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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SECOND-ORDER NONLINEAR HYPERBOLIC SYSTEM

RODICA LUCA

Abstract. We study the existence, uniqueness and asymptotic behaviour of solutions to a second-order nonlinear hyperbolic system of equations. The spatial variable is in the positive half-axis.

1. Introduction

We study the second-order nonlinear hyperbolic system

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + \frac{\partial^2 v}{\partial x^2}(t,x) + \alpha(x,u) = f(t,x), \\ &\frac{\partial v}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) + \beta(x,v) = g(t,x), \quad t > 0,\ x > 0, \end{aligned} \tag{1.1}$$

with the boundary condition

$$\begin{pmatrix} \operatorname{col}\big(-\frac{\partial u}{\partial x}(t,0), u(t,0)\big) \\ w'(t) \end{pmatrix} \in -G\left(\begin{pmatrix} \operatorname{col}\big(v(t,0), \frac{\partial v}{\partial x}(t,0)\big) \\ w(t) \end{pmatrix}\right) + B(t), \quad t > 0, \tag{1.2}$$

and the initial condition

$$\begin{gathered} u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x > 0, \\ w(0) = w_0. \end{gathered} \tag{1.3}$$

The unknowns u, v and f and g are the vectorial functions depending on $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$ with values in $\mathbb{R}^n$, the unknown w is a vectorial function depending on $t \in \mathbb{R}_+$ with values in $\mathbb{R}^m$. In the system (1.1), the functions α and β are of the form $\alpha(x,u) = \operatorname{col}(\alpha_1(x,u_1), \dots, \alpha_n(x,u_n))$, $\beta(x,v) = \operatorname{col}(\beta_1(x,v_1), \dots, \beta_n(x,v_n))$, G is an operator in $\mathbb{R}^{2n+m}$ and $B(t) = \operatorname{col}(b_1(t), \dots, b_{2n+m}(t)) \in \mathbb{R}^{2n+m}$, for all $t > 0$.

This problem is a generalization of the case studied by Luca [8], where $B(t) \equiv 0$. The methods we shall use to prove the main results in this article are different from that used in Luca [8]. We also mention the articles [6, 7, 9], where we have investigated a n-order hyperbolic system, for spatial variable $x \in (0,1)$ and $t > 0$, subject to some nonlinear boundary conditions.

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In the present work, we shall prove the existence, uniqueness, some regularity properties and the asymptotic behaviour of the strong and weak solutions for the problem (1.1), (1.2), (1.3). For the basic notation, concepts and results in the theory of monotone operators and nonlinear evolution equations of monotone type in Hilbert spaces we refer the reader to Barbu [2], Brezis [3], Lakshmikantham et al [5].

Now we introduce the assumptions to be used in this article.

(A1) (a) The functions $x \to \alpha_k(x,p)$ and $x \to \beta_k(x,p)$ are measurable on $\mathbb{R}_+$, for any fixed $p \in \mathbb{R}$. Besides, the functions $p \to \alpha_k(x,p)$ and $p \to \beta_k(x,p)$ are continuous and nondecreasing from $\mathbb{R}$ into $\mathbb{R}$, for a.a. $x \in \mathbb{R}_+$, $k = \overline{1,n}$.

(b) There exist $a_k > 0$, $b_k > 0$, $k = \overline{1,n}$ and the functions $c_k^1, c_k^2 \in L^2(\mathbb{R}_+)$ such that

$$|\alpha_k(x,p)| \le a_k|p| + c_k^1(x), \quad |\beta_k(x,p)| \le b_k|p| + c_k^2(x),$$

for a.a. $x \in \mathbb{R}_+$, for all $p \in \mathbb{R}$, $k = \overline{1,n}$.

(c) There exist $\chi_1 > 0$, $\chi_2 > 0$ such that

$$\begin{gathered}(\alpha_k(x,p_1) - \alpha_k(x,p_2))(p_1 - p_2) \ge \chi_1(p_1 - p_2)^2, \\ (\beta_k(x,p_1) - \beta_k(x,p_2))(p_1 - p_2) \ge \chi_2(p_1 - p_2)^2,\end{gathered}$$

for a.a. $x \in \mathbb{R}_+$, for all p_1, $p_2 \in \mathbb{R}$, $k = \overline{1,n}$.

(A2) (a) $G : D(G) \subset \mathbb{R}^{2n+m} \to \mathbb{R}^{2n+m}$ is a maximal monotone operator (possibly multivalued). Moreover, $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ with

$$\begin{gathered}G_{11} : D(G_{11}) \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad G_{12} : D(G_{12}) \subset \mathbb{R}^{m} \to \mathbb{R}^{2n}, \\ G_{21} : D(G_{21}) \subset \mathbb{R}^{2n} \to \mathbb{R}^{m}, \quad G_{22} : D(G_{22}) \subset \mathbb{R}^{m} \to \mathbb{R}^{m},\end{gathered}$$

where

$$G((u_1,\dots,u_{2n+m})^T) = \begin{pmatrix} G_{11}((u_1,\dots,u_{2n})^T) + G_{12}((u_{2n+1},\dots,u_{2n+m})^T) \\ G_{21}((u_1,\dots,u_{2n})^T) + G_{22}((u_{2n+1},\dots,u_{2n+m})^T) \end{pmatrix}.$$

(b) There exists $\zeta_1 > 0$ such that for all $x, y \in D(G)$, $x = \operatorname{col}(x^a, x^b)$, $y = \operatorname{col}(y^a, y^b) \in \mathbb{R}^{2n} \times \mathbb{R}^m$ and for all $w_1 \in G(x)$, $w_2 \in G(y)$ we have

$$\langle w_1 - w_2, x - y \rangle_{\mathbb{R}^{2n+m}} \ge \zeta_1 \|x^b - y^b\|^2_{\mathbb{R}^m}.$$

(c) There exists $\zeta_2 > 0$ such that for all $x, y \in D(G)$ and for all $w_1 \in G(x)$, $w_2 \in G(y)$ we have

$$\langle w_1 - w_2, x - y \rangle_{\mathbb{R}^{2n+m}} \ge \zeta_2 \|x - y\|^2_{\mathbb{R}^{2n+m}}.$$

The operator G is a generalization of the matrix case. It also covers some general boundary conditions for (1.1). For example, if $G_{12} = 0$ and $G_{21} = 0$, then the boundary condition (1.2) becomes

$$\operatorname{col}(-u_x(t,0), u(t,0)) \in -G_{11}(\operatorname{col}(v(t,0), v_x(t,0))) + B_1(t), \tag{1.4}$$

$$w'(t) \in -G_{22}(w(t)) + B_2(t), \tag{1.5}$$

where $B(t) = \operatorname{col}(B_1(t), B_2(t))$.

