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THREE POSITIVE SOLUTIONS FOR m-POINT BOUNDARY-VALUE PROBLEMS WITH ONE-DIMENSIONAL p-LAPLACIAN

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Abstract. In this article, we study the multipoint boundary value problem for the one-dimensional p-Laplacian

$$(\phi_p(u'))' + q(t)f(t, u(t), u'(t)) = 0, \quad t \in (0, 1),$$

subject to the boundary conditions

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(1) = \beta u'(0).$$

Using a fixed point theorem due to Avery and Peterson, we provide sufficient conditions for the existence of at least three positive solutions to the above boundary value problem. The interesting point is that the nonlinear term f involves the first derivative of the unknown function.

1. Introduction

In this article, we study the existence of multiple positive solutions to the boundary value problem (BVP for short) for the one-dimensional p-Laplacian

$$(\phi_p(u'))' + q(t)f(t, u(t), u'(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(1) = \beta u'(0), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. use the following assumptions:

- (H1) $a_i \in [0, 1)$ satisfies $\sum_{i=1}^{m-2} a_i < 1$, $\beta \in (0, 1)$;
- (H2) $f \in C([0, 1] \times [0, +\infty) \times \mathbb{R}, [0, +\infty))$;
- (H3) $q \in L^1[0, 1]$ is nonnegative on $(0, 1)$ and q is not identically zero on any subinterval of $(0, 1)$. Furthermore, q satisfies $0 < \int_0^1 q(t)dt < \infty$.

Multipoint boundary value problems of ordinary differential equations arise in a variety of areas of applied mathematics and physics. For example, the vibrations of

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a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multipoint boundary value problem (see [15]). The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [9]. Since then there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see [1, 3, 4, 5, 6, 7, 8, 11, 12, 13, 16, 17, 18]. However, to the best knowledge of the authors, no work has been done for (1.1), (1.2). The aim of this paper is to fill this gap in the relevant literature.

Karakostas [10] proved the existence of positive solutions for the multipoint boundary-value problem

$$x''(t) - sign(1-\alpha)q(t)f(x,x')x' = 0,\ t \in (0,1),$$

with one of the following sets of boundary conditions:

$$x(0) = 0, \quad x'(1) = \alpha x'(0),$$

or

$$x(1) = 0, \quad x'(1) = \alpha x'(0),$$

where $\alpha > 0$, $\alpha \neq 1$. By using indices of convergence of the nonlinearities at 0 and at $+\infty$, the author provide a priori upper and lower bounds for the slope of the solutions.

Ma [14] proved the existence of positive solutions for the multipoint boundary-value problem

$$\begin{gathered}
x''(t) - q(t)f(x,x')x' = 0, \quad t \in (0,1), \\
x(0) = \sum_{i=1}^{n-2} b_i x(\xi_i), \quad x'(1) = \alpha x'(0),
\end{gathered}$$

where $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $b_i \in [0,1)$, $\alpha > 1$. They provided sufficient conditions for the existence of multiple positive solutions to the above BVP by applying the fixed point theorem in cones.

Motivated by these results, our purpose of this paper is to show the existence of at least three positive solutions to multipoint BVP (1.1) and (1.2).

2. Preliminaries

For the convenience of readers, we provide some background material from the theory of cones in Banach spaces. We also state in this section the Avery-Peterson's fixed point theorem.

Definition 2.1. Let E be a real Banach space over $\mathbb{R}$. A nonempty closed set $P \subset E$ is said to be a cone provide that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0, b \geq 0$; and
(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map γ is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\gamma : P \to [0, \infty)$ is continuous and

$$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on a cone P, α be a nonnegative continuous concave functional on a cone P, and ψ a nonnegative continuous functional on a cone P. Then for positive real numbers a, b, c, d, we define the following convex sets:

$$\begin{gathered}
P(\gamma, d) = \{u \in P : \gamma(u) < d\}, \\
P(\gamma, \alpha, b, d) = \{u \in P : b \leq \alpha(u), \gamma(u) \leq d\}, \\
P(\gamma, \theta, \alpha, b, c, d) = \{u \in P : b \leq \alpha(u), \theta(u) \leq c, \gamma(u) \leq d\}, \\
R(\gamma, \psi, a, d) = \{u \in P : a \leq \psi(u), \gamma(u) \leq d\}.
\end{gathered}$$

To prove our results, we need the following fixed point theorem due to Avery and Peterson [2].

