COMPACTNESS RESULTS FOR QUASILINEAR PROBLEMS WITH VARIABLE EXPONENT ON THE WHOLE SPACE

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Abstract. In this work we give a compactness result which allows us to prove the point-wise convergence of the gradients of a sequence of solutions to a quasilinear inequality and for an arbitrary open set. This result suggests solutions to many problems, notably nonlinear elliptic problems with critical exponent.

1. Introduction and preliminary results

In their recent work El Hamidi and Rakotoson [5] gave a compactness result to prove the point-wise convergence of the gradients of a sequence of solutions to a general quasilinear inequality and for an arbitrary open set. They proved the following result.

Lemma 1.1. Let $\hat{a}$ be a Carathéodory function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^N$ satisfying the usual Leray-Lions growth and monotonicity conditions. Let $(u_n)$ be a bounded sequence of $W^{1,p}_{\text{loc}}(\mathbb{R}^N) = \{v \in L^p_{\text{loc}}(\mathbb{R}^N), |\nabla v| \in L^p_{\text{loc}}(\mathbb{R}^N)\}, \text{ with } 1 < p < +\infty$, $(f_n)$ be a bounded sequence of $L^1_{\text{loc}}(\mathbb{R}^N)$ and $(g_n)$ be a sequence of $W^{-1,p'}_{\text{loc}}(\mathbb{R}^N)$ tending strongly to zero. Assume that $(u_n)$ satisfies:

$$\int_{\mathbb{R}^N} \hat{a}(x, u_n(x), \nabla u_n(x)).\nabla \phi \, dx = \int_{\mathbb{R}^N} f_n \phi \, dx + \langle g_n, \phi \rangle,$$

for all $\phi \in W^{1,p}_{\text{comp}}(\mathbb{R}^N) = \{v \in W^{1,p}(\mathbb{R}^N), \text{ with compact support}\}, \phi \text{ bounded}$. Then:

1. there exists a function $u$ such that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$;
2. $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$;
3. there exists a subsequence, still denoted $(u_n)$, such that $\nabla u_n(x) \to \nabla u(x)$ a.e. in $\mathbb{R}^N$.

In the present work, we generalize Lemma 1.1 for the $p(x)$-Laplace operator. Our principal result can be applied to a large class of quasilinear elliptic problems where there holds a lack of compactness, especially for the critical exponent equations.

In the sequel, we start with some preliminary basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. We refer to the book by Musielak
For any $p \in L^2$, the papers by Kovacik and Rakosnik \[13\] and by Fan et al. \[4, 7, 8\]. Set 
$$C_+(\Omega) = \{ h \in C(\Omega) : h(x) > 1 \text{ for all } x \in \Omega \}.$$

For any $h \in C_+(\Omega)$ we define 
$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

For any $p \in C_+(\Omega)$, we define the variable exponent Lebesgue space 
$$L^{p(x)}(\Omega) = \{ u : u \text{ is a Borel real-valued function on } \Omega, \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \}.$$

We define on $L^{p(x)}$, the so-called Luxemburg norm, by the formula 
$$|u|_{p(x)} := \inf \{ \mu > 0 : \int_{\Omega} \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are separable and Banach spaces \[13\] Theorem 2.5; Corollary 2.7 and the Hölder inequality holds \[13\] Theorem 2.1]. The inclusions between Lebesgue spaces are also naturally generalized \[13\] Theorem 2.8]: if $0 < |\Omega| < \infty$ and $r_1$, $r_2$ are variable exponents so that $r_1(x) \leq r_2(x)$ almost everywhere in $\Omega$ then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality 
$$| \int_{\Omega} uv \, dx | \leq \left( \frac{1}{p} + \frac{1}{p'} \right) |u|_{p(x)} |v|_{p'(x)}, \quad (1.1)$$
is held.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ defined by 
$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} \, dx.$$
The space $W^{1,p(x)}(\Omega)$ is equipped by the norm 
$$\| u \| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$
We recall that if $(u_n), u, \in W^{1,p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold: 
$$\min(|u|_{p(x)}^p, |u|_{p(x)}^p) \leq \rho_{p(x)}(u) \leq \max(|u|_{p(x)}^p, |u|_{p(x)}^p), \quad (1.2)$$
$$\min(|\nabla u|_{p(x)}^p, |\nabla u|_{p(x)}^p) \leq \rho_{p(x)}(|\nabla u|) \leq \max(|\nabla u|_{p(x)}^p, |\nabla u|_{p(x)}^p), \quad (1.3)$$
$$\lim_{n \to \infty} |u_n - u|_{p(x)} = 0 \iff \lim_{n \to \infty} \rho_{p(x)}(u_n) = 0, \quad (1.4)$$
$$|u_n|_{p(x)} \to \infty \iff \rho_{p(x)}(u_n) \to \infty.$$

We define also $W^{1,p(x)}_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm 
$$\| u \|_{p(x)} = |\nabla u|_{p(x)}.$$
The space $(W^{1,p(x)}_0(\Omega), \| \cdot \|)$ is a separable and reflexive Banach space.
Next, we recall some embedding results regarding variable exponent Lebesgue-Sobolev spaces. We note that if \( s(x) \in C_+ (\Omega) \) and \( s(x) < p^*(x) \) for all \( x \in \Omega \) then the embedding \( W^{1, p(x)}_0 (\Omega) \hookrightarrow L^{s(x)}(\Omega) \) is compact and continuous, where \( p^*(x) = Np(x)/(N - p(x)) \) if \( p(x) < N \) or \( p^*(x) = +\infty \) if \( p(x) \geq N \). We refer to [13] for more properties of Lebesgue and Sobolev spaces with variable exponent. We also refer to the recent papers [1, 3, 4, 10, 11, 16, 17, 18] for the treatment of nonlinear boundary value problems in Lebesgue-Sobolev spaces with variable exponent. For relevant applications and related results we refer to the recent books by Ghergu and Rădulescu [9] and Kristály, Rădulescu and Varga [12].