The condition (1.5) with (1.3) give us, by integration, the function w. For (1.4), by making suitable choices of G_{11}, we deduce many classical boundary conditions. Here are some examples in the case $G_{11} = \partial l$, the subdifferential of $l : \mathbb{R}^{2n} \to (-\infty, +\infty]$:

(a) If
$$l\begin{pmatrix}u\\v\end{pmatrix}=\begin{cases}0, & \text{if } u=a,\ v=b\\ +\infty, & \text{otherwise.}\end{cases}$$
then (1.4) becomes $v(t,0)=a$, $v_x(t,0)=b$.

(b) For $n=1$ and $l\begin{pmatrix}u\\v\end{pmatrix}=\begin{cases}bv, & \text{if } u=a\\ \infty, & \text{otherwise,}\end{cases}$ we obtain $u(t,0)=B_2(t)-b$, $v(t,0)=a$.

(c) For $n=1$ and $l\begin{pmatrix}u\\v\end{pmatrix}=\begin{cases}au, & \text{if } v=b\\ \infty, & \text{otherwise,}\end{cases}$ we obtain $u_x(t,0)=-B_1(t)+a$, $v_x(t,0)=b$.

(d) For $n=1$ and $l\begin{pmatrix}u\\v\end{pmatrix}=au+bv$, then (1.4) becomes $u_x(t,0)=-B_1(t)+a$, $u(t,0)=B_2(t)-b$.

2. Preliminary results

Firstly we consider the case $B(t)\equiv b_0$ (a constant vector). In this situation, we can replace G by $\widetilde{G}$, defined by $\widetilde{G}w=Gw-b_0$, which is under the assumption (A2)a, a maximal monotone operator. So, we suppose without loss of generality that $B(t)\equiv 0$. In what follows, we shall recall some results from Luca [8] relating to the existence and uniqueness of the solutions of our problem (1.1), (1.2), (1.3). We consider the spaces $X=(L^2(\mathbb{R}_+;\mathbb{R}^n))^2$, $\mathbb{R}^m$ and $Y=X\times\mathbb{R}^m$ with the corresponding scalar products
$$\langle f,g\rangle_X=\langle f_1,g_1\rangle_{L^2(\mathbb{R}_+;\mathbb{R}^n)}+\langle f_2,g_2\rangle_{L^2(\mathbb{R}_+;\mathbb{R}^n)},\quad f=\begin{pmatrix}f_1\\f_2\end{pmatrix},\quad g=\begin{pmatrix}g_1\\g_2\end{pmatrix}\in X,$$
$$\langle x,y\rangle_{\mathbb{R}^m}=\sum_{i=1}^m x_iy_i,\quad x,y\in\mathbb{R}^m,$$
$$\Big\langle\begin{pmatrix}f\\x\end{pmatrix},\begin{pmatrix}g\\y\end{pmatrix}\Big\rangle_Y=\langle f,g\rangle_X+\langle x,y\rangle_{\mathbb{R}^m},\quad \begin{pmatrix}f\\x\end{pmatrix},\quad \begin{pmatrix}g\\y\end{pmatrix}\in Y.$$
We define the operator $\mathcal{A}:D(\mathcal{A})\subset Y\to Y$,
$$\begin{gathered}D(\mathcal{A})=\{y\in Y, y=\operatorname{col}(u,v,w);\ u,v\in H^2(\mathbb{R}_+;\mathbb{R}^n),\ w\in\mathbb{R}^m,\\ \operatorname{col}(\gamma_0v,w)\in D(G),\ \gamma_1u\in -G_{11}(\gamma_0v)-G_{12}(w)\},\end{gathered}$$
where $\gamma_0v=\operatorname{col}(v(0),v'(0))$, $\gamma_1u=\operatorname{col}(-u'(0),u(0))$,
$$\mathcal{A}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}v''\\-u''\\G_{21}(\gamma_0v)+G_{22}(w)\end{pmatrix},\quad \begin{pmatrix}u\\v\\w\end{pmatrix}\in D(\mathcal{A}).$$
We also define the operator $\mathcal{B}:D(\mathcal{B})\subset Y\to Y$,
$$\mathcal{B}(y)=\operatorname{col}(\alpha(\cdot,u),\beta(\cdot,v),0),\ D(\mathcal{B})=\{y\in Y,\ y=\operatorname{col}(u,v,w);\ \mathcal{B}(y)\in Y\}.$$
Under assumption (A2)a, we have $D(\mathcal{A})\neq\emptyset$, $\overline{D(\mathcal{A})}=X\times\overline{D(G_{12})\cap D(G_{22})}$ and, under the assumptions (A1)ab, we obtain $D(\mathcal{B})=Y$.

Lemma 2.1. *If* (A2)a *holds, then the operator $\mathcal{A}$ is maximal monotone.*

Lemma 2.2. *If* (A1)ab *hold, then the operator $\mathcal{B}$ is maximal monotone.*

Remark 2.3. By Lemma 2.1, Lemma 2.2 and Rockafellar's theorem (see Barbu [2, Theorem 1.7, Chapter II]), it follows that, under the assumptions (A1)ab and (A2)a, the operator $\mathcal{A}+\mathcal{B}: D(\mathcal{A}) \subset Y \to Y$ is maximal monotone in the space Y.

Using the operators $\mathcal{A}$ and $\mathcal{B}$, our problem (1.1), (1.2), (1.3) can be equivalently expressed as the following Cauchy problem in the space Y

$$\begin{gathered}\frac{dy}{dt}(t)+\mathcal{A}(y(t))+\mathcal{B}((y(t)) \ni F(t,\cdot), \quad ,t>0, \\ y(0)=y_0,\end{gathered} \tag{2.1}$$

where $y(t)=\operatorname{col}(u(t),v(t),w(t))$, $F(t,\cdot)=\operatorname{col}(f(t,\cdot),g(t,\cdot),0)$, $y_0=\operatorname{col}(u_0,v_0,w_0)$.

