EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC SYSTEMS IN THE EUCLIDEAN PLANE

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Abstract. We study the semilinear elliptic system
\[ \Delta u = \lambda p(x)f(v), \quad \Delta v = \lambda q(x)g(u), \]
in an unbounded domain \( D \) in \( \mathbb{R}^2 \) with compact boundary subject to some Dirichlet conditions. We give existence results according to the monotonicity of the nonnegative continuous functions \( f \) and \( g \). The potentials \( p \) and \( q \) are nonnegative and required to satisfy some hypotheses related on a Kato class.

1. Introduction

Semilinear elliptic systems of the form
\[ \begin{align*}
\Delta u &= F(u, v), \\
\Delta v &= G(u, v),
\end{align*} \tag{1.1} \]
in \( \mathbb{R}^n \) have been extensively treated recently. Lair and Wood \cite{9} studied the semilinear elliptic system
\[ \begin{align*}
\Delta u &= p(|x|)v^\alpha, \\
\Delta v &= q(|x|)u^\beta,
\end{align*} \tag{1.2} \]
in \( \mathbb{R}^n (n \geq 3) \). They showed the existence of entire positive radial solutions. More precisely, for the sublinear case where \( \alpha, \beta \in (0, 1) \), they proved the existence of bounded solutions of (1.2) if \( p \) and \( q \) satisfy the decay conditions
\[ \int_0^\infty tp(t)dt < \infty, \quad \int_0^\infty tq(t)dt < \infty, \tag{1.3} \]
and the existence of large solutions if
\[ \int_0^\infty tp(t)dt = \infty, \quad \int_0^\infty tq(t)dt = \infty. \tag{1.4} \]
For the superlinear case, where \( \alpha, \beta \in (1, +\infty) \). The authors proved the existence of an entire large positive solution of problem (1.2), provided that the functions \( p \) and \( q \) satisfy (1.3).

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Peng and Song [13] considered the semilinear elliptic system
\begin{align*}
\Delta u &= p(|x|)f(v), \\
\Delta v &= q(|x|)g(u),
\end{align*}
(1.5)
in \(\mathbb{R}^n (n \geq 3)\), under the assumptions:

(A1) The functions \(p\) and \(q\) satisfy condition (1.3).

(A2) The functions \(f\) and \(g\) are positive nondecreasing, satisfying the Keller-Osserman condition [8, 12]
\begin{align*}
\int_1^\infty \frac{1}{\sqrt{\int_0^s f(t)dt}}ds < \infty, \quad \int_1^\infty \frac{1}{\sqrt{\int_0^s g(t)dt}}ds < \infty.
\end{align*}
(1.6)

(A3) The functions \(f\) and \(g\) are convex on \([0, +\infty)\).

The authors proved the existence of an entire large positive solution of problem (1.5). We remark that Peng and Song extended their results to the superlinear case in [9].

Cirstea and Radulescu [5] gave existence results for system (1.5). They adopted the assumptions (A1)-(A2) and the assumption

\(f,g \in C^1[0, +\infty), \ f(0) = g(0) = 0, \lim_{t \to +\infty} \inf \frac{f(t)}{g(t)} > 0\),

(A3') to prove the existence of entire large positive solutions.

Recently, Ghanmi et al [7] considered the semilinear elliptic system
\begin{align*}
\Delta u &= \lambda p(x)f(v), \\
\Delta v &= \mu q(x)g(u),
\end{align*}
in a domain \(D\) of \(\mathbb{R}^n (n \geq 3)\) with compact boundary subject to some Dirichlet conditions. They assumed that the functions \(f, g\) are nonnegative continuous monotone on \((0, \infty)\), the nonnegative potentials \(p\) and \(q\) are required to satisfy some hypotheses related to a Kato class [3, 10]. In particular, in the case where \(f\) and \(g\) are nondecreasing and for given positive constants \(\lambda_0\), \(\mu_0\), they showed that for each \(\lambda \in [0, \lambda_0]\) and \(\mu \in [0, \mu_0]\), there exists a positive bounded solution \((u, v)\) satisfying the boundary conditions
\begin{align*}
u|_{\partial \times D} &= 1_{\partial D} + a1_{\{\infty\}}, \quad v|_{\partial \times D} = \psi 1_{\partial D} + b1_{\{\infty\}}
\end{align*}
where \(\varphi\) and \(\psi\) are nontrivial nonnegative continuous functions on \(\partial D\).