2. NOTATION AND COMPACTNESS RESULT

Let \( \Omega \) be an arbitrary open set of \( \mathbb{R}^N \), we shall denote by \( \omega \subset \subset \Omega \) any relatively compact open subset \( \omega \) of \( \Omega \) (that is \( \overline{\omega} \subset \subset \Omega \), where \( \overline{\omega} \) is the closure of \( \omega \)). Let \( 1 < p(x) < +\infty \), we set

\[
W^{1, p(x)}_{\text{loc}}(\Omega) = \{ v \in L^{p(x)}_{\text{loc}}(\Omega); \nabla v \in L^{p(x)}_{\text{loc}}(\Omega) \}.
\]

For a given \( q(x) \in (1, +\infty) \), we denote by \( q'(x) := \frac{q(x)}{q(x) - 1} \) its conjugate exponent. We shall use the following globally real Lipschitz functions: For \( \epsilon > 0, \sigma \in \mathbb{R} \), let

\[
\sigma_k(\sigma) = \begin{cases} 
\sigma & \text{if } |\sigma| \leq \epsilon \\
\text{sign}(\sigma) & \text{otherwise},
\end{cases}
\]

and \( \sigma^k := \sigma_k(\sigma) \) for \( k \geq 1 \).

We shall consider a nonlinear map \( \hat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) satisfying the following conditions:

(L1) \( \hat{a}(x, \cdot, \cdot) \) is a continuous map for almost every \( x \) and for all \((\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N\), \( \hat{a}(x, \sigma, \xi) \) is measurable (such a property is called Carathéodory property),

(L2) \( \hat{a} \) maps bounded sets of \( W^{1, p(x)}_{\text{loc}}(\Omega) \) into bounded sets of \( L^{p(x)}_{\text{loc}}(\Omega) \), and for almost all \( x \in \Omega \), for all \((\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N\), \( \hat{a}(x, \sigma, \xi) \cdot \xi \geq 0 \), for almost every \( x \in \Omega \) and for all \( v \in W^{1, p(x)}_{\text{loc}}(\Omega) \), the mapping \( u \mapsto \hat{a}(x, u, \nabla v) \) is continuous from \( W^{1, p(x)}(\omega) \)-weak into \( L^{p(x)}(\omega) \)-strong, for all \( \omega \subset \subset \Omega \),

(L3) for almost every \( x \in \Omega \), for all \((\sigma_i, \xi_i) \in \mathbb{R} \times \mathbb{R}^N, i = 1, 2, \)

\[
|\hat{a}(x, \sigma_1) - \hat{a}(x, \sigma_2)||\xi_1 - \xi_2| > 0, \quad \text{for } \xi_1 \neq \xi_2.
\]

(L4) if for some \( x \in \Omega \), there is a sequence \((\sigma_n, \xi_{1n}) \in \mathbb{R} \times \mathbb{R}^N, \xi_2 \in \mathbb{R}^N \) such that \( |\hat{a}(x, \sigma_n, \xi_{1n}) - \hat{a}(x, \sigma_n, \xi_2)||\xi_{1n} - \xi_2| \) and \( \sigma_n \) are bounded as \( n \to +\infty \) then \( \xi_{1n} \) remains in a bounded set of \( \mathbb{R}^N \) as \( n \to +\infty \).

As a corollary of our main result, we state the following result.

Lemma 2.1. Let \( \hat{a} \) be a Carathéodory function from \( \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \) into \( \mathbb{R}^N \) satisfying the usual Leray-Lions growth and monotonicity conditions. Let \((u_n)\) be a bounded sequence of

\[
W^{1, p(x)}_{\text{loc}}(\mathbb{R}^N) = \{ v \in L^{p(x)}_{\text{loc}}(\mathbb{R}^N), |\nabla v| \in L^{p(x)}_{\text{loc}}(\mathbb{R}^N) \},
\]

with \( 1 < p(x) < +\infty \), \((f_n)\) be a bounded sequence of \( L^1_{\text{loc}}(\mathbb{R}^N) \) and \((g_n)\) be a sequence of \( W^{1, p(x)}_{\text{loc}}(\mathbb{R}^N) \) tending strongly to zero. Assume that \((u_n)\) satisfies

\[
\int_{\mathbb{R}^N} a(x, u_n(x), \nabla u_n(x)) \cdot \nabla \phi \, dx = \int_{\mathbb{R}^N} f_n \phi \, dx + \langle g_n, \phi \rangle,
\]

for all \( \phi \in W^{1, p(x)}_{\text{loc}}(\mathbb{R}^N) \).
Theorem 2.2. Let \( \phi \) be a bounded sequence of \( W^{1,p(x)}_{\text{loc}}(\Omega) \). Then:

(i) There is a subsequence still denoted \((u_n)\) and a function \( u \in W^{1,p(x)}_{\text{loc}}(\Omega) \) such that
\[
\lim_{n \to +\infty} u_n(x) = u(x) \quad \text{a.e. in } \Omega.
\]

(ii) If furthermore, we assume \((L1)-(L4)\) and that for all \( \phi \in C^\infty_c(\Omega) \), and all \( k \geq k_0 > 0 \):
\[
\limsup_{n \to +\infty} \int_{\Omega} \left| \hat{a}(x, u_n(x), \nabla u_n(x)) \cdot \nabla(\phi \sigma(u_n - u^k)) \right| \leq o(1)
\]
as \( \epsilon \to 0 \) then there exists a subsequence still denoted \((u_n)\) such that
\[
\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega \text{ as } n \to +\infty.
\]

Remark 2.3. (1) The term \( o(1) \) in (ii) might depend on \( k \) and \( \phi \).

(2) \((L2)\) is satisfied if for all \( \omega \subset \subset \Omega \), there is a constant \( c_\omega > 0 \) and a function \( a_0 \in L^{p'(x)}(\omega) \) such that for almost every \( x \in \omega \), for all \( (\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N \):
\[
|\hat{a}(x, \sigma, \xi)| \leq c_\omega |\sigma|^{p(x)-1} + |\xi|^{p(x)-1} + a_0(x).
\]
and \((L4)\) is true if \( \hat{a}(x, \sigma, \xi) \cdot \xi \geq c^1_\omega |\xi|^{p(x)} - c^2_\omega, c^3_\omega > 0 \).