Theorem 2.3. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers $\overline{M}$ and d,*

$$\alpha(u) \leq \psi(u) \quad \text{and} \quad \|u\| \leq \overline{M}\gamma(u) \tag{2.1}$$

for all $u \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b and c with $a < b$ such that

- (S1) *the set $\{u \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(u) > b\} \neq \phi$ and $\alpha(Tu) > b$ for all u in $P(\gamma, \theta, \alpha, b, c, d)$;*
- (S2) *$\alpha(Tu) > b$ for $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > c$;*
- (S3) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tu) < a$ for $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$.*

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, d)}$, such that $\gamma(u_i) \leq d$ for $i = 1, 2, 3$, $b < \alpha(u_1)$, $a < \psi(u_2)$, with $\alpha(u_2) < b$, $\psi(u_3) < a$.

3. Related lemmas

Let the Banach space $E = C^1[0, 1]$ be endowed with the norm

$$\|u\| = \max\big\{\max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)|\big\}.$$

Define the cone $P \subset E$ by $P = \{u \in E : u(t) \geq 0,\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u'(1) = \beta u'(0),\ u \text{ is concave on } [0, 1]\} \subset E$.

It follows from (H3) that there exists a natural number $k > \max\{\frac{1}{\xi_1}, \frac{1}{1-\xi_{m-2}}\}$ such that $0 < \int_{1/k}^{1-(1/k)} q(t)dt < \infty$.

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\gamma(u) = \max_{0\leq t\leq 1} |u'(t)|, \quad \psi(u) = \theta(u) = \max_{0\leq t\leq 1} |u(t)|,$$

$$\alpha(u) = \min_{1/k \le t \le 1-(1/k)} |u(t)| \quad \text{for } u \in P.$$

Lemma 3.1. *Assume that* (H1)–(H3) *hold. Then, for any* $x \in C^+[0,1] =: \{x \in C^1[0,1] : x(t) \ge 0\}$,

$$(\phi_p(u'))' + q(t)f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \tag{3.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \; u'(1) = \beta u'(0), \tag{3.2}$$

has the unique solution

$$\begin{aligned} u(t) = &\int_0^t \phi_p^{-1}\Big(\int_s^1 q(\tau)f(\tau, x(\tau), x'(\tau))d\tau \\ &+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\Big)ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\Big(\int_s^1 q(\tau)f(\tau, x(\tau), x'(\tau))d\tau \\ &+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\Big)ds. \end{aligned} \tag{3.3}$$

Proof. For an $x \in C^+[0,1]$, suppose u is a solution of (3.1), (3.2). By integration of (3.1), it follows that

$$u'(t) = \phi_p^{-1}\Big(\phi_p(u'(0)) - \int_0^t q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\Big),$$
$$u(t) = u(0) + \int_0^t \phi_p^{-1}\Big(\phi_p(u'(0)) - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\Big)ds.$$

Using the boundary condition (3.2), we can easily show that

$$\begin{aligned} u(t) = &\int_0^t \phi_p^{-1}\Big(\int_s^1 q(\tau)f(\tau, x(\tau), x'(\tau))d\tau \\ &+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\Big)ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\Big(\int_s^1 q(\tau)f(\tau, x(\tau), x'(\tau))d\tau \\ &+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\Big)ds. \end{aligned}$$

One the other hand, it is easy to verify that if u is the solution of (3.3), then u is a solution of (3.1) and (3.2). □

Lemma 3.2. *Assume that* (H1)–(H3) *hold. If* $x \in C^+[0,1]$, *then the unique solution* $u(t)$ *of* (3.1) *and* (3.2) *is concave and* $u(t) \ge 0$, $u'(t) \ge 0$, $t \in [0,1]$.

Proof. From the fact that $(\phi_p(u'))'(t) = -q(t)f(t, x(t), x'(t)) \le 0$, we have $\phi_p(u'(t))$ is nonincreasing. It follows that $u'(t)$ is also nonincreasing. Thus, we know that the graph of $u(t)$ is concave down on $(0,1)$. Then the concavity of u together with boundary condition $u'(1) = \beta u'(0)$ implies that $u'(t) \ge 0$ for $t \in [0,1]$.