Lemma 2.4. *Assume that* (A1)ab, (A2)a *hold. If* $f,g \in W^{1,1}(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ *(with* $T>0$ *fixed),* $u_0,v_0 \in H^2(\mathbb{R}_+;\mathbb{R}^n)$, $\operatorname{col}(\gamma_0 v_0,w_0) \in D(G)$, $\gamma_1 u_0$ *belongs to* $-G_{11}(\gamma_0 v_0)-G_{12}(w_0)$*, then problem* (1.1), (1.2), (1.3) *has a unique strong solution* $\operatorname{col}(u,v,w) \in W^{1,\infty}(0,T;Y)$*. Moreover* $u,v \in L^\infty(0,T;H^2(\mathbb{R}_+;\mathbb{R}^n))$.

Remark 2.5. For all $t \in [0,T)$, the above functions $u(t,\cdot),v(t,\cdot)$ satisfy the system (1.1) for a.a. $x \in \mathbb{R}_+$ (with $\partial^+u/\partial t$, $\partial^+v/\partial t$ instead of $\partial u/\partial t$, $\partial v/\partial t$), and together with $w(t)$ verify the boundary condition (1.2) (with d^+w/dt instead of dw/dt) and the initial data (1.3).

Lemma 2.6. *Assume that* (A1)ab, (A2)a *hold. If* $f,g \in L^1(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ *(with* $T>0$ *fixed),* $u_0,v_0 \in L^2(\mathbb{R}_+;\mathbb{R}^n)$, $w_0 \in \overline{D(G_{12}) \cap D(G_{22})}$ *then the problem* (1.1), (1.2), (1.3) *has a unique weak solution* $\operatorname{col}(u,v,w) \in C([0,T];Y)$.

For the proofs of Lemmas 2.1–2.6 see Luca [8].

In what follows we shall present an existence result for the stationary problem associated to (2.1).

Lemma 2.7. *If* (A1)abc, (A2)ab *hold, then the stationary problem*

$$\mathcal{A}(y)+\mathcal{B}(y) \ni 0 \tag{2.2}$$

has a unique solution $y=\operatorname{col}(u,v,w) \in D(\mathcal{A})$.

Proof. By Remark 2.3, the operator $\mathcal{A}+\mathcal{B}$ is maximal monotone in Y. In addition, it is strongly monotone. Indeed, for all $y=\operatorname{col}(u,v,w)$, $\widetilde{y}=\operatorname{col}(\widetilde{u},\widetilde{v},\widetilde{w}) \in D(\mathcal{A})$, $h \in (\mathcal{A}+\mathcal{B})(y)$, $\widetilde{h} \in (\mathcal{A}+\mathcal{B})(\widetilde{y})$ we have

$$\begin{aligned}
&\langle h-\widetilde{h}, y-\widetilde{y}\rangle_Y \\
&= \langle g-\widetilde{g}, z-\widetilde{z}\rangle_{\mathbb{R}^{2n+m}} + \sum_{k=1}^n \int_0^\infty [\alpha_k(x,u_k(x))-\alpha_k(x,\widetilde{u}_k(x))][u_k(x)-\widetilde{u}_k(x)]dx \\
&\quad + \sum_{k=1}^n \int_0^\infty [\beta_k(x,v_k(x))-\beta_k(x,\widetilde{v}_k(x))][v_k(x)-\widetilde{v}_k(x)]dx \\
&\ge \zeta_1\|w-\widetilde{w}\|_{\mathbb{R}^m}^2 + \sum_{k=1}^n \chi_1\|u_k-\widetilde{u}_k\|_{L^2(\mathbb{R}_+)}^2 + \sum_{k=1}^n \chi_2\|v_k-\widetilde{v}_k\|_{L^2(\mathbb{R}_+)}^2 \\
&\ge \chi_0\|y-\widetilde{y}\|_Y^2,
\end{aligned}$$

where $z=\operatorname{col}(\gamma_0 v,w)$, $\widetilde{z}=\operatorname{col}(\gamma_0\widetilde{v},\widetilde{w})$, $g \in G(z)$, $\widetilde{g} \in G(\widetilde{z})$ and

$$\chi_0=\min\{\chi_1,\chi_2,\zeta_1/s_i,,i=\overline{1,m}\}.$$

Therefore, this operator is coercive and then $R(\mathcal{A}+\mathcal{B}) = Y$. So we deduce that equation (2.2) has a unique solution $y = \operatorname{col}(u, v, w) \in D(\mathcal{A})$. $\square$

Now using Remark 2.3, Lemma 2.6, Lemma 2.7 and Brezis [3, Theorem 3.9], we deduce the following result.

Lemma 2.8. *Assume that* (A1)abc, (A2)ab *hold,* $f, g \in L^1_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}_+;\mathbb{R}^n))$ *verify the conditions* $\lim_{t\to\infty} f(t) = f^0$, $\lim_{t\to\infty} g(t) = g^0$, *strongly in* $L^2(\mathbb{R}_+;\mathbb{R}^n)$, *and* $\delta = \operatorname{col}(p, q, r)$ *is the unique solution of equation* (2.2). *Then* $\lim_{t\to\infty} y(t) = \delta$, *strongly in* Y, *where* $y(t) = \operatorname{col}(u(t), v(t), w(t))$, $t \geq 0$ *is an arbitrary weak solution of equation* $(2.1)_1$. *More precisely*

$$\|y(t)-\delta\|_Y \leq e^{-\chi_0 t}\|y(0)-\delta\|_Y + \int_0^t e^{\chi_0(s-t)}\|F(s)-F^0\|_Y ds, \quad t \geq 0,$$

where $F^0 = \operatorname{col}(f^0, g^0, 0)$.