(3) Bounded sets in \( W^{1,p(x)}_{\text{loc}}(\Omega) \) will be bounded in
\[
W^{1,p(x)}(\omega) = \{ v \in L^{p(x)}(\omega), \nabla v \in L^{p(x)}(\omega) \}, \quad \text{for every } \omega \subset \subset \Omega.
\]

Proof of theorem 2.2.

(i) Let \((w_j)_{j \geq 0}\) be a sequence of bounded relatively compact subsets of \( \Omega \) such that \( \omega_j \subset \omega_{j+1} \) and \( \bigcup_{j=0}^{+\infty} \omega_j = \Omega \). Since \((u_n)_{n}\) is bounded in \( W^{1,p(x)}(\omega_j) \), by the usual embeddings, we deduce that there is a subsequence \( u_{n_j} \)
and a function \( u \) in \( W^{1,p(x)}(\omega_j) \) such that \( u_{n_j}(x) \to u(x) \) as \( n \to \infty \). We conclude with the usual diagonal Cantor process.

(ii) Let \( \phi \in C^\infty_c(\Omega) \), \( 0 \leq \phi \leq 1 \), \( \phi = 1 \) on \( \omega_j \) and \( \text{supp}(\phi) \subset \omega_{j+1} \), and set
\[
\Delta(u_n, u)(x) = [\hat{a}(x, u_n(x), \nabla u_n(x)) - \hat{a}(x, u_n(x), \nabla u(x))]\nabla(u_n - u)(x).
\]
Then one has:

(ii.1) \( \Delta(u_n, u)(x) \geq 0 \) a.e. on \( \Omega \) (due to \((L3)\)).

(ii.2) \( \sup_n \int_{\omega_j \to +1} \Delta(u_n, u) \, dx \) is finite (since \((u_n)\) is in a bounded set of \( W^{1,p(x)}_{\text{loc}}(\Omega) \) and the growth condition \((L2)\)).

Let us show that \( \lim_n \int_{\Omega} \phi \Delta(u_n, u) \, dx = 0 \). On one hand,
\[
\int_{\Omega} \phi \Delta(u_n, u) \, dx = \int_{\{ |u| > k \}} \phi \Delta(u_n, u) \, dx + \int_{\{ |u| \leq k \}} \phi \Delta(u_n, u) \, dx.
\]

(2.1)
By the Hölder inequality
\[ \int_{\{|u|>k\}} \Delta(u, u) \frac{1}{p(x)} \phi \, dx \leq |\Delta(u, u) \frac{1}{p(x)}|_{p(x)} |\phi|_{\frac{p(x)}{p(x)-1}} \leq a_1(j) |\phi|_{\frac{p(x)}{p(x)-1}}, \]

where \(a_s(j)\) are different constants depending on \(j\) but independent of \(n, \epsilon\) and \(k\). Noticing that
\[ \text{meas}\{x \in w_{j+1} : |u| > k\} \leq \frac{c_1(j)}{kp^*}, \]
one deduces that
\[ \rho \frac{\phi}{p(x)} (\phi) = \int_{\{|u|>k\}} \phi \frac{1}{p(x)} \, dx \leq \frac{c_1(j)}{kp^*} \tag{2.2} \]

where \(c_m(j)\) are different constants depending on \(j\) and \(\phi\) but independent of \(n, \epsilon\) and \(k\). We conclude that
\[ \limsup_{n \to \infty} \int_{\{|u|>k\}} \Delta(u, u) \frac{1}{p(x)} \phi \, dx \leq o(1) \quad \text{as } k \to \infty \tag{2.3} \]

While for the second integral, we have
\[ \int_{\{|u| \leq k\}} \Delta(u, u) \frac{1}{p(x)} \phi \, dx = \int_{\{|u| \leq k\} \cap \{|u_n - u| \leq \epsilon\}} \Delta(u, u) \frac{1}{p(x)} \phi \, dx \]
\[ + \int_{\{|u| \leq k\} \cap \{|u_n - u| > \epsilon\}} \Delta(u, u) \frac{1}{p(x)} \phi \, dx. \tag{2.4} \]

Moreover, the second term in the right hand side in the last inequality satisfies
\[ \int_{\{|u| \leq k\} \cap \{|u_n - u| > \epsilon\}} \Delta(u, u) \frac{1}{p(x)} \phi \, dx \leq |\Delta(u, u) \frac{1}{p(x)}|_{p(x)} |\phi|_{\frac{p(x)}{p(x)-1}} \leq a_2(j) |\phi|_{\frac{p(x)}{p(x)-1}}, \]

and
\[ \rho \frac{\phi}{p(x)} (\phi) \leq a_2(\phi) \text{meas}\{x \in w_{j+1} : |u_n - u| > \epsilon\}. \]

Since \((u_n)\) converges to \(u\) in measure, we deduce that, for \(n\) sufficiently large, \(\text{meas}\{x \in w_{j+1} : |u_n - u| > \epsilon\} \leq \epsilon\). It follows that
\[ \limsup_{n \to \infty} \int_{\{|u| \leq k\} \cap \{|u_n - u| > \epsilon\}} \Delta(u, u) \frac{1}{p(x)} \phi \, dx \leq o(1) \quad \text{as } \epsilon \to 0. \tag{2.5} \]

Setting \(A_{n,k} = w_{j+1} \cap \{|u| \leq k\} \cap \{|u_n - u| \leq \epsilon\}\), we obtain from the Hölder inequality
\[ \int_{A_{n,k}} \Delta(u, u) \frac{1}{p(x)} \phi \, dx \leq c_2(j) |\Delta(u, u) \frac{1}{p(x)} \phi \frac{1}{p(x)}|_{p(x)}, \tag{2.6} \]

and
\[ \rho \frac{\phi}{p(x)} (\Delta(u, u) \frac{1}{p(x)} \phi \frac{1}{p(x)}) = I_{n,k}^1(\epsilon) - I_{n,k}^2(\epsilon), \]

with
\[ I_{n,k}^1(\epsilon) = \int_{A_{n,k}} \tilde{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u) \phi \, dx, \]
\[ I_{n,k}^2(\epsilon) = \int_{\{|u| \leq k\}} \tilde{a}(x, u_n, \nabla u) \cdot \nabla S_{\epsilon}(u_n - u) \phi \, dx. \]