From $u'(t) \geq 0$, we know that $u(\xi_i) \geq u(0)$, for $i = 1, 2, \dots, m-2$. This implies

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i u(0).$$

By $1 - \sum_{i=1}^{m-2} a_i > 0$, it is obvious that $u(0) \geq 0$. Hence, we know that $u(t) \geq 0$, $t \in [0, 1]$. □

Lemma 3.3. *If $u \in P$, then $\max_{0 \leq t \leq 1} |u(t)| \leq \overline{M} \max_{0 \leq t \leq 1} |u'(t)|$, where $\overline{M} = 1 + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i}$.*

Proof. For $u \in P$, by the concavity of u and that $u'(t) \geq 0$, one arrives at

$$u(1) - u(0) \leq u'(0) = \max_{0 \leq t \leq 1} |u'(t)|.$$

On the other hand,

$$\begin{aligned}
(1 - \sum_{i=1}^{m-2} a_i) u(0) &= u(0) - \sum_{i=1}^{m-2} a_i u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i u(0) \\
&= \sum_{i=1}^{m-2} a_i (u(\xi_i) - u(0)) = \sum_{i=1}^{m-2} a_i \xi_i u'(\eta_i),
\end{aligned}$$

where $\eta_i \in (0, \xi_i)$. So

$$u(0) = \frac{\sum_{i=1}^{m-2} a_i \xi_i u'(\eta_i)}{1 - \sum_{i=1}^{m-2} a_i} \leq \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i} \max_{0 \leq t \leq 1} |u'(t)|.$$

Thus one has

$$\max_{0 \leq t \leq 1} |u(t)| = u(1) \leq \Big(1 + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i}\Big) u'(0) = \overline{M} \max_{0 \leq t \leq 1} |u'(t)|.$$

With Lemma 3.3 and the concavity of u, for all $u \in P$, we obtain

$$\frac{1}{k} \theta(u) \leq \alpha(u) \leq \theta(u) = \psi(u), \quad \|u\| = \max\{\theta(u), \gamma(u)\} \leq \overline{M} \gamma(u). \tag{3.4}$$

□

Lemma 3.4. *Define an operator $T : P \to P$,*

$$\begin{aligned}
(Tu)(t) = \int_0^t & \phi_p^{-1}\Big(\int_s^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau l \\
&+ \frac{\phi_p(\beta)}{1 - \phi_p(\beta)} \int_0^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau\Big) ds \\
&+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\Big(\int_s^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \\
&+ \frac{\phi_p(\beta)}{1 - \phi_p(\beta)} \int_0^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau\Big) ds.
\end{aligned} \tag{3.5}$$

Then $T : \ P \to P$ is completely continuous.

Proof. According to the definition of T and Lemma 3.2, it is easy to show that $T(P) \subset P$. By similar arguments in [10, 17], $T : P \to P$ is completely continuous. □

4. Existence of positive solutions

We are now ready to apply the Avery-Peterson's fixed point theorem to the operator T to give sufficient conditions for the existence of at least three positive solutions to (1.1), (1.2). For convenience we introduce following notation. Let

$$K = \frac{4k^2(1-\sum_{i=1}^{m-2} a_i)}{(k\beta+\beta+3k-1)(1-\sum_{i=1}^{m-2} a_i)+(k^2+k)\sum_{i=1}^{m-2} a_i[2\xi_i-(1-\beta)\xi_i^2]},$$

$$L = \phi_p^{-1}\Big(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)d\tau\Big),$$

$$\begin{aligned} M = &\int_{1/k}^{1-(1/k)} \phi_p^{-1}\Big(\int_s^{1-(1/k)} q(\tau)d\tau + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_{1/k}^{1-(1/k)} q(\tau)d\tau\Big)ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\int_{1/k}^{\xi_i} \phi_p^{-1}\Big(\int_s^{1-(1/k)} q(\tau)d\tau \\ &+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_{1/k}^{1-(1/k)} q(\tau)d\tau\Big)ds, \end{aligned}$$

$$\begin{aligned} N = &\int_0^1 \phi_p^{-1}\Big(\int_s^1 q(\tau)d\tau + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)d\tau\Big)ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} \phi_p^{-1}\Big(\int_s^1 q(\tau)d\tau + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)d\tau\Big)ds. \end{aligned}$$