If $\frac{dF}{dt} \in L^1(\mathbb{R}_+; Y)$ *and* $y(0) \in D(\mathcal{A})$ *then* $\lim_{t\to\infty} \|\frac{d^+y}{dt}(t)\|_Y = 0$ *strongly in* Y *and*

$$\int_0^\infty \|\frac{d^+y}{dt}(t)\|_Y dt \leq \frac{1}{\chi_0}\|((\mathcal{A}+\mathcal{B})(y(0)) - F(0))^0\|_Y + \frac{1}{\chi_0}\int_0^\infty \|\frac{dF}{dt}(t)\|_Y dt.$$

3. Existence, uniqueness and asymptotic behaviour of solutions

In the general case $B(t)$ is not constant, we make a change of variables $u_k = \widetilde{u}_k + \widetilde{\widetilde{u}}_k$, where

$$\widetilde{\widetilde{u}}_k(t,x) = (1+x)e^{-x}b_{n+k}(t) - xe^{-x}b_k(t), \quad k = \overline{1,n}.$$

Our problem (1.1), (1.2), (1.3) can be written as

$$\begin{gathered} \frac{\partial \widetilde{u}}{\partial t}(t,x) + \frac{\partial^2 v}{\partial x^2}(t,x) + \alpha(x, \widetilde{u} + \widetilde{\widetilde{u}}(t,x)) = \widetilde{f}(t,x), \\ \frac{\partial v}{\partial t}(t,x) - \frac{\partial^2 \widetilde{u}}{\partial x^2}(t,x) + \beta(x, v) = \widetilde{g}(t,x), \quad t > 0,\ x > 0, \end{gathered} \tag{3.1}$$

with the boundary condition

$$\begin{pmatrix} -\frac{\partial \widetilde{u}}{\partial x}(t,0) \\ \widetilde{u}(t,0) \\ w'(t) \end{pmatrix} \in -G\begin{pmatrix} v(t,0) \\ \frac{\partial v}{\partial x}(t,0) \\ w(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ B_2(t) \end{pmatrix}, \quad t > 0, \tag{3.2}$$

and the initial data

$$\begin{gathered} \widetilde{u}(0,x) = \widetilde{u}_0(x), \quad v(0,x) = v_0(x), \quad x > 0, \\ w(0) = w_0, \end{gathered} \tag{3.3}$$

where

$$\begin{gathered} \widetilde{f}_k(t,x) = f_k(t,x) - \frac{\partial \widetilde{\widetilde{u}}_k}{\partial t}(t,x) = f_k(t,x) - (1+x)e^{-x}b'_{n+k}(t) + xe^{-x}b'_k(t), \\ \widetilde{g}_k(t,x) = g_k(t,x) + \frac{\partial^2 \widetilde{\widetilde{u}}_k}{\partial x^2}(t,x) = g_k(t,x) + (x-1)e^{-x}b_{n+k}(t) - (x-2)e^{-x}b_k(t), \\ x > 0, \quad t > 0, \quad k = \overline{1,n}, \\ \widetilde{u}_{k0}(x) = u_{k0}(x) - (1+x)e^{-x}b_{n+k}(0) + xe^{-x}b_k(0), \quad x > 0, \quad k = \overline{1,n}, \\ B_2(t) = \operatorname{col}(b_{2n+1}(t), \dots, b_{2n+m}(t)). \end{gathered}$$

Using the operators $\mathcal{A}$ and $\mathcal{B}$, the problem (3.1), (3.2), (3.3) can be equivalently formulated as a time dependent Cauchy problem in the space Y,

$$\begin{gathered}
\frac{d}{dt}\begin{pmatrix}\widetilde{u}\\ v\\ w\end{pmatrix} + \mathcal{A}\begin{pmatrix}\widetilde{u}\\ v\\ w\end{pmatrix} + \mathcal{B}\begin{pmatrix}\widetilde{u}+\widetilde{\widetilde{u}}(t)\\ v\\ w\end{pmatrix} \ni \begin{pmatrix}\widetilde{f}(t,\cdot)\\ \widetilde{g}(t,\cdot)\\ B_2(t)\end{pmatrix}\\
\begin{pmatrix}\widetilde{u}(0)\\ v(0)\\ w(0)\end{pmatrix} = \begin{pmatrix}\widetilde{u}_0\\ v_0\\ w_0\end{pmatrix},
\end{gathered} \tag{3.4}$$

where $\widetilde{f} = \operatorname{col}(\widetilde{f}_1,\dots,\widetilde{f}_n)$, $\widetilde{g} = \operatorname{col}(\widetilde{g}_1,\dots,\widetilde{g}_n)$, $\widetilde{u}_0 = \operatorname{col}(\widetilde{u}_{10},\dots,\widetilde{u}_{n0})$.

Theorem 3.1. *Assume that* (A1)ab, (A2)ac *hold,* $f, g \in W^{1,1}(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ *(*$T>0$ *fixed),* $b_k \in W^{1,2}(0,T)$*,* $k=\overline{1,2n+m}$*,* $u_0, v_0 \in H^2(\mathbb{R}_+;\mathbb{R}^n)$*,* $w_0 \in \mathbb{R}^m$*,* $\operatorname{col}(\gamma_0 v_0, w_0) \in D(G)$ *and* $B_1(0) \in \gamma_1 u_0 + G_{11}(\gamma_0 v_0) + G_{12}(w_0)$*. Then problem* (3.4) *(equivalently problem* (3.1), (3.2), (3.3)*) has a unique strong solution* $\operatorname{col}(u,v,w) \in W^{1,\infty}(0,T;Y)$*. Moreover* $u, v \in L^\infty(0,T;H^2(\mathbb{R}_+;\mathbb{R}^n))$*,* $(B_1(t) = \operatorname{col}(b_1(t),\dots,b_{2n}(t)))$*.*

Proof. We shall use some similar techniques as those used in Luca [9]. We assume in a first stage that $f, g \in W^{1,\infty}(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$, $b_k \in W^{2,\infty}(0,T)$, $k=\overline{1,2n}$, $b_j \in W^{1,\infty}(0,T)$, $j=\overline{2n+1,2n+m}$, and the functions $\alpha_k(x,\cdot)$, $k=\overline{1,n}$ are Lipschitz continuous with Lipschitz constant L independent of x. We consider the operators $\mathcal{C}(t)$, $t\in[0,T]$, defined by $D(\mathcal{C}(t)) = D(\mathcal{A})$ and

$$\mathcal{C}(t)\begin{pmatrix}\widetilde{u}\\ v\\ w\end{pmatrix} = \mathcal{A}\begin{pmatrix}\widetilde{u}\\ v\\ w\end{pmatrix} + \mathcal{B}\begin{pmatrix}\widetilde{u}+\widetilde{\widetilde{u}}(t)\\ v\\ w\end{pmatrix} - \begin{pmatrix}\widetilde{f}(t,\cdot)\\ \widetilde{g}(t,\cdot)\\ B_2(t)\end{pmatrix}, \quad \begin{pmatrix}\widetilde{u}\\ v\\ w\end{pmatrix} \in D(\mathcal{A}).$$