Since \(\tilde{a}(x, u_n, \nabla u) \to \tilde{a}(x, u, \nabla u)\) strongly in \(L^{p'(x)}(w_{j+1})\) (by the last statement of (L2)) and \(\nabla S_{\epsilon}(u_n - u) \to 0\) in \(L^{p(x)}(w_{j+1})\)-weak, we deduce that
\[ \lim_{n \to +\infty} I_{n,k}^2(\epsilon) = 0, \tag{2.7} \]
while for the term $I_{n,k}^1(\epsilon)$, we obtain

$$I_{n,k}^1(\epsilon) \leq \int_{\Omega} \hat{a}(x, u_n, \nabla u_n) \cdot \nabla (\phi\mathcal{S}_\epsilon(u_n - u^k)) - \int_{\Omega} \hat{a}(x, u_n, \nabla u_n) \cdot \nabla \phi\mathcal{S}_\epsilon(u_n - u^k) dx.$$  \hspace{1cm} (2.8)

Since

$$\left| \int_{\Omega} \hat{a}(x, u_n, \nabla u_n) \cdot \nabla \phi\mathcal{S}_\epsilon(u_n - u^k) dx \right| \leq c_3(j)\epsilon;$$  \hspace{1cm} (2.9)

then assumption (ii) implies

$$\limsup_{n \to +\infty} I_{n,k}^1(\epsilon) \leq c_3(j)\epsilon + o(1) \quad \text{as } \epsilon \to 0.$$  \hspace{1cm} (2.10)

Combining relations (2.6), (2.7) and (2.10), it follows that

$$\limsup_{n \to +\infty} \int_{\Omega} \Delta(u_n, u) \frac{1}{p(x)} \phi dx \leq o(1) \quad \text{as } \epsilon \to 0.$$  \hspace{1cm} (2.11)

Letting first $\epsilon \to 0$ and then $k$ to infinity, by relations (2.1), (2.3), (2.4), (2.5) and (2.11), we deduce

$$\lim_{n \to +\infty} \int_{\Omega} \Delta(u_n, u) \frac{1}{p(x)} \phi dx = 0.$$  

We then obtain that for a subsequence $(u_{j_n})$,

$$\Delta(u_{j_n}, u)(x) \to 0 \quad \text{a.e. on } w_j.$$  

Arguing as Leray-Lions [14, 15], we deduce from (L4) that $\nabla u_{j_n}(x) \to \nabla u(x)$ a.e. on $w_j$. The proof is achieved by the diagonal process of Cantor. \hfill \Box

**Proof of lemma 2.1**. Since $(u_n)$ belongs to a bounded set of $W^{1,p(x)}_{\text{loc}}(\mathbb{R}^N)$, statement (i) of Theorem 2.2 implies that there is a function $u$ and a subsequence still denoted by $(u_n)$ such that

$$u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N, \text{ as } n \to \infty$$  

and

$$u \in W^{1,p(x)}_{\text{loc}}(\mathbb{R}^N).$$

Then for all $\phi \in C_c^\infty(\mathbb{R}^N)$, $\phi\mathcal{S}_\epsilon(u_n - u^k)$ is an element of $W^{1,p'(x)}_{\text{comp}}(\mathbb{R}^N)$ and

$$\left| \int_{\mathbb{R}^N} f_n \phi\mathcal{S}_\epsilon(u_n - u^k) dx \right| \leq \epsilon|\phi|_\infty |f_n|_{L^1(\omega)} \leq c(\phi)\epsilon;$$  \hspace{1cm} (2.12)

(for every $\phi$ such that supp($\phi$) $\subset \omega$, $\omega$ is a compact of $\mathbb{R}^N$), and

$$|\langle g_n, \phi\mathcal{S}_\epsilon(u_n - u^k) \rangle| \leq |g_n|_{W^{-1,p'(x)}(\omega)} |\phi\mathcal{S}_\epsilon(u_n - u^k)|_{W^{1,p'(x)}(\mathbb{R}^N)}.$$  

Using the fact that $|\phi\mathcal{S}_\epsilon(u_n - u^k)|_{W^{1,p'(x)}(\mathbb{R}^N)}$ is bounded independently of $\epsilon$, $n$, $k$ and that $|g_n|_{W^{-1,p'(x)}(\omega)} \to 0$, it holds:

$$\limsup_n \int_{\mathbb{R}^N} \hat{a}(x, u_n, \nabla u_n) \cdot \nabla (\phi\mathcal{S}_\epsilon(u_n - u^k)) dx \leq O(\epsilon).$$

Finally, by Theorem 2.2 we complete the proof. \hfill \Box
3. Examples of Compactness Results

In this section, we are interested in the existence of solutions to the problem

\[ - \text{div} \left( \left| \nabla u(x) \right|^{p(x) - 2} \nabla u(x) \right) = \lambda f(u) + g(u) \quad \text{for } x \in \Omega, \]

\[ u \geq 0 \quad \text{for } x \in \Omega, \]

\[ u = 0 \quad \text{for } x \in \partial \Omega, \]

where \( \Omega \subset \mathbb{R}^N \), \( N \geq 3 \) is a bounded domain with smooth boundary, \( \lambda \) is a positive real number and \( p \) is a continuous function on \( \Omega \) with \( p^+ < N \).

In the first result, we assume that \( f \) and \( g \) are continuous and satisfy the following hypotheses (see [2]):

(F1) There exist positive constants \( C_1, C_2 > 0 \) and \( q : \Omega \rightarrow \mathbb{R} \) a continuous function such that

\[ C_1 t^{q(x) - 1} \leq f(t) \leq C_2 t^{q(x) - 1}, \quad \forall t \geq 0. \]

(G1) There exists a positive constant \( C_3 > 0 \) such that

\[ |g(t)| \leq C_3 |t|^{p^+(x) - 1}, \quad \forall t \in \mathbb{R}. \]

(G2) There exists \( \gamma \in (p^+, p^-] \) such that

\[ 0 < \gamma G(t) \leq tg(t), \quad \forall t \in \mathbb{R}, \]

where \( G(t) = \int_0^t g(s)ds \).

We prove the following result.

**Theorem 3.1.** If \( 1 < q^+ \leq p^+, \quad q^- \leq p^- \), and (F1), (G1), (G2) hold, then there exists \( \lambda^* \) such that for all \( \lambda \in (0, \lambda^*) \), problem (3.1) has a non-trivial solution.