Theorem 4.1. *Assume that* (H1)–(H3) *hold. Let* $0 < a < b \le \min\{\frac{1}{K}, \frac{\overline{M}}{k}\}d$, *and suppose that* f *satisfies the following conditions:*

- (A1) $f(t,u,v) \le \phi_p(d/L)$ *for* $(t,u,v) \in [0,1]\times[0,\overline{M}d]\times[0,d]$;
- (A2) $f(t,u,v) \ge \phi_p(kb/M)$ *for* $(t,u,v) \in [1/k, 1-1/k]\times[b,kb]\times[0,d]$;
- (A3) $f(t,u,v) < \phi_p(a/N)$ *for* $(t,u,v) \in [0,1]\times[0,a]\times[0,d]$.

Then boundary-value problem (1.1), (1.2) *has at least three positive solutions* u_1, u_2, u_3 *such that* $\max_{0\le t\le 1}|u_i'(t)| \le d$, *for* $i = 1,2,3$, *and*

$$b < \min_{1/k\le t\le 1-(1/k)}|u_1(t)|, \quad \max_{0\le t\le 1}|u_1(t)| \le \overline{M}d, \quad a < \max_{0\le t\le 1}|u_2(t)| < kb,$$

with

$$\min_{1/k\le t\le 1-(1/k)}|u_2(t)| < b, \quad \max_{0\le t\le 1}|u_3(t)| < a.$$

Proof. Recall that (1.1), (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$. Thus we set out to verify that the operator T satisfies Avery-Peterson's fixed point theorem which will prove the existence of three fixed points of T. The proof is divided into four steps.

(1) We will show that (A1) implies

$$T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}. \tag{4.1}$$

In fact, for $u \in \overline{P(\gamma,d)}$, there is $\gamma(u) = \max_{0\le t\le 1}|u'(t)| \le d$. With Lemma 3.3, we have $\max_{0\le t\le 1}|u(t)| \le \overline{M}d$. Then condition (A1) implies $f(t,u,v) \le \phi_p(d/L)$. On the other hand, one has $Tu \in P$ for $u \in P$, then Tu is concave and $\max_{0\le t\le 1}|(Tu)'(t)| = (Tu)'(0)$, so

$$\gamma(Tu) = \max_{0\le t\le 1}|(Tu)'(t)| = (Tu)'(0)$$

$$
\begin{aligned}
&= \phi_p^{-1}\Big(\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)\\
&= \phi_p^{-1}\Big(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)\\
&\le \phi_p^{-1}\Big(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)d\tau\phi_p(d/L)\Big)\\
&= \frac{d}{L}\phi_p^{-1}\Big(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)d\tau\Big) = \frac{d}{L}L = d.
\end{aligned}
$$

Thus, (4.1) holds.

(2) We show that the condition (S1) in Theorem 2.3 holds. We take

$$
u_0(t) = -\frac{2k^2b(1-\sum_{i=1}^{m-2}a_i)[(1-\beta)t^2-2t] + 2k^2b\sum_{i=1}^{m-2}a_i[(1-\beta)\xi_i^2-2\xi_i]}{(k\beta+\beta+3k-1)(1-\sum_{i=1}^{m-2}a_i) + (k^2+k)\sum_{i=1}^{m-2}a_i[2\xi_i-(1-\beta)\xi_i^2]},
$$

for $t \in [0,1]$. It is easy to see that $u_0(0) = \sum_{i=1}^{m-2}a_iu_0(\xi_i)$, $u_0'(1) = \beta u_0'(0)$, $u_0(t) \ge 0$ and is concave on $[0,1]$ so that $u_0(t) \in P$. At the same time,

$$
\begin{aligned}
\alpha(u_0) &= \min_{1/k\le t\le 1-(1/k)}|u_0| = u_0(\frac{1}{k})\\
&= \frac{2b\Big((1-\sum_{i=1}^{m-2}a_i)(2k+\beta-1) + k^2\sum_{i=1}^{m-2}a_i[2\xi_i-(1-\beta)\xi_i^2]\Big)}{(k\beta+\beta+3k-1)(1-\sum_{i=1}^{m-2}a_i) + (k^2+k)\sum_{i=1}^{m-2}a_i[2\xi_i-(1-\beta)\xi_i^2]}\\
&> b,
\end{aligned}
$$