In this article, we consider an unbounded domain \(D\) in \(\mathbb{R}^2\) with compact non-empty boundary \(\partial D\) consisting of finitely many Jordan curves. We are concerned with the semilinear elliptic system
\begin{align*}
\Delta u &= \lambda p(x)f(v), \quad \text{in } D \\
\Delta v &= \mu q(x)g(u), \quad \text{in } D \\
u|_{\partial D} &= a\varphi, \quad \psi|_{\partial D} = b\psi,
\end{align*}
(1.7)
where \(a, b, \alpha\) and \(\beta\) are nonnegative constants such that \(a + \alpha > 0, \ b + \beta > 0\). The functions \(\varphi\) and \(\psi\) are nontrivial nonnegative and continuous on \(\partial D\). We will give two existence results according to the monotonicity of the functions \(f\) and \(g\).
Throughout this paper, we denote by $H_D\varphi$ the bounded continuous solution of the Dirichlet problem
\[
\Delta w = 0 \quad \text{in } D,
\]
\[
w|_{\partial D} = \varphi, \quad \lim_{|x| \to +\infty} \frac{w(x)}{\ln |x|} = 0,
\]
where $\varphi$ is a nonnegative continuous function on $\partial D$.

We remark that the solution $H_D\varphi$ of (1.8) belongs to $C(D \cup \{\infty\})$ and satisfies
\[
\lim_{|x| \to +\infty} H_D\varphi(x) = C > 0 \text{ (See [6, p. 427]).}
\]

For the sake of simplicity we denote
\[
\tilde{\varphi} := aH_D\varphi + \alpha h, \quad \tilde{\psi} := bH_D\psi + \beta h,
\]
where $h$ is the harmonic function defined by (2.2), below.

The outline of this paper is as follows. In section 2, we will give some notions related to the Green function $G_D$ of the domain $D$ associated to the Laplace operator $\Delta$ and properties of the functions belonging to a some Kato class $K(D)$ (See [10, 14]). In section 3, we will first give an example and then we give the proof of the existence result for the problem (1.7). More precisely, we adopt in section 3 the following hypotheses

(H1) The functions $f, g : [0, \infty) \to [0, \infty)$ are nondecreasing and continuous.

(H2) The functions $\tilde{p} := p\tilde{\varphi}$ and $\tilde{q} := q\tilde{\psi}$ belong to the Kato class $K(D)$.

(H3) $\lambda_0 := \inf_{x \in D} \frac{\tilde{\varphi}(x)}{V(\tilde{p})(x)} > 0$ and $\mu_0 := \inf_{x \in D} \frac{\tilde{\psi}(x)}{V(\tilde{q})(x)} > 0$, where $V$ is the Green kernel defined by (2.1) below.

We prove the following result.

**Theorem 1.1.** Assume (H1)–(H3), then for each $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$, problem (1.7) has a positive continuous solution $(u, v)$ satisfying, on $D$,
\[
(1 - \frac{\lambda}{\lambda_0})[aH_D\varphi + \alpha h] \leq u \leq aH_D\varphi + \alpha h,
\]
\[
(1 - \frac{\mu}{\mu_0})[bH_D\psi + \beta h] \leq v \leq bH_D\psi + \beta h.
\]

In the last section, we fix $\lambda = \mu = 1$ and a nontrivial nonnegative continuous function $\Phi$ on $\partial D$ and we note $h_0 = H_D\Phi$. Then we give an existence result for problem (1.7) with $a = 1$ and $b = 1$, under the following hypotheses:

(H4) The functions $f, g : [0, \infty) \to [0, \infty)$ are nonincreasing and continuous.