Using Remark 2.3, we deduce that the operators $\mathcal{C}(t)$, $t\in[0,T]$ are maximal monotone in Y. By the above assumption on the functions α_k, $k=\overline{1,n}$, we have

$$\begin{aligned}
&|\alpha_k(x,\widetilde{u}_k+\widetilde{\widetilde{u}}_k(t,x)) - \alpha_k(x,\widetilde{u}_k+\widetilde{\widetilde{u}}_k(s,x))|\\
&\le L|\widetilde{\widetilde{u}}_k(t,x) - \widetilde{\widetilde{u}}_k(s,x)|\\
&\le L[(1+x)e^{-x}|b_{n+k}(t)-b_{n+k}(s)| + xe^{-x}|b_k(t)-b_k(s)|],
\end{aligned}$$

for all t, $s\in[0,T]$ for almost all $x>0$, $k=\overline{1,n}$. Therefore, we deduce

$$\begin{aligned}
&\|\alpha_k(\cdot,\widetilde{u}_k+\widetilde{\widetilde{u}}_k(t,\cdot)) - \alpha_k(\cdot,\widetilde{u}_k+\widetilde{\widetilde{u}}_k(s,\cdot))\|^2_{L^2(\mathbb{R}_+)}\\
&\le 2L^2|b_{n+k}(t)-b_{n+k}(s)|^2\int_0^\infty (1+x)^2e^{-2x}dx + 2L^2|b_k(t)-b_k(s)|^2\int_0^\infty x^2e^{-2x}dx\\
&= \frac{5L^2}{2}|b_{n+k}(t)-b_{n+k}(s)|^2 + \frac{L^2}{2}|b_k(t)-b_k(s)|^2, \quad \forall t,s\in[0,T],\ k=\overline{1,n}.
\end{aligned} \tag{3.5}$$

On the other hand for the functions $\widetilde{f}_k$, $k=\overline{1,n}$ we obtain the inequality

$$\begin{aligned}
|\widetilde{f}_k(t,x) - \widetilde{f}_k(s,x)| &\le |f_k(t,x)-f_k(s,x)| + (1+x)e^{-x}|b'_{n+k}(t)-b'_{n+k}(s)|\\
&\quad + xe^{-x}|b'_k(t)-b'_k(s)|, \quad \forall t,s\in[0,T],\ x>0,\ k=\overline{1,n},
\end{aligned}$$

and so

$$\begin{aligned}\|\widetilde{f}_k(t,\cdot)-\widetilde{f}_k(s,\cdot)\|^2_{L^2(\mathbb{R}_+)} &\le 3\|f_k(t,\cdot)-f_k(s,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\ &\quad+\frac{15}{4}|b'_{n+k}(t)-b'_{n+k}(s)|^2+\frac{3}{4}|b'_k(t)-b'_k(s)|^2,\end{aligned}\tag{3.6}$$

for all $t,s\in[0,T]$, $k=\overline{1,n}$. For the functions $\widetilde{g}_k$, $k=\overline{1,n}$ we deduce

$$\begin{aligned}|\widetilde{g}_k(t,x)-\widetilde{g}_k(s,x)| &\le |g_k(t,x)-g_k(s,x)|+|x-1|e^{-x}|b_{n+k}(t)-b_{n+k}(s)|\\ &\quad+|x-2|e^{-x}|b_k(t)-b_k(s)|,\quad \forall t,\ s\in[0,T],\ x>0,\ k=\overline{1,n},\end{aligned}$$

and so

$$\begin{aligned}\|\widetilde{g}_k(t,\cdot)-\widetilde{g}_k(s,\cdot)\|^2_{L^2(\mathbb{R}_+)} &\le 3\|g_k(t,\cdot)-g_k(s,\cdot)\|^2_{L^2(\mathbb{R}_+)}+\frac{3}{4}|b_{n+k}(t)-b_{n+k}(s)|^2\\ &\quad+\frac{15}{4}|b_k(t)-b_k(s)|^2,\quad \forall t,s\in[0,T],\ k=\overline{1,n}.\end{aligned}\tag{3.7}$$

Therefore, we obtain the following inequality for the operators $\mathcal{C}(t)$,

$$\begin{aligned}&\|h_t-h_s\|_Y\\ &\le \|\alpha(\cdot,\widetilde{u}+\widetilde{\widetilde{u}}(t,\cdot))-\alpha(\cdot,\widetilde{u}+\widetilde{\widetilde{u}}(s,\cdot))\|_{L^2(\mathbb{R}_+;\mathbb{R}^n)}+\|\widetilde{f}(t,\cdot)-\widetilde{f}(s,\cdot)\|_{L^2(\mathbb{R}_+;\mathbb{R}^n)}\\ &\quad+\|\widetilde{g}(t,\cdot)-\widetilde{g}(s,\cdot)\|_{L^2(\mathbb{R}_+;\mathbb{R}^n)}+\|B_2(t)-B_2(s)\|_{\mathbb{R}^m},\end{aligned}$$

for all $t,s\in[0,T]$, all $\widetilde{y}=\mathrm{col}(\widetilde{u},v,w)\in D(\mathcal{A})$, all $h_t\in\mathcal{C}(t)(\widetilde{y})$, all $h_s\in\mathcal{C}(s)(\widetilde{y})$.

Using now the relations (3.5)–(3.7) and the assumptions on the functions f,g,b_k, $k=\overline{1,2n+m}$, from the last inequality we deduce that there exists $L_1>0$ such that

$$\|h_t-h_s\|_Y\le L_1|t-s|,\quad \forall t,s\in[0,T],\ \forall\widetilde{y}\in D(\mathcal{A}),\ \forall h_t\in\mathcal{C}(t)(\widetilde{y}),\ h_s\in\mathcal{C}(s)(\widetilde{y}).$$

Therefore, the operator family $\{\mathcal{C}(t);t\in[0,T]\}$ verifies the conditions of Kato's Theorem (see Kato [4]). By the assumptions of our theorem, we deduce that $\widetilde{y}_0=\mathrm{col}(\widetilde{u}_0,v_0,w_0)\in D(\mathcal{A})$. It follows that the problem (3.4) has a unique strong solution $\widetilde{y}=\mathrm{col}(\widetilde{u},v,w)\in W^{1,\infty}(0,T;Y)$, $\mathrm{col}(\widetilde{u}(t),v(t),w(t))\in D(\mathcal{A})$, for all $t\in[0,T]$. Moreover $\widetilde{y}$ is everywhere differentiable from right on $[0,T)$ and

$$\frac{d^+}{dt}\begin{pmatrix}\widetilde{u}(t)\\ v(t)\\ w(t)\end{pmatrix}+\mathcal{A}\begin{pmatrix}\widetilde{u}(t)\\ v(t)\\ w(t)\end{pmatrix}+\mathcal{B}\begin{pmatrix}\widetilde{u}(t)+\widetilde{\widetilde{u}}(t)\\ v(t)\\ w(t)\end{pmatrix}\ni\begin{pmatrix}\widetilde{f}(t,\cdot)\\ \widetilde{g}(t,\cdot)\\ B_2(t)\end{pmatrix}$$
$$\begin{pmatrix}\widetilde{u}(0)\\ v(0)\\ w(0)\end{pmatrix}=\begin{pmatrix}\widetilde{u}_0\\ v_0\\ w_0\end{pmatrix}.$$