In the second result, we are concerned with the special case \( f(u) = -|u|^{q(x) - 2}u \) and \( g(u) = |u|^{p^+(x) - 2}u \). We prove the following result.

**Theorem 3.2.** For any \( \lambda > 0 \) problem (3.1) has infinitely many weak solutions provided that \( p^+ > \max(p^+, q^+) \).

**Proof of Theorem 3.1.** Let \( E \) denote the generalized Sobolev space \( W_0^{1,p(x)}(\Omega) \). The energy functional corresponding to (3.1) is \( J_\lambda : E \rightarrow \mathbb{R} \), defined as

\[ J_\lambda(u) := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \lambda \int_\Omega F(u_+)dx - \int_\Omega G(u_+)dx, \]

where \( u_+(x) = \max\{u(x), 0\} \) and \( F \) is defined by \( F(t) = \int_0^t f(s)ds \).

**Remark 3.3.** Assume that condition (G1) is fulfilled, it is clear that for every \( t \geq 0 \), we obtain

\[ -\frac{C_3}{p^+ - 1}t^{p^+(x)} \leq G(t) \leq \frac{C_3}{p^+ - 1}t^{p^+(x)} \]

**Proposition 3.4.** The functional \( J_\lambda \) is well-defined on \( E \) and \( J_\lambda \in C^1(E, \mathbb{R}) \).

**Proof.** We have the following continuous embedding (see [2], Theorem 2.8)

\[ W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^+(x)}(\Omega) \]

using the fact that \( \Omega \) is bounded, we obtain the continuous embedding

\[ W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega), \quad s \in [1, p^+], \]
which implies that \( J_\lambda \) is well-defined on \( E \) and \( J_\lambda \in C^1(E, \mathbb{R}) \), with the derivative given by

\[
\langle dJ_\lambda(u), v \rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v - \lambda f(u)v - g(u)v) \, dx, \quad \forall v \in E.
\]

\( \square \)

The proof of Theorem 3.1 is related to Ekeland’s variational principle. In order to apply it we need the following lemmas:

**Lemma 3.5.** Under hypotheses of theorem 3.1 there exists \( M_2 > 0 \) such that for all \( \rho \in (0, 1) \) for all \( C_3 < \frac{q^-}{p^+ M_2^{q^-}} \rho^{p^+ - q^-} \), there exists \( \lambda^* > 0 \) and \( r > 0 \) such that, for all \( u \in E \) with \( \|u\| = \rho \), \( J_\lambda(u) \geq r > 0 \) for all \( \lambda \in (0, \lambda^*) \).

**Proof.** Since \( E \hookrightarrow L^{q(x)}(\Omega) \) and \( E \hookrightarrow L^{p^+}(\Omega) \) are continuous, there exists \( M_1 > 0 \) and \( M_2 > 0 \) such that

\[
|u|_{q(x)} \leq M_1 \|u\| \quad \text{and} \quad |u|_{p^+(x)} \leq M_2 \|u\|, \quad \forall u \in E. \tag{3.2}
\]

We fix \( \rho \in (0, 1) \) such that \( \rho < \min(1, 1/M_1, 1/M_2) \). Then for all \( u \in E \), with \( \|u\| = \rho \), we deduce that

\[
|u|_{q(x)} < 1 \quad \text{and} \quad |u|_{p^+(x)} < 1.
\]

Furthermore, by (1.2) for all \( u \in E \) with \( \|u\| = \rho \), we have

\[
\int_\Omega |u|^{q(x)} \, dx \leq |u|^{q^-}_{q(x)}, \quad \text{and} \quad \int_\Omega |u|^{p^+(x)} \, dx \leq |u|^{p^-}_{p(x)}.
\]

The above inequality and relation (3.2) imply that for all \( u \in E \) with \( \|u\| = \rho \),

\[
\int_\Omega |u|^{q(x)} \, dx \leq M_1^{q^-} \|u\|^{q^-}, \quad \text{and} \quad \int_\Omega |u|^{p^+(x)} \, dx \leq M_2^{p^-} \|u\|^{p^-}. \tag{3.3}
\]

Using relation (3.3) we deduce that, for any \( u \in E \) with \( \|u\| = \rho \), the following inequalities hold:

\[
J_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} C_2 M_1^{q^-} \|u\|^{q^-} - \frac{C_3}{p^+ M_2^{q^-}} \|u\|^{p^-},
\]

\[
\geq \frac{1}{p^+} \rho^{p^+} - \frac{\lambda}{q^-} C_2 M_1^{q^-} \rho^{q^-} - \frac{C_3}{p^+ M_2^{q^-}} \rho^{p^-}.
\]

By the above inequality we remark that if we define for all \( C_3 < \frac{q^-}{p^+ M_2^{q^-}} \rho^{p^+ - q^-} \)

\[
\lambda^* = \frac{q^-}{2C_2 M_1^{q^-}} \left[ \frac{1}{p^+} \rho^{p^+ - q^-} - \frac{C_3}{q^- M_2^{q^-}} \right], \tag{3.4}
\]

then for any \( \lambda \in (0, \lambda^*) \), there exists \( r > 0 \) such that \( J_\lambda(u) \geq r > 0 \) for all \( u \in E \) with \( \|u\| = \rho \). The proof is complete. \( \square \)

**Lemma 3.6.** There exists \( \phi \in E \) such that \( \phi \geq 0 \), \( \phi \neq 0 \) and \( J_\lambda(t\phi) < 0 \), for \( t > 0 \) small enough.