(H5) The functions $p_0 := p\frac{\Phi(h_0)}{h_0}$ and $q_0 := q\frac{\Phi(h_0)}{h_0}$ belong to the Kato class $K(D)$.

More precisely, we obtain the following result.

**Theorem 1.2.** Assume (H4)–(H5), then there exists a constant $c > 1$ such that if $\varphi \geq c\Phi$ and $\psi \geq c\Phi$ on $\partial D$, then problem (1.7) with $a = 1$ and $b = 1$ has a positive continuous solution $(u, v)$ satisfying, on $D$,
\[
h_0 + \alpha h \leq u \leq H_D\varphi + \alpha h,
\]
\[
h_0 + \beta h \leq v \leq H_D\psi + \beta h.
\]

Note that this result generalizes those by Athreya [2] and Toumi and Zeddini [14], stated for semilinear elliptic equations.
2. Preliminaries

In the reminder of this paper, we will adopt the following notation.
\( \mathcal{C}(\mathcal{D} \cup \{\infty\}) = \{f \in \mathcal{C}(\mathcal{D}) : \lim_{|x| \to +\infty} f(x) \text{ exists}\}. \) We note that \( \mathcal{C}(\mathcal{D} \cup \{\infty\}) \) is a Banach space endowed with the uniform norm \( \|f\|_\infty = \sup_{x \in D} |f(x)|. \)

For \( x \in D \), we denote by \( \delta_D(x) \) the distance from \( x \) to \( \partial D \), by \( \rho_D(x) := \min\{1, \delta_D(x)\} \) and by \( \lambda_D(x) := \delta_D(x)(1 + \delta_D(x)) \).

Let \( f \) and \( g \) be two positive functions on a set \( S \). We denote \( f \sim g \) if there exists a constant \( c > 0 \) such that
\[
\frac{1}{c} g(x) \leq f(x) \leq cg(x) \quad \text{for all } x \in S.
\]

For a Borel measurable and nonnegative function \( f \) on \( D \), we denote by \( Vf \) the Green kernel of \( f \) defined on \( D \) by
\[
Vf(x) = \int_D G_D(x, y)f(y)dy. \tag{2.1}
\]
We recall that if \( f \in L^1_{\text{loc}}(D) \) and \( Vf \in L^1_{\text{loc}}(D) \), then we have \( \Delta(Vf) = -f \) in \( D \), in the distributional sense (See [14, p 52]).

We note that the Green function satisfies
\[
G_D(x, y) \sim \ln(1 + \frac{\lambda_D(x)\lambda_D(y)}{|x - y|^2})
\]
on \( D^2 \) (See [11]).

**Definition 2.1.** A Borel measurable function \( q \) in \( D \) belongs to the Kato class \( K(D) \) if \( q \) satisfies
\[
\lim_{\alpha \to 0} \left( \sup_{x \in D} \int_{D \cap B(x, \alpha)} \frac{\rho_D(y)}{\rho_D(x)} G_D(x, y)|q(y)|dy \right) = 0,
\]
and
\[
\lim_{M \to +\infty} \left( \sup_{x \in D} \int_{D \cap \{|y| \geq M\}} \frac{\rho_D(y)}{\rho_D(x)} G_D(x, y)|q(y)|dy \right) = 0.
\]

**Example 2.2.** Let \( p > 1 \) and \( \gamma, \theta \in \mathbb{R} \) such that \( \gamma < 2 - \frac{2}{p} < \theta \). Then using the H"{o}lder inequality and the same arguments as in [14, Proposition 3.4] it follows that for each \( f \in L^p(D) \), the function defined in \( D \) by
\[
(1 + |x|)^{\frac{f(x)}{\gamma(\delta_D(x))^\gamma}}
\]
belongs to \( K(D) \).