Hence $y(t)=\mathrm{col}(u(t),v(t),w(t))$ solves the problem

$$\begin{gathered}\frac{d^+y}{dt}(t)+\mathcal{A}(y(t))+\mathcal{B}(y(t))\ni F_1(t,\cdot),\quad 0\le t<T,\quad \text{in } Y\\ \gamma_1u(t)\in-G_{11}(\gamma_0v(t))-G_{12}(w(t))+B_1(t),\quad 0\le t<T\\ y(0)=y_0,\end{gathered}$$

where $F_1(t,\cdot)=\mathrm{col}(f(t,\cdot),g(t,\cdot),B_2(t))$. We deduce that $y=\mathrm{col}(u,v,w)$ is a solution of the problem (1.1), (1.2), (1.3).

In a second stage, we suppose that $\alpha_k(x,\cdot)$, $k=\overline{1,n}$ are not Lipschitz continuous and we replace the functions $\alpha_k(x,\cdot)$ by the Yosida approximations $\alpha_k^\lambda(x,\cdot)$, $k=$

$\overline{1,n}$, $\lambda > 0$. Using the above reasoning, we deduce that the problem (3.4) with α_k^λ instead of α_k has a unique strong solution $\mathrm{col}(\widetilde{u}^\lambda, v^\lambda, w^\lambda) \in W^{1,\infty}(0,T;Y)$. Then $y^\lambda = \mathrm{col}(u^\lambda, v^\lambda, w^\lambda)$ solves the problem

$$
\begin{gathered}
\frac{d^+ y^\lambda}{dt}(t) + \mathcal{A}(y^\lambda(t)) + \mathcal{B}_\lambda(y^\lambda(t)) \ni F_1(t,\cdot), \quad 0 \le t < T, \quad \text{in } Y \\
\gamma_1 u^\lambda(t) \in -G_{11}(\gamma_0 v^\lambda(t)) - G_{12}(w^\lambda(t)) + B_1(t), \quad 0 \le t < T \\
y^\lambda(0) = y_0,
\end{gathered}
\tag{3.8}
$$

with

$$
\mathcal{B}_\lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \mathrm{col}(\alpha_1^\lambda(\cdot,u_1),\dots,\alpha_n^\lambda(\cdot,u_n)) \\ \mathrm{col}(\beta_1(\cdot,v_1),\dots,\beta_n(\cdot,v_n)) \\ 0 \end{pmatrix}, \quad \lambda > 0.
$$

We write the first relation in (3.8) for $t+h$ and t, we subtract the relations and we multiply the obtained relation by $y^\lambda(t+h) - y^\lambda(t)$ in the space Y. We obtain after some computations

$$
\begin{aligned}
&\frac{1}{2}\frac{d^+}{dt}\|y^\lambda(t+h) - y^\lambda(t)\|_Y^2 + \langle g_{t+h} - g_t, z_{t+h} - z_t\rangle_{\mathbb{R}^{2n+m}} \\
&- \langle B_1(t+h) - B_1(t), \gamma_0 v^\lambda(t+h) - \gamma_0 v^\lambda(t)\rangle_{\mathbb{R}^n} \\
&\le \langle f(t+h,\cdot) - f(t,\cdot), u^\lambda(t+h) - u^\lambda(t)\rangle_{L^2(\mathbb{R}_+;\mathbb{R}^n)} \\
&\quad + \langle g(t+h,\cdot) - g(t,\cdot), v^\lambda(t+h) - v^\lambda(t)\rangle_{L^2(\mathbb{R}_+;\mathbb{R}^n)} \\
&\quad + \langle B_2(t+h) - B_2(t), w^\lambda(t+h) - w^\lambda(t)\rangle_{\mathbb{R}^m},
\end{aligned}
$$

where $z_t = col(\gamma_0 v^\lambda(t), w^\lambda(t))$, $z_{t+h} = col(\gamma_0 v^\lambda(t+h), w^\lambda(t+h))$, $g_t \in G(z_t)$, $g_{t+h} \in G(z_{t+h})$.

Using (A2)c, the above inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d^+}{dt}\|y^\lambda(t+h) - y^\lambda(t)\|_Y^2 + \zeta_2\|\gamma_0 v^\lambda(t+h) - \gamma_0 v^\lambda(t)\|_{\mathbb{R}^{2n}}^2 \\
&+ \zeta_2\|w^\lambda(t+h) - w^\lambda(t)\|_{\mathbb{R}^m}^2 \\
&\le \frac{1}{\zeta_0}\|B_1(t+h) - B_1(t)\|_{\mathbb{R}^{2n}}^2 + \zeta_0\|\gamma_0 v^\lambda(t+h) - \gamma_0 v^\lambda(t)\|_{\mathbb{R}^n}^2 \\
&\quad + \frac{1}{\zeta_0}\|B_2(t+h) - B_2(t)\|_{\mathbb{R}^m}^2 + \zeta_0\|w^\lambda(t+h) - w^\lambda(t)\|_{\mathbb{R}^m}^2 \\
&\quad + \|F_0(t+h,\cdot) - F_0(t,\cdot)\|_X \cdot \|y^\lambda(t+h) - y^\lambda(t)\|_Y,
\end{aligned}
$$

for $0 \le t < t+h < T$, $\lambda > 0$, where $F_0(t,\cdot) = \mathrm{col}(f(t,\cdot), g(t,\cdot))$.