$$
\begin{aligned}
\theta(u_0) &= \max_{0\le t\le 1}|u_0(t)| = u_0(1)\\
&= \frac{2k^2b\Big((1-\sum_{i=1}^{m-2}a_i)(\beta+1) + \sum_{i=1}^{m-2}a_i[2\xi_i-(1-\beta)\xi_i^2]\Big)}{(k\beta+\beta+3k-1)(1-\sum_{i=1}^{m-2}a_i) + (k^2+k)\sum_{i=1}^{m-2}a_i[2\xi_i-(1-\beta)\xi_i^2]}\\
&< kb,
\end{aligned}
$$

$$
\begin{aligned}
\gamma(u_0) &= \max_{0\le t\le 1}|u'(t)| = u_0'(0)\\
&= \frac{4k^2b(1-\sum_{i=1}^{m-2}a_i)}{(k\beta+\beta+3k-1)(1-\sum_{i=1}^{m-2}a_i) + (k^2+k)\sum_{i=1}^{m-2}a_i[2\xi_i-(1-\beta)\xi_i^2]}\\
&= Kb \le d.
\end{aligned}
$$

So $u_0(t) \in P(\gamma,\theta,\alpha,b,kb,d)$ and $\{u \in P(\gamma,\theta,\alpha,b,kb,d) \mid \alpha(u) > b\} \ne \phi$. Thus for $u \in P(\gamma,\theta,\alpha,b,kb,d)$, there is $b \le u(t) \le kb$, $0 \le u'(t) \le d$, for $1/k \le t \le 1-1/k$. Hence by condition (A2) of this theorem, one has $f(t,u,u'(t)) \ge \phi_p(kb/M)$, for $t \in [1/k, 1-1/k]$. Noting $(Tu)(1) \ge 0$ from Lemma 3.2 and combining the conditions on α and P, one arrives at

$$
\begin{aligned}
\alpha(Tu) &= \min_{1/k\le t\le 1-(1/k)}|(Tu)(t)| \ge \frac{1}{k}\max_{0\le t\le 1}|(Tu)(t)| = \frac{1}{k}(Tu)(1)\\
&= \frac{1}{k}\int_0^1 \phi_p^{-1}\Big(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\\
&\quad + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{k(1-\sum_{i=1}^{m-2} a_i)} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\Big(\int_s^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \\
&+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \Big) ds \\
> &\frac{1}{k} \int_{1/k}^{1-(1/k)} \phi_p^{-1}\Big(\int_s^{1-(1/k)} q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \\
&+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_{1/k}^{1-(1/k)} q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \Big) ds \\
&+ \frac{1}{k(1-\sum_{i=1}^{m-2} a_i)} \sum_{i=1}^{m-2} a_i \int_{1/k}^{\xi_i} \phi_p^{-1}\Big(\int_s^{1-(1/k)} q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \\
&+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_{1/k}^{1-(1/k)} q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \Big) ds \\
\geq &\frac{1}{k} \int_{1/k}^{1-(1/k)} \phi_p^{-1}\Big(\int_s^{1-(1/k)} q(\tau) d\tau \phi_p(\frac{kb}{M}) \\
&+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_{1/k}^{1-(1/k)} q(\tau) d\tau \phi_p(\frac{kb}{M}) \Big) ds \\
&+ \frac{1}{k(1-\sum_{i=1}^{m-2} a_i)} \sum_{i=1}^{m-2} a_i \int_{1/k}^{\xi_i} \phi_p^{-1}\Big(\int_s^{1-(1/k)} q(\tau) d\tau \phi_p(\frac{kb}{M}) \\
&+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_{1/k}^{1-(1/k)} q(\tau) d\tau \phi_p(\frac{kb}{M}) \Big) ds \\
= &\frac{1}{k} \frac{kb}{M} M = b.
\end{aligned}
$$

Therefore, $\alpha(Tu) > b$, for all $u \in P(\gamma, \theta, \alpha, b, kb, d)$. Consequently, condition (S1) in Theorem 2.3 holds.

(3) We now prove (S2) in Theorem 2.3 holds. With (3.4) we have

$$
\alpha(Tu) \geq \frac{1}{k} \theta(Tu) > \frac{1}{k} kb = b,
$$

for $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > kb$. Hence, condition (S2) in Theorem 2.3 is satisfied.