 Throughout this article, \( h \) will be the function defined on \( D \) by
\[
h(x) = 2\pi \lim_{|y| \to +\infty} G_D(x, y) \tag{2.2}
\]

**Proposition 2.3** ([15]). The function \( h \) defined by (2.2) is harmonic positive in \( D \) and satisfies
\[
\lim_{x \to z \in \partial D} h(x) = 0, \quad \lim_{|x| \to +\infty} \frac{h(x)}{\ln |x|} = 1.
\]

In the sequel, we use the notation
\[
\|[q]_D = \sup_{x \in D} \int_D \frac{\rho_D(y)}{\rho_D(x)} G_D(x, y)|q(y)|dy, \tag{2.3}
\]
and
\[
\alpha_q = \sup_{x,y \in D} \int_D \frac{G_D(x, z)G_D(z, y)}{G_D(x, y)}|q(z)|dz. \tag{2.4}
\]
It is shown in [14], that if $q \in K(D)$, then $\|q\|_D < \infty$, and $\alpha_q \sim \|q\|_D$. For stating our results we need the following result.

**Proposition 2.4** ([14]). Let $q$ in $K(D)$, then the following assertions hold

(i) For any nonnegative superharmonic function $w$ in $D$, we have

$$V(wq)(x) = \int_D G_D(x,y)w(y)|q(y)|dy \leq \alpha_q w(x), \forall x \in D. \quad (2.5)$$

(ii) The potential $Vq \in C(\overline{D} \cup \infty)$ and $\lim_{x \to z \in \partial D} Vq(x) = 0$.

(iii) Let $\Lambda_q = \{p \in K(D) : |p| \leq q\}$. Then the family of functions

$$\mathfrak{G}_q = \int_D G_D(.,y)h_0(y)p(y)dy : p \in \Lambda_q$$

is uniformly bounded and equicontinuous in $D \cup \{\infty\}$. Consequently, it is relatively compact in $C(D \cup \{\infty\})$.

### 3. Proof of Theorem 1.1

Before stating the proof, we give an example where (H2) and (H3) are satisfied.

**Example 3.1.** Let $D = \overline{B}(0,1)^c$ be the exterior of the unit closed disk. Let $\alpha = b = 1$ and $\beta = a = 0$. Assume that $\psi \geq c_1 > 0$ on $\partial D$. Let $p_1, \tilde{q}$ be nonnegative functions in $K(D)$ such that the function $p := p_1 h$ is in $K(D)$. Then using the fact that the function $f$ is continuous and $H_D \psi$ is bounded on $D$ we obtain that $\tilde{p} := pf(H_D \psi) \in K(D)$ and so the hypothesis (H2) is satisfied. Now, since $\tilde{p}_1 := p_1 f(H_D \psi) \in K(D)$ then by Proposition 2.4 (i) we obtain

$$V(\tilde{p}) \leq \alpha \tilde{p}_1 h.$$

Therefore, for each $x \in D$

$$\frac{h(x)}{V(\tilde{p})(x)} \geq \frac{1}{\alpha \tilde{p}_1} > 0;$$

that is, $\lambda_0 > 0$. On the other hand we have

$$\frac{H_D \psi(x)}{V(\tilde{q})(x)} \geq \frac{c_1}{\alpha \tilde{q}} > 0,$$

which yields $\mu_0 > 0$. Thus the hypothesis (H3) is satisfied.

**Proof of Theorem 1.1.** Let $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$. We intend to prove that the problem (1.7) has a positive continuous solution. To this aim we define the sequences $(u_k)_{k \in \mathbb{N}}$ and $(v_k)_{k \in \mathbb{N}}$ as follows:

$$v_0 = \tilde{\psi},$$

$$u_k = \tilde{\varphi} - \lambda V(pf(v_k)),$$

$$v_{k+1} = \tilde{\psi} - \mu V(qg(u_k)),$$

where $\tilde{\varphi}$ and $\tilde{\psi}$ are defined by (1.9). We shall prove by induction that for each $k \in \mathbb{N},$

$$0 < (1 - \frac{\lambda}{\lambda_0}) \tilde{\varphi} \leq u_k \leq u_{k+1} \leq \tilde{\varphi},$$