**Proof.** Since \( q^- < p^- \), then let \( \epsilon_0 > 0 \) be such that \( q^- + \epsilon_0 < p^- \). On the other hand, since \( q \in C(\Omega) \) it follows that there exists an open set \( \Omega_0 \subset \subset \Omega \) such that \( |q(x) - q^-| < \epsilon_0 \) for all \( x \in \Omega_0 \). Thus, we conclude that \( q(x) \leq q^- + \epsilon_0 < p^- \) for all...
Therefore, \( J_\lambda(t\phi) < 0 \), for \( t < \delta^{1/(p^+ - q^-) - \epsilon_0} \) with

\[
0 < \delta < \min \left\{ 1, \frac{p^- \lambda|\Omega_0|}{q^+ \left[ \int_{\Omega} |\nabla \phi|^{p(x)} dx + C_3 \int_{\Omega} |\phi|^{q^+(x)} dx \right]} \right\}.
\]

Finally, we point out that \( \int_{\Omega} |\nabla \phi|^{p(x)} dx + C_3 \int_{\Omega} |\phi|^{q^+(x)} dx > 0 \). In fact if

\[
\int_{\Omega} |\nabla \phi|^{p(x)} dx + C_3 \int_{\Omega} |\phi|^{q^+(x)} dx = 0,
\]

then \( \int_{\Omega} |\phi|^{q^+(x)} dx = 0 \). Using relation (1.2), we deduce that \( |\phi|_{p^+(x)} = 0 \) and consequently \( \phi = 0 \) in \( \Omega \) which is a contradiction. The proof is complete. \( \square \)

**Proof of theorem 3.1.** Let \( \lambda^* \) be defined as in (3.4) and \( \lambda \in (0, \lambda^*) \). By Lemma 3.6 it follows that on the boundary of the ball centered at the origin and of radius \( \rho \) in \( E \), denoted by \( B_\rho(0) \), we have

\[
\inf_{\partial B_\rho(0)} J_\lambda > 0. \tag{3.5}
\]

On the other hand, by Lemma 3.6 there exists \( \phi \in E \) such that \( J_\lambda(t\phi) < 0 \), for all \( t > 0 \) small enough. Moreover, relations (1.2) and (3.2) imply, that for any \( u \in B_\rho(0) \), we have

\[
J_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^+} \left[ C_{\lambda}^2 M_1^{q^+} \|u\|^{q^+} - C_{\lambda}^3 M_2^{q^+} \|u\|^{q^+} - C_{\lambda}^4 M_3^{q^+} \|u\|^{q^+} \right].
\]

It follows that

\[
-\infty < J_\infty := \inf_{B_\rho(0)} J_\lambda < 0.
\]

We let now \( 0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda \). Using the above information, the functional \( J_\lambda : B_\rho(0) \to \mathbb{R} \), is lower bounded on \( B_\rho(0) \) and \( J_\lambda \in C^1(B_\rho(0), \mathbb{R}) \). Then by Ekeland’s variational principle there exists \( u_\epsilon \in \overline{B_\rho(0)} \) such that

\[
J_\lambda \leq J_\lambda(u_\epsilon) \leq J_\infty + \epsilon,
\]

\[
0 < J_\lambda(u) - J_\lambda(u_\epsilon) + \epsilon \|u - u_\epsilon\|, \quad u \neq u_\epsilon.
\]

Since

\[
J_\lambda(u_\epsilon) \leq \inf_{B_\rho(0)} J_\lambda + \epsilon \leq \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda,
\]

we have that \( u_\epsilon \to u \) as \( \epsilon \to 0 \).
we deduce that \( u_e \in B_p(0) \).

Now, we define \( I_\lambda : B_p(0) \to \mathbb{R} \) by \( I_\lambda(u) = J_\lambda(u) + \epsilon \cdot \| u - u_e \| \). It is clear that \( u_e \) is a minimum point of \( I_\lambda \) and thus
\[
\frac{I_\lambda(u_e + t \cdot v) - I_\lambda(u_e)}{t} \geq 0,
\]
for small \( t > 0 \) and any \( v \in B_1(0) \). The above relation yields
\[
\frac{J_\lambda(u_e + t \cdot v) - J_\lambda(u_e)}{t} + \epsilon \cdot \| v \| \geq 0.
\]
Letting \( t \to 0 \) it follows that \( \langle dJ_\lambda(u_e), v \rangle + \epsilon \cdot \| v \| \geq 0 \) we have \( \| dJ_\lambda(u_e) \| \leq \epsilon \). We deduce that there exists a sequence \( \{ w_n \} \subset B_p(0) \) such that
\[
J_\lambda(u_n) \to J_\infty \quad \text{and} \quad dJ_\lambda(u_n) \to 0_{E^*}.
\]

From where we can conclude that \( \{ u_n \} \) is a bounded \((PS)_{J_\infty} \) sequence to \( J_\lambda \). By a subsequence still denoted by \( \{ u_n \} \), we may assume that \( \{ u_n \} \) has a weak limit \( u_\lambda \in W^{1,p(x)}_0(\Omega) \). Moreover, from the definition of the functional \( J_\lambda \), we can assume that \( \{ u_n \} \) is a sequence of non negative functions. Now, we need the following lemma.

**Lemma 3.7.** The weak limit \( u_\lambda \) of \( \{ u_n \} \) is a non negative solution to (3.1) for \( \lambda \in (0, \lambda^*) \).

**Proof.** In what follows, we will show \( dJ_\lambda(u_\lambda) = 0 \) and \( u_\lambda \neq 0, \forall \lambda \in (0, \lambda^*) \) which imply that lemma 3.7 holds true. Firstly note that
\[
J_\lambda(u_n) = \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \lambda \int_\Omega F(u_n)dx - \int_\Omega G(u_n)dx,
\]
\[
\langle dJ_\lambda(u_n), u_n \rangle = \int_\Omega |\nabla u_n|^{p(x)}dx - \lambda \int_\Omega f(u_n)u_n dx - \int_\Omega g(u_n)u_n dx.
\]
Then \( J_\lambda(u_n) - \frac{1}{\gamma} \langle dJ_\lambda(u_n), u_n \rangle = J_\infty + o_n(1) \). Thus, since \( uf(u) \geq 0 \) for every \( u \geq 0 \), we obtain
\[
\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \frac{1}{\gamma} \int_\Omega |\nabla u_n|^{p(x)}dx - \lambda \int_\Omega F(u_n)dx + \frac{\lambda}{\gamma} \int_\Omega f(u_n)u_n dx - \int_\Omega G(u_n)dx + \frac{1}{\gamma} \int_\Omega g(u_n)u_n dx
\]
\[
\geq \left( \frac{1}{p^*} - \frac{1}{\gamma} \right) \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \lambda \int_\Omega F(u_n)dx + \frac{1}{\gamma} \left( \int_\Omega (g(u_n)u_n - \gamma G(u_n))dx \right).
\]
Since \( \gamma > p^* \) and applying \((G2)\) we have
\[
J_\lambda(u_n) - \frac{1}{\gamma} \langle dJ_\lambda(u_n), u_n \rangle \geq \left( \frac{1}{p^*} - \frac{1}{\gamma} \right) \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \lambda \int_\Omega F(u_n)dx,
\]
\[
\geq -\lambda \int_\Omega F(u_n)dx,
\]
\[
\geq -\lambda C_2 \int_\Omega \frac{1}{q(x)}u_n^{q(x)}dx.
\]
Using (1.2) we deduce that \(-\frac{\lambda C_2}{q} |u_n|_{q(x)}^{q^+} \leq J_\infty + o_n(1)\). Moreover, \(W^{1,p(x)}_0(\Omega) \hookrightarrow L^{q(x)}(\Omega)\) is compact and passing to the limit as \(n \to \infty\), we obtain