When we choose $0 < \zeta_0 < \zeta_2$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d^+}{dt}\|y^\lambda(t+h) - y^\lambda(t)\|_Y^2 \\
&\le \frac{1}{\zeta_0}\|B(t+h) - B(t)\|_{\mathbb{R}^{2n+m}}^2 + \|F_0(t+h,\cdot) - F_0(t,\cdot)\|_X \cdot \|y^\lambda(t+h) - y^\lambda(t)\|_Y,
\end{aligned}
$$

for $0 \le t < t+h < T$, $\lambda > 0$. We integrate the above inequality over $[0,t]$ and we deduce that

$$
\frac{1}{2}\|y^\lambda(t+h) - y^\lambda(t)\|_Y^2
$$

$$
\begin{aligned}
&\le \frac{1}{2}\Big(\|y^\lambda(h)-y^\lambda(0)\|_Y^2+\frac{2}{\zeta_0}\int_0^T\|B(s+h)-B(s)\|^2_{\mathbb{R}^{2n+m}}ds\Big)\\
&\quad+\int_0^t\|F_0(s+h,\cdot)-F_0(s,\cdot)\|_X\cdot\|y^\lambda(s+h)-y^\lambda(s)\|_Y ds,
\end{aligned}
$$

for $0\le t<t+h<T$, $\lambda>0$. Using a variant of Gronwal's lemma, we obtain

$$
\begin{aligned}
&\|y^\lambda(t+h)-y^\lambda(t)\|_Y\\
&\le\|y^\lambda(h)-y^\lambda(0)\|_Y+\sqrt{\frac{2}{\zeta_0}}\Big(\int_0^T\|B(s+h)-B(s)\|^2_{\mathbb{R}^{2n+m}}ds\Big)^{1/2}\\
&\quad+\int_0^t\|F_0(s+h,\cdot)-F_0(s,\cdot)\|_X ds,\quad 0\le t<t+h<T,\ \lambda>0.
\end{aligned}
$$

We deduce from the above inequality that

$$
\begin{aligned}
\|\frac{d^+y^\lambda}{dt}(t)\|_Y&\le\|\frac{d^+y^\lambda}{dt}(0)\|_Y+\sqrt{\frac{2}{\zeta_0}}\Big(\int_0^T\|\frac{dB}{ds}(s)\|^2_{\mathbb{R}^{2n+m}}ds\Big)^{1/2}\\
&\quad+\int_0^T\|\frac{df}{ds}(s,\cdot)\|_{L^2(\mathbb{R}_+;\mathbb{R}^n)}ds+\int_0^T\|\frac{dg}{ds}(s,\cdot)\|_{L^2(\mathbb{R}_+;\mathbb{R}^n)}ds,
\end{aligned}
\tag{3.9}
$$

for $0\le t<T$, $\lambda>0$. Because $\sup\big\{\|\frac{d^+y^\lambda}{dt}(0)\|_Y;\lambda>0\big\}$ is a positive constant independent of λ, using the assumptions of the theorem, the inequality (3.9) gives us

$$
\sup\big\{\|\frac{dy^\lambda}{dt}(t)\|_Y;\lambda>0,\ 0<t<T\big\}\le\text{const.}
$$

and then $\sup\{\|y^\lambda(t)\|_Y;\lambda>0,\ 0<t<T\}\le$ const. Hence

$$
\begin{gathered}
\{u^\lambda;\lambda>0\},\ \{v^\lambda;\lambda>0\}\text{ are bounded in }L^\infty(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n)),\\
\{w^\lambda;\lambda>0\}\text{ is bounded in }L^\infty(0,T;\mathbb{R}^m),
\end{gathered}
$$

and, using the assumption (A1)b, we deduce that

$$
\{\mathcal{B}_\lambda(y^\lambda(t));\lambda>0\}\text{ is bounded in }L^\infty(0,T;Y). \tag{3.10}
$$

By (3.8), we have

$$
\frac{1}{2}\frac{d}{dt}\|y^\lambda(t)-y^\mu(t)\|_Y^2\le-\langle\mathcal{B}_\lambda(y^\lambda(t))-\mathcal{B}_\mu(y^\mu(t)),y^\lambda(t)-y^\mu(t)\rangle_Y, \tag{3.11}
$$

for $0<t<T$, $\lambda>0$. Using now the relations (3.10) and (3.11), we obtain

$$
\|y^\lambda(t)-y^\mu(t)\|_Y\le\text{const.}(\lambda+\mu)^{1/2},\quad 0\le t\le T,\ \lambda,\ \mu>0.
$$

Therefore, the sequence $\{y^\lambda;\lambda>0\}$ converges to some function $y=\mathrm{col}(u,v,w)$ in $C([0,T];Y)$ as $\lambda\to0$. Using Lebesgue's Dominated Convergence Theorem, we obtain $\mathcal{B}_\lambda(y^\lambda)\to\mathcal{B}(y)$, as $\lambda\to0$, strongly in $L^2(0,T;Y)$. By letting $\lambda\to0$ in (3.8), $\mathcal{A}$ and G being demi-closed operators, we obtain that y is a strong solution of the problem (1.1), (1.2), (1.3).

In the third stage (general case), we approximate $f,g\in W^{1,1}(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ by $\{f^j\}_{j\ge1}$, $\{g^j\}_{j\ge1}\subset W^{1,\infty}(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ in $W^{1,1}(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$, and $b_k\in W^{1,2}(0,T)$ by $\{b_k^j\}_{j\ge1}\subset W^{2,\infty}(0,T)$, $k=\overline{1,2n}$, $b_i\in W^{1,2}(0,T)$ by $\{b_i^j\}_{j\ge1}\subset W^{1,\infty}(0,T)$, $i=\overline{2n+1,2n+m}$, in $W^{1,2}(0,T)$.

Fixing $y_0=\mathrm{col}(u_0,v_0,w_0)\in Y$ with $\widetilde{y}_0=\mathrm{col}(\widetilde{u}_0,v_0,w_0)\in D(\mathcal{A})$, we deduce after some considerations (see also Luca [9]) that the sequence of the corresponding strong solutions $\{y^j = \operatorname{col}(u^j, v^j, w^j)\}_{j\ge 1}$ converges as $j \to \infty$ to $y = \operatorname{col}(u, v, w)$, which is a strong solution of our problem.