$$0 < (1 - \frac{\mu}{\mu_0}) \tilde{\psi} \leq v_k \leq v_{k+1} \leq \tilde{\psi}.$$
First, using hypothesis (H3) we obtain, on $D$, 
\[ \lambda_0 V(pf(\tilde{\psi})) \leq \tilde{\varphi}. \]
Then by the monotonicity of $f$, it follows that 
\[ \tilde{\varphi} \geq u_0 = \tilde{\varphi} - \lambda V(pf(\tilde{\psi})) \geq (1 - \frac{\lambda}{\lambda_0})\tilde{\varphi} > 0. \]
So 
\[ v_1 - v_0 = -\mu V(qg(u_0)) \leq 0 \]
and consequently 
\[ u_1 - u_0 = \lambda V(pf(v_0) - f(v_1)) \geq 0. \]
Moreover, the hypothesis (H3) yields 
\[ \mu_0 V(qg(\tilde{\varphi})) \leq \tilde{\psi}. \]
Then using the fact that the function $g$ is nondecreasing we have 
\[ v_1 \geq \tilde{\psi} - \mu V(qg(\tilde{\varphi})) \geq (1 - \frac{\mu}{\mu_0})\tilde{\psi} > 0. \]
In addition, we have $u_1 \leq \tilde{\varphi}$, then it follows that 
\[ u_0 \leq u_1 \leq \tilde{\varphi} \quad \text{and} \quad v_1 \leq v_0 \leq \tilde{\psi}. \]
Suppose that 
\[ u_k \leq u_{k+1} \leq \tilde{\varphi} \quad \text{and} \quad (1 - \frac{\mu}{\mu_0})\tilde{\psi} \leq v_{k+1} \leq v_k. \]
Therefore, 
\[ v_{k+2} - v_{k+1} = \mu V(q[g(u_k) - g(u_{k+1})]) \leq 0, \]
\[ u_{k+2} - u_{k+1} = \lambda V(f(v_{k+1}) - f(v_{k+2})) \geq 0. \]
Furthermore, since $u_{k+1} \leq \tilde{\varphi}$ the monotonicity of the function $g$ yields 
\[ v_{k+2} \geq \tilde{\psi} - \lambda V(qg(\tilde{\varphi})) \geq (1 - \frac{\mu}{\mu_0})\tilde{\psi} > 0. \]
Thus, we obtain 
\[ u_{k+1} \leq u_{k+2} \leq \tilde{\varphi} \quad \text{and} \quad (1 - \frac{\mu}{\mu_0})\tilde{\psi} \leq v_{k+2} \leq v_{k+1}. \]
Hence, the sequences $(u_k)$ and $(v_k)$ converge respectively to two functions $u$ and $v$ satisfying 
\[ 0 < (1 - \frac{\lambda}{\lambda_0})\tilde{\varphi} \leq u \leq \tilde{\varphi}, \quad 0 < (1 - \frac{\mu}{\mu_0})\tilde{\psi} \leq v \leq \tilde{\psi}. \]
Furthermore, for each $k \in \mathbb{N}$, we have 
\[ f(v_k) \leq f(\tilde{\psi}), \quad g(u_k) \leq g(\tilde{\varphi}). \quad (3.1) \]
Therefore, using hypothesis (H2) and Proposition 2.4(ii) we deduce by Lebesgue’s theorem that $V(pf(u_k))$ and $V(qg(u_k))$ converge respectively to $V(pf(v))$ and $V(qg(u))$ as $k$ tends to infinity. Then, on $D$, $(u, v)$ satisfies 
\[ u = \tilde{\varphi} - \lambda V(pf(v)) \]
\[ v = \tilde{\psi} - \mu V(qg(u)). \quad (3.2) \]
Moreover, by (3.2) and the monotonicity of the functions $f$ and $g$ we obtain 
\[ pf(v) \leq \tilde{p} \quad \text{and} \quad qg(u) \leq \tilde{q}. \]
So $pf(v), qg(u) \in K(D)$ and consequently by Proposition 2.4(ii) we have $V(pf(v)), V(qg(u)) \in C(D \cup \{\infty\})$. Now using the fact
that the functions $\tilde{\phi}$ and $\tilde{\psi}$ are continuous we conclude that $u$ and $v$ are continuous and satisfy in the distributional sense $\Delta u = \lambda pf(v)$ and $\Delta v = \mu qg(u)$ in $\bar{D}$. Now, since $H_D \varphi = \varphi$ on $\partial D$, $\lim_{x \to z \in \partial D} h(x) = 0$, and $\lim_{x \to z \in \partial D} V(\tilde{p})(x) = 0$, we conclude that $\lim_{x \to z \in \partial D} u(x) = a \varphi(z)$. By similar arguments we have $\lim_{x \to z \in \partial D} v(x) = b \psi(z)$. Furthermore, by Proposition 2.4 (ii) and Proposition 2.3, we have $\lim_{|x| \to +\infty} \frac{1}{h(x)} V(pf(v)) = 0$ and $\lim_{|x| \to +\infty} \frac{1}{h(x)} V(qg(u)) = 0$. Hence $(u, v)$ is a continuous positive solution of the problem (1.7), which completes the proof. □