\[-\frac{\lambda C_2}{q} |u_\lambda|_{q(x)}^{q^+} \leq J_\infty < 0.\]

We deduce that \(u_\lambda \neq 0\). To conclude that \(u_\lambda\) is a solution to (3.1), we use Theorem 2.2 which implies \(\nabla u_n(x) \to \nabla u_\lambda(x)\) a.e. in \(\Omega\) as \(n \to \infty\).

**Proof of Theorem 3.2.** Now, we are concerned with the special case of problem (3.1),

\[-\text{div} \left(\left|\nabla u(x)\right|^{p(x)-2} \nabla u(x)\right) = -\lambda |u|^{q(x)-2} u + |u|^{p(x)-2} u \quad \text{for } x \in \Omega,\]

\[u \geq 0 \quad \text{for } x \in \Omega,\]

\[u = 0 \quad \text{for } x \in \partial \Omega,\]

where \(\Omega \subset \mathbb{R}^N, (N \geq 3)\) is a bounded domain with smooth boundary, \(\lambda\) is a positive real number and \(p\) is a continuous function on \(\Omega\). The proof of Theorem 3.2 is based on a \(Z_2\)-symmetric version for even functionals of the Mountain pass Theorem (see [20] Theorem 9.12).

The energy functional corresponding to the problem (3.7) is \(J_\lambda : E \to \mathbb{R}\), defined as

\[J_\lambda(u) := \int \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \lambda \int \frac{1}{q(x)} |u|^{q(x)} \, dx - \int \frac{1}{p^*(x)} |u|^{p^*(x)} \, dx,\]

It is clear that the functional \(J_\lambda\) is well-defined on \(E\) and \(J_\lambda \in C^1(E, \mathbb{R})\), with the derivative given by

\[\langle dJ_\lambda(u), v \rangle = \int (|\nabla u|^{p(x)-2} \nabla u \nabla v + \lambda |u|^{q(x)-2} uv - |u|^{p^*(x)-2} uv) \, dx, \quad \forall v \in E.\]

To use the mountain pass theorem, we need the following lemmas:

**Lemma 3.8.** *For any \(\lambda > 0\) there exists \(r, a > 0\) such that \(J_\lambda(u) \geq a > 0\) for any \(u \in E\) with \(|u| = r\).*

**Proof.** Recall that \(E\) is continuously embedded in \(L^{p^*(x)}(\Omega)\). So there exists a positive constant \(C_4\) such that, for all \(u \in E\),

\[|u|_{p^*(x)} \leq C_4 \|u\|.\]

(3.8)

Suppose that \(|u| < \min(1, \frac{1}{C_4})\), then for all \(u \in E\) with \(|u| = \rho\) we have \(|u|_{p^*(x)} < 1\). Furthermore, relation (1.2) yields for all \(u \in E\) with \(|u| = \rho\) we have

\[\int_\Omega |u|^{p^*(x)} \, dx \leq |u|^{p^* \cdot}.\]

The above inequality and relation (3.8) imply that for all \(u \in E\) with \(|u| = \rho\), we have

\[\int_\Omega |u|^{p^*(x)} \, dx \leq C_4^{p^* \cdot} \|u\|^{p^* \cdot}.\]

(3.9)

Then using relation (3.9), we deduce that, for any \(u \in E\) with \(|u| = \rho\), the following inequalities hold

\[J_\lambda(u) \geq \frac{1}{p^*} \int_\Omega |\nabla u|^{p(x)} \, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*(x)} \, dx,\]
Then we have
\[ \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{1}{p^+} C_4^{p^+} - \lambda \|u\|^{p^+}. \]

Let \( h(t) = \frac{1}{p^+} t^{p^+} - \frac{1}{p^+} C_4^{p^+} t^{p^+}, \) \( t > 0. \) It is easy to see that \( h(t) > 0 \) for all \( t \in (0, t_1), \) where \( t_1 < \left( \frac{p^+}{p^+ C_4^{p^+}} \right)^{\frac{1}{p^2 - p^+}}. \)

So for any \( \lambda > 0, \) we can choose \( r, a > 0 \) such that \( J_\lambda(u) \geq a > 0 \) for all \( u \in E \) with \( \|u\| = r. \) The proof is complete.

\[ \square \]

**Lemma 3.9.** If \( E_1 \subseteq E \) is a finite dimensional subspace, the set \( S = \{ u \in E_1 \colon J_\lambda(u) \geq 0 \} \) is bounded in \( E. \)

**Proof.** We have
\[ \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \leq K_1(\|u\|^{p^+} + \|u\|^{p^+}) \quad \forall u \in E, \tag{3.10} \]
where \( K_1 \) is a positive constant. Also we have
\[ \int_{\Omega} |u|^{q(x)} \, dx \leq |u|^{q^-} + |u|^{q^+} \quad \forall u \in E. \tag{3.11} \]

The fact that \( E \) is continuously embedded in \( L^{q(x)}(\Omega), \) assures the existence of a positive constant \( C_5 \) such that
\[ |u|^{q(x)} \leq C_5 \|u\| \quad \forall u \in E. \tag{3.12} \]

The last two inequalities show that there exists a positive constant \( K_2(\lambda) \) such that
\[ \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx \leq K_2(\lambda) (\|u\|^{q^-} + \|u\|^{q^+}) \quad \forall u \in E. \tag{3.13} \]