By system (1.1), we deduce u_{xx}, $v_{xx} \in L^\infty(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ and, using the inequality

$$\|z'\|_{L^2(\mathbb{R}_+)} \le C(\|z''\|_{L^2(\mathbb{R}_+)} + \|z\|_{L^2(\mathbb{R}_+)}), \quad \text{for } z \in H^2(\mathbb{R}_+),$$

(see Adams [1]), we get that u_x, $v_x \in L^\infty(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$. So we obtain u, $v \in L^\infty(0,T;\, H^2(\mathbb{R}_+;\mathbb{R}^n))$. □

Theorem 3.2. *Assume that* (A1)ab, (A2)ac *hold. If $f, g \in L^1(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ where T is fixed positive value, $b_k \in L^2(0,T)$, $k = \overline{1, 2n+m}$, $u_0, v_0 \in L^2(\mathbb{R}_+;\mathbb{R}^n)$, $w_0 \in \overline{D(G_{12}) \cap D(G_{22})}$, then problem* (1.1), (1.2), (1.3) *has a unique weak solution* $\operatorname{col}(u,v,w) \in C([0,T];Y)$.

Proof. By the assumptions of the theorem, it follows that $y_0 = \operatorname{col}(u_0, v_0, w_0) \in \overline{D(\mathcal{A})}$. We consider $\{y_0^j\}_{j\ge1} \subset Y$ such that $\widetilde{y}_0^j \in D(\mathcal{A})$ and $y_0^j \to y_0$, as $j \to \infty$, in Y. Also let the sequences $\{f^j\}_{j\ge1}$, $\{g^j\}_{j\ge1} \subset W^{1,1}(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ be such that $f^j \to f$, $g^j \to g$, as $j \to \infty$, in $L^1(0,T;L^2(\mathbb{R}_+;\mathbb{R}^n))$ and the sequences $\{b_k^j\}_{j\ge1} \subset W^{1,2}(0,T)$ be such that $b_k^j \to b_k$, as $j \to \infty$, in $L^2(0,T)$, $k = \overline{1, 2n+m}$. Then the corresponding strong solutions $y^j = \operatorname{col}(u^j, v^j, w^j) \in W^{1,\infty}(0,T;Y)$ of problem (1.1), (1.2), (1.3), given by Theorem 3.1, satisfy the inequality

$$\begin{aligned} \|y^j(t) - y^l(t)\|_Y \le \|y_0^j - y_0^l\|_Y + \sqrt{\frac{2}{\zeta_0}}\Big(\int_0^T \|B^j(s) - B^l(s)\|^2_{\mathbb{R}^{2n+m}} ds\Big)^{1/2} \\ + \int_0^t \|F_0^j(s,\cdot) - F_0^l(s,\cdot)\|_X^2 ds, \quad 0 \le t \le T, \quad \forall j, l \in \mathbb{N}, \end{aligned}$$

where $F_0^j = \operatorname{col}(f^j, g^j)$, $j \ge 1$, which leads us to the conclusion. □

Theorem 3.3. *Assume that* (A1)abc, (A2)ac *hold. If $f, g \in L^1_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}_+;\mathbb{R}^n))$, $b_k \in L^2(\mathbb{R}_+)$, $k = \overline{1, 2n+m}$ such that $\lim_{t\to\infty} f(t) = f^0$, $\lim_{t\to\infty} g(t) = g^0$, strongly in $L^2(\mathbb{R}_+;\mathbb{R}^n)$ and $\delta = \operatorname{col}(p,q,r)$ is the unique solution of* (2.2). *Then $\lim_{t\to\infty} y(t) = \delta$, strongly in Y, where $y(t) = \operatorname{col}(u(t), v(t), w(t))$, $t \ge 0$ is an arbitrary weak solution of* (3.4).

Proof. By Lemma 2.7, the operator $\mathcal{A}+\mathcal{B}$ is strongly monotone and equation (2.2) has a unique solution $\delta = \operatorname{col}(p,q,r) \in D(\mathcal{A})$. We define for any $l \in \mathbb{N}$ the function

$$B^l(t) = \begin{cases} B(t), & \text{for } 0 \le t \le l \\ 0, & \text{for } t > l. \end{cases}$$

Let $y_0 = \operatorname{col}(u_0, v_0, w_0) \in \overline{D(\mathcal{A})}$; we denote by $y(t)$, $y^l(t)$, $t \ge 0$ the weak solutions of problem (1.1), (1.2), (1.3) corresponding to data $\{B, f, g, y_0\}$, respectively $\{B^l, f, g, y_0\}$, given by Theorem 3.2. Then we have

$$\|y^l(t) - y(t)\|_Y \le \text{const.}\Big(\int_l^\infty \|B(s)\|^2_{\mathbb{R}^{2n+m}} ds\Big)^{1/2}, \quad t > l. \tag{3.12}$$

Because for $t > l$, y^l is the weak solution corresponding to $B(t) \equiv 0$, by Lemma 2.8, we deduce that $y^l(t) \to \delta$, as $t \to \infty$ in Y, $(l \in \mathbb{N})$. Hence this last conclusion with (3.12) and the inequality

$$\|y(t) - \delta\|_Y \le \|y(t) - y^l(t)\|_Y + \|y^l(t) - \delta\|_Y$$

give us that $y(t) \to \delta$, as $t \to \infty$, in Y. $\square$

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References

[1] R. A. Adams; *Sobolev spaces*, Academic Press, New York (1975).
[2] V. Barbu; *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden (1976).
[3] H. Brezis; *Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam (1973).
[4] T. Kato; *Nonlinear semigroups and evolution equations*; J. Math. Soc. Japan, **19** (1967), 508-520.
[5] V. Lakshmikantham, S. Leela; *Nonlinear Differential Equations in Abstract Spaces*, International Series in Nonlinear Mathematics: Theory, Methods and Applications, Vol.2, Pergamon Press (1981).
[6] R. Luca; *Time periodic solutions for a higher order nonlinear hyperbolic system*, Libertas Math., Vol. **XIX** (1999), 83-93.
[7] R. Luca; *Existence and uniqueness results for a higher order hyperbolic system*, Mathematical Reports (Stud. Cerc. Mat.), Vol. **1(51)** (1999), 551-571.
[8] R. Luca; *Boundary value problems for a second-order nonlinear hyperbolic system*, Comm. Appl. Anal., **8** (2004), 305-321.
[9] R. Luca; *An aymptotic result for the solutions of a higher order hyperbolic problem*, PanAmer. Math. J., **14** (2004), 99-109.
[10] R. Luca; *Monotone boundary conditions for a class of nonlinear hyperbolic systems*, Intern. J. Pure Appl. Math., **32**, No.1 (2006), 83-103.

Rodica Luca
Department of Mathematics, Gh. Asachi Technical University, 11 Blvd. Carol I, Iasi 700506, Romania
E-mail address: rluca@math.tuiasi.ro