4. PROOF OF THEOREM 1.2

In the sequel, we recall that $h_0 = H_D \Phi$ is a fixed positive harmonic function in $D$ and $h$ is the function defined by (2.2).

Proof. Let $\alpha_{p_0}$ and $\alpha_{q_0}$ be the constants defined by (2.4) associated respectively to the functions $p_0$ and $q_0$ given in the hypothesis (H5). Put $c = 1 + \alpha_{p_0} + \alpha_{q_0}$.

Suppose that $\varphi(x) \geq c\Phi(x)$ and $\psi(x) \geq c\Phi(x)$, $\forall x \in \partial D$.

Then by the maximum principle it follows that for each $x \in D$

$$H_D \varphi(x) \geq ch_0(x), \quad (4.1)$$

$$H_D \psi(x) \geq ch_0(x). \quad (4.2)$$

Consider the nonempty convex set $\Omega$ given by

$$\Omega := \{ w \in C(\bar{D} \cup \{ \infty \}) : h_0 \leq w \leq H_D \varphi \}.$$

Let $T$ be the operator defined on $\Omega$ by

$$Tw := H_D \varphi - V(pf[\beta h + H_D \psi - V(qg(w + \alpha h))]).$$

We shall prove that the operator $T$ has a fixed point. First, let us prove that the operator $T$ maps $\Omega$ to its self. Let $w \in \Omega$. Since $w + \alpha h \geq h_0$, then from hypothesis (H4) we deduce that

$$V(qg(w + \alpha h)) \leq V(qg(h_0)). \quad (4.3)$$

Therefore, using (4.2) and (4.3) we obtain

$$v := \beta h + H_D \psi - V(qg(w + \alpha h)) \geq \beta h + H_D \psi - V(qg(h_0)) \geq \beta h + (c - \alpha_{q_0})h_0.$$

This yields

$$v \geq h_0 > 0. \quad (4.4)$$

So, $Tw \leq H_D \varphi$. On the other hand, by (4.4), the monoticity of $f$ and Proposition 2.4 (i), we obtain

$$V(pf(v)) \leq V(pf(h_0)) = V(p_0 h_0) \leq \alpha_{p_0} h_0. \quad (4.5)$$

Then, by (4.1) and (4.5), we have

$$Tw \geq H_D \varphi - \alpha_{p_0} h_0 \geq (1 + \alpha_{q_0})h_0 \geq h_0.$$
Hence $T\Omega \subseteq \Omega$. Next, let us prove that the set $T\Omega$ is relatively compact in $C(\overline{D} \cup \{\infty\})$. Let $w \in \Omega$, then by (H4), (H5) and using Proposition 2.4(iii) it follows that the family of functions

$$\{ \int_D G(\cdot,y)p(y)f[\beta h + H_D\psi - V(qg(w + \alpha h)](y)dy : w \in \Omega \}$$

is relatively compact in $C(\overline{D} \cup \{\infty\})$. Since $H_D\varphi \in C(\overline{D} \cup \{\infty\})$ we deduce that $T\Omega$ is relatively compact in $C(\overline{D} \cup \{\infty\})$.