By inequalities (3.10) and (3.13), we obtain
\[ J_\lambda(u) \leq K_1(\|u\|^{p^+} + \|u\|^{p^+}) + K_2(\lambda) (\|u\|^{q^-} + \|u\|^{q^+}) - \frac{1}{p^+} \int_{\Omega} |u|^{p^+} \, dx, \tag{3.14} \]
for all \( u \in E. \)

Let \( u \in E \) be arbitrary but fixed. We define
\[ \Omega_\prec = \{ x \in \Omega \mid |u(x)| < 1 \}, \quad \Omega_\succ = \Omega \setminus \Omega_\prec. \]

Then we have
\[ J_\lambda(u) \leq K_1(\|u\|^{p^+} + \|u\|^{p^+}) + K_2(\lambda) (\|u\|^{q^-} + \|u\|^{q^+}) - \frac{1}{p^+} \int_{\Omega} |u|^{p^+} \, dx, \]
\[ \leq K_1(\|u\|^{p^+} + \|u\|^{p^+}) + K_2(\lambda) (\|u\|^{q^-} + \|u\|^{q^+}) - \frac{1}{p^+} \int_{\Omega_\succ} |u|^{p^+} \, dx, \]
\[ \leq K_1(\|u\|^{p^+} + \|u\|^{p^+}) + K_2(\lambda) (\|u\|^{q^-} + \|u\|^{q^+}) - \frac{1}{p^+} \int_{\Omega_\succ} |u|^{p^-} \, dx, \]
\[ \leq K_1(\|u\|^{p^+} + \|u\|^{p^+}) + K_2(\lambda) (\|u\|^{q^-} + \|u\|^{q^+}) - \frac{1}{p^+} \int_{\Omega_\prec} |u|^{p^-} \, dx, \]
\[ + \frac{1}{p^+} \int_{\Omega_\prec} |u|^{p^-} \, dx. \]

But there exists positive constant \( K_3 \) such that
\[ \frac{1}{p^+} \int_{\Omega_\prec} |u|^{p^-} \, dx \leq K_3 \quad \forall u \in E. \]
The functional $|.|_{p^*} : E \to \mathbb{R}$ defined by

$$|u|_{p^*} = \left( \int |u|^{p^*} \, dx \right)^{1/p^*}$$

is a norm in $E$. In the finite dimensional subspace $E_1$ the norm $|u|_{p^*}$ and $\|u\|$ are equivalent, so there exists a positive constant $K = K(E_1)$ such that

$$\|u\| \leq K \|u\|_{p^*}, \quad \forall u \in E_1.$$

So that there exists a positive constant $K_4$ such that

$$J_3(u) \leq K_1(\|u\|^{p^*} + \|u\|^{p^*_0}) + K_2(\lambda)(\|u\|^{q^*} + \|u\|^{q^*_0}) + K_3 - K_4 \|u\|^{p^*}, \quad \forall u \in E_1.$$

Hence

$$K_1(\|u\|^{p^*} + \|u\|^{p^*_0}) + K_2(\lambda)(\|u\|^{q^*} + \|u\|^{q^*_0}) + K_3 - K_4 \|u\|^{p^*} \geq 0, \quad \forall u \in S.$$

And since $p^* > \max(p^*_0, q^*_0)$, we conclude that $S$ is bounded in $E$.

**Lemma 3.10.** If $\{u_n\} \subset E$ is a sequence which satisfies the properties

$$J_3(u_n) < C_6, \quad dJ_3(u_n) \to 0 \quad \text{as} \quad n \to \infty,$$

where $C_6$ is a positive constant, then $\{u_n\}$ possesses a convergent subsequence.

**Proof.** First we show that $\{u_n\}$ is bounded in $E$. If not, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\| > 1$ for any integer $n$. Using (3.16) it follows that there exists $N_1 > 0$ such that for any $n > N_1$ we have

$$\|dJ_3(u_n)\| \leq 1.$$

On the other hand, for all $n > N_1$ fixed, the application $E \ni v \to \langle dJ_3(u_n), v \rangle$ is linear and continuous. The above information yield that

$$|\langle dJ_3(u_n), v \rangle| \leq \|dJ_3(u_n)\||v| \leq \|v\|, \quad \forall v \in E, \quad n > N_1.$$

Setting $v = u_n$, we have

$$-\|u_n\| \leq \int \nabla u_n \cdot |p(x)| dx - \int |u_n|^{p^*(x)} dx + \lambda \int |u_n|^{q(x)} dx \leq \|u_n\|,$$

for all $n > N_1$. We obtain

$$-\|u_n\| - \int \nabla u_n \cdot |p(x)| dx - \lambda \int |u_n|^{q(x)} dx \leq - \int |u_n|^{p^*(x)} dx,$$

for all $n > N_1$. Provided that $\|u_n\| > 1$ relation (3.15) and (3.17) imply

$$C_6 > J_3(u_n) \geq \left( \frac{1}{p^*} - \frac{1}{p^*} \right) \int |u_n|^{p^*(x)} dx + \lambda \left( \frac{1}{q^*} - \frac{1}{p^*} \right) \int |u_n|^{q(x)} dx - \frac{1}{p^*} \|u_n\|,$$

for all $n > N_1$. Provided that $\|u_n\| > 1$ relation (3.15) and (3.17) imply

$$C_6 > J_3(u_n) \geq \left( \frac{1}{p^*} - \frac{1}{p^*} \right) \|u_n\|^{p^*} dx - \frac{1}{p^*} \|u_n\|.$$

Letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in $E$. And we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u \in E$ such that $\{u_n\}$ converges weakly to $u$ in $E$. Now by Theorem 2.2 we have

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \ \mathbb{R}^N \quad \text{as} \quad n \to \infty.$$
Proof of Theorem 3.2. It is clear that the functional $J_\lambda$ is even and verifies $J_\lambda(0) = 0$. Lemma 3.8, Lemma 3.9 and Lemma 3.10 implies that $J_\lambda$ satisfies the the Mountain Pass Theorem condition. Thus we conclude that problem (3.7) has infinitely many weak solutions in $E$. The proof is complete. □

References

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