(4) Finally, we show that (S3) in Theorem 2.3 is satisfied. Since $\psi(0) = 0 < a$, so $0 \notin R(\gamma, \psi, a, d)$. Suppose that $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$. Then by the condition (A3) of this theorem,

$$
\begin{aligned}
\psi(Tu) &= \max_{0 \leq t \leq 1} |(Tu)(t)| = (Tu)(1) \\
&= \int_0^1 \phi_p^{-1}\Big(\int_s^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \\
&\quad + \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \Big) ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\Big(\int_s^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \Big) ds \\
&< \int_0^1 \phi_p^{-1} \Big(\int_s^1 q(\tau) d\tau \phi_p(a/N) + \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau) d\tau \phi_p(a/N) \Big) ds \\
&+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1} \Big(\int_s^1 q(\tau) d\tau \phi_p(a/N) \\
&+ \frac{\phi_p(\beta)}{1-\phi_p(\beta)} \int_0^1 q(\tau) d\tau \phi_p(a/N) \Big) ds \\
&= \frac{a}{N} N = a.
\end{aligned}
$$

Thus, condition (S3) in Theorem 2.3 holds.

Then Theorem 2.3 implies that (1.1), (1.2) has at least three positive solutions satisfying the statement in Theorem 4.1. The proof is complete. □

5. Example

Let $p = 3$, $q(t) = 1$ in (1.1) and $m = 4$, $\beta = 1/2$, $\xi_1 = 1/3$, $\xi_2 = 2.3$, $a_1 = 1/2$, $a_2 = 1/4$ in (1.2). Now we consider the boundary-value problem

$$
(|u'(t)|u'(t))' + f(t, u(t), u'(t)) = 0, \quad t \in (0,1), \tag{5.1}
$$

$$
u(0) = \frac{1}{2} u(\frac{1}{3}) + \frac{1}{4} u(\frac{2}{3}), \quad u'(1) = \frac{1}{2} u'(0), \tag{5.2}
$$

where

$$
f(t,u,v) = \begin{cases} \frac{t}{20} + 1.8 \times 10^4 \cdot u^{40} + \frac{1}{100} (\frac{v}{3 \times 10^{21}})^4 + \frac{1}{1000}, & u \le 6; \\ \frac{t}{20} + 1.8 \times 10^4 \cdot 6^{40} + \frac{1}{100} (\frac{v}{3 \times 10^{21}})^4 + \frac{1}{1000}, & u > 6. \end{cases}
$$

Choose $a = 2/3$, $b = 1$, $k = 6$, $d = 3 \times 10^{21}$, we note that $K = 4/5$, $L = 2\sqrt{3}/3$, $\overline{M} = 7/3$, $M \doteq 1.185$, $N \doteq 2.281$. Consequently, $f(t,u,v)$ satisfies

(1) $f(t,u,v) < 2.407 \times 10^{35} < \phi_3(d/L) \doteq 6.75 \times 10^{42}$ for $(t,u,v) \in [0,1] \times [0, 7 \times 10^{21}] \times [0, 3 \times 10^{21}]$;
(2) $f(t,u,v) > 1.8 \times 10^4 > \phi_3(kb/M) \doteq 25.637$ for $(t,u,v) \in [1/6, 5/6] \times [1,6] \times [0, 3 \times 10^{21}]$;
(3) $f(t,u,v) < 0.063 < \phi_3(a/N) \doteq 0.085$ for $(t,u,v) \in [0,1] \times [0, 2/3] \times [0, 3 \times 10^{21}]$.

Then all conditions of Theorem 4.1 hold. Therefore, (5.1), (5.2) has at least three positive solutions u_1, u_2, u_3 satisfying

$$
\max_{0 \le t \le 1} |u_i'(t)| \le 3 \times 10^{21} \quad \text{for } i = 1,2,3,
$$

$$
1 < \min_{1/k \le t \le 1-(1/k)} |u_1(t)|, \quad \max_{0 \le t \le 1} |u_1(t)| \le 7 \times 10^{21}, \quad \frac{2}{3} < \max_{0 \le t \le 1} |u_2(t)| < 6,
$$

with $\min_{1/k \le t \le 1-(1/k)} |u_2(t)| < 1$, $\max_{0 \le t \le 1} |u_3(t)| < 2/3$.

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