Now we prove the continuity of the operator $T$ in $\Omega$ in the supremum norm. Let $(w_k)$ be a sequence in $\Omega$ which converges uniformly to a function $w$ in $\Omega$. Then using (4.4) and the monotonicity of $f$ we have, for each $x$ in $D$,

$$p(x)|f(\beta h + H_D\psi - V(qg(w + \alpha h)))(x) - f(\beta h + H_D\psi - V(qg(w + \alpha h)))(x)| \leq 2f(h_0)p(x) \leq 2\|h_0\|_\infty p_0(x)$$

Using the fact that $Vp_0$ is bounded, we conclude by the continuity of $f$ and the dominated convergence theorem that for all $x \in D$, $Tw_k(x) \to Tw(x)$ as $k \to +\infty$. Consequently, as $T\Omega$ is relatively compact in $C(\overline{D} \cup \{\infty\})$, we deduce that the pointwise convergence implies the uniform convergence; that is,

$$\|Tw_k - Tw\|_\infty \quad \text{as} \quad k \to +\infty$$

Therefore, $T$ is a continuous mapping of $\Omega$ to itself. So, since $T\Omega$ is relatively compact in $C(\overline{D} \cup \{\infty\})$, it follows that $T$ is compact mapping on $\Omega$. Thus, the Schauder fixed-point theorem yields the existence of $w \in \Omega$ such that

$$w = H_D\varphi - V(pf[\beta h + H_D\psi - V(qg(w + \alpha h))]).$$

Put $u(x) = w(x) + \alpha h(x)$ and $v(x) = \beta h(x) + H_D\psi(x) - V(qg(u))(x)$ for $x \in D$. Then $(u, v)$ is a positive continuous solution of (1.7) with $a = 1, b = 1$, for the same arguments as in the proof of Theorem 1.1. \hfill $\square$

**Example 4.1.** Let $D = B(0,1)$ be the exterior of the unit closed disk, $0 < \theta < 1$ and $0 < \gamma < 1$. Let $p, q$ be two nonnegative functions such that the functions $(\frac{|x|}{|x|^1 - 1})^{1+\theta}p(x)$ and $(\frac{|x|}{|x|^1 - 1})^{1+\gamma}q(x)$ are in $K(D)$. Suppose that the functions $\varphi$ and $\psi$ are nonnegative continuous on $\partial D$. Then for a fixed nontrivial nonnegative continuous function $\Phi$ in $\partial D$, there exists a constant $c > 1$ such that if $\varphi \geq c\Phi$ and $\psi \geq c\Phi$ on $\partial D$, the problem

$$\Delta u = p(x)v^{-\gamma}, \quad \text{in} \quad D
$$
$$\Delta v = q(x)u^{-\theta}, \quad \text{in} \quad D$$

$$u|_{\partial D} = \varphi, \quad v|_{\partial D} = \psi,$$

$$\lim_{|x| \to +\infty} \frac{u(x)}{\ln |x|} = \alpha \geq 0, \quad \lim_{|x| \to +\infty} \frac{v(x)}{\ln |x|} = \beta \geq 0,$$

has a positive continuous solution $(u, v)$ satisfying

$$H_D\Phi(x) + \alpha h(x) \leq u(x) \leq H_D\varphi(x) + \alpha h(x),$$
$$H_D\Phi(x) + \beta h(x) \leq v(x) \leq H_D\psi(x) + \beta h(x),$$

for each $x \in D$. Indeed, from [11] there exists $c_0 > 0$ such that for each $x \in D$,

$$c_0 \frac{|x|^{-1}}{|x|} \leq H_D\Phi(x).$$
It follows that $p_0 := p \left( \frac{H_D \Phi(x)}{H_D \Phi(x)} \right)^{-\delta} \in K(D)$. In a similar way we have $q_0 \in K(D)$. Thus the hypothesis (H5) is satisfied.

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