)} \right| \leq \left| \frac{e^{Q_2(z)}}{f^n(z)} \right|^2.
\]

Thus by (3.9), we obtain \( I_1 \leq S(r) \). For \( \theta \in E_2 \), it follows from \( |e^{z^k}| \leq 1 \) and \( \alpha_k + \beta_k \geq 0 \) that \( |e^{(\alpha_k + \beta_k)z^k(1 + o(1))}| \leq 1 \). Therefore,

\[
\left| \frac{e^{Q_1(z) + Q_2(z)}}{f^{2n-2}(z)} \right| \leq \frac{1}{|f^{2n-2}(z)|}.
\]
Then by (3.3), we obtain \( I_2 \leq S(r) \). For \( \theta \in E_3 \), we have \(|f^2(z)| > |e^{Q_2(z)} - Q_1(z)|\). Thus

\[
-\frac{|e^{Q_1(z)} + Q_2(z)|}{f^{2n-2}(z)} < \frac{|e^{Q_1(z)} + Q_2(z)|}{|e^{(n-1)(Q_2(z) - Q_1(z))}|} = \frac{1}{|e^{(n-2)Q_2(z) - nQ_1(z)}|} = \frac{1}{|e^{(n-2)\beta e^{-\alpha_k}z^r(1+o(1))}|} \leq 1.
\]

It follows that \( I_3 \leq S(r) \). Hence we have

\[
m(r, \frac{e^{Q_1 + Q_2}}{f^{2n-2}}) = S(r, f).
\]

Multiplying (3.5) by (3.6) gives

\[
f^{2n-2} \varphi + Q(f) = -\beta^2 e^{Q_1 + Q_2},
\]

where \( Q(f) \) is a differential polynomial in \( f \) of degree at most \( 2n - 2 \), and

\[
\varphi = ((P_1' + Q_1 P_1)f - nP_1 f')( (P_2' + Q_2 P_2)f - nP_2 f').
\]

From (3.12) and by Lemma 2.1 we obtain \( m(r, \varphi) = S(r, f) \). Therefore, \( T(r, \varphi) = S(r, f) \).

If \( (P_1' + Q_1 P_1)f - nP_1 f' \equiv 0 \), then by integration we obtain \( f^n = cP_1 e^{Q_1} \), for a nonzero constant \( c \). Therefore, \( f = a e^{Q_1} \) for a small function \( a \) of \( f \). Thus we see that the left-hand side of (1.4) is a polynomial in \( e^{Q_1} \) of degree \( n \). However, the right-hand side of (1.4) cannot be a polynomial in \( e^{Q_1} \). Hence \( (P_1' + Q_1 P_1)f - nP_1 f' \neq 0 \). Similarly, we have \( (P_2' + Q_2 P_2)f - nP_2 f' \neq 0 \). Therefore, \( \varphi \neq 0 \). Let

\[
(P_2' + Q_2 P_2)f - nP_2 f' = h.
\]

Then we have

\[
(P_1' + Q_1 P_1)f - nP_1 f' = \frac{\varphi}{h}.
\]

By eliminating \( f' \) and \( f \), respectively from (3.14) and (3.15), we obtain

\[
f = \frac{P_1}{\beta} h - \frac{\varphi P_2}{\beta} \frac{1}{h},
\]

\[
f' = \frac{P_1'}{n\beta} h - \frac{P_2'}{n\beta} \frac{\varphi}{h},
\]

where \( \beta = P_1 P_2' - P_2 P_1' + (Q_2' - Q_1') P_1 P_2 \) which is a small function of \( f \), and cannot vanish identically. From (3.16), we see that

\[
2T(r, h) = T(r, f) + S(r, f).
\]

Therefore, any small function of \( f \) is also a small function of \( h \). And from the definition of \( \varphi \) we see that \( h \) is a function in family \( \mathcal{F} \). Thus \( \frac{h'}{h} \) is a small function of \( f \). By taking derivative in both sides of (3.16), we obtain

\[
f' = ((\frac{P_1'}{\beta})' + \frac{P_1}{\beta} \frac{h'}{h}) h - ((\frac{\varphi P_2}{\beta})' - \frac{\varphi P_2}{\beta} \frac{h'}{h}) \frac{1}{h}.
\]
Comparing the coefficients of the right-hand side of (3.17) and (3.18), we deduce that
\[ \frac{P'_1 + Q'_1 P_1}{n\beta} = \left( \frac{P_1}{\beta} \right)' + \frac{P_1 h'}{h}, \] (3.19)
\[ \frac{(P'_2 + Q'_2 P_2)\varphi}{n\beta} = \left( \frac{\varphi P_2}{\beta} \right)' - \frac{\varphi P_2 h'}{h}. \] (3.20)
By integrating (3.19) and (3.20), respectively, we obtain
\[ P_1 e^{Q_1} = d_1 \left( \frac{P_1}{\beta} h \right)^n, \quad P_2 e^{Q_2} = d_2 \left( \frac{\varphi P_2}{\beta} \right)^n, \] (3.21)
where \( d_1 \) and \( d_2 \) are two nonzero constants. From the above two equations, we see that there exist two small functions \( \beta_1 \) and \( \beta_2 \) of \( e^z \) such that
\[ P_i = \beta_i e^{Q_i}, \quad i = 1, 2, \] (3.22)
The right-hand side of the above equation is a small function of \( f \), and thus a small function of \( e^z \). Therefore, the above equation holds only when \( \alpha_k + \beta_k \equiv 0 \).
Furthermore, from (3.21), we see that there exist two nonzero constants \( c_1 \) and \( c_2 \) such that
\[ P_1 h = c_1 \beta_1 e^{Q_1}, \quad P_2 \varphi = -c_2 \beta_2 e^{Q_2}. \] (3.23)
Finally, from (3.16), we obtain (1.8).

**Proof of Theorem 1.9** If \( f \) is a transcendental entire solution of (1.4), then by Theorem 1.7 there exists a small function \( \gamma \) of \( f \) such that (1.5) holds. And thus \( N(r, f - \gamma) = S(r, f) \), i.e., \( \gamma \) is an exceptional small function of \( f \). Equation (1.5) also shows that there exist two small functions \( \omega_1 \) and \( \omega_2 \) of \( f \) such that \( f' = \omega_1 f + \omega_2 \).
By substituting this equation into (1.4), we see that \( P_1 e^{Q_1} \) is a polynomial in \( f \) of degree \( t < n \). By Lemma 2.2, there exist two small functions \( a \) and \( \gamma_1 \) of \( f \) such that
\[ \frac{P_1}{\beta} h = c_1 \beta_1 e^{Q_1}, \quad \frac{P_2 \varphi}{\beta} = -c_2 \beta_2 e^{Q_2}. \] (3.24)
The right-hand side of the above equation is small function of \( f \), and thus a small function of \( e^z \). Hence we obtain \( nQ_1 - tQ_2 \equiv 0 \). Therefore, \( \lim_{z \to \infty} \frac{Q_1}{Q_2} = \frac{\alpha_k}{\beta_k} = \frac{t}{n} \) must be a rational number, which contradicts the assumption. This also completes the proof of Theorem 1.9.

**Acknowledgements.** The authors would like to express their hearty thanks to Professor Hongxun Yi for his valuable advice and helpful information.
References


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