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EXISTENCE RESULTS FOR A P-LAPLACIAN PROBLEM WITH COMPETING NONLINEARITIES AND NONLINEAR BOUNDARY CONDITIONS

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Abstract. By using the fibering method we study the existence of non-negative solutions for a class of quasilinear elliptic problems in the presence of competing subcritical nonlinearities.

1. Introduction

In this paper we study the problem

$$\begin{gathered}\Delta_p u = a(x)|u|^{p-2}u - b(x)|u|^{q-2}u \quad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda c(x)|u|^{p-2}u \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $1 < q < p < N$, $a(\cdot)$, $b(\cdot) \in L^\infty(\Omega)$ with $a(x) > \theta > 0$, $b(x) > 0$ a.e., $c(x) \in L^\infty(\partial\Omega)$, with $c(x) > 0$ a.e. As usual, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator.

When $b \equiv 0$, problem (1.1) appears naturally in the study of the Sobolev trace inequality. Since the embedding $W^{1,p}(\Omega) \subseteq L^p(\Omega)$ is compact there exists a constant λ_1 such that

$$\lambda_1^{1/p}\|u\|_{L^p(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}.$$

The functions at which equality holds; that is,

$$\lambda_1 := \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{W^{1,p}(\Omega)}}{\|u\|^p_{L^p(\partial\Omega)}}, \tag{1.2}$$

are called extremals and are the solutions to the problem

$$\begin{gathered}\Delta_p u = a(x)|u|^{p-2}u \quad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda_1 c(x)|u|^{p-2}u \quad \text{on } \partial\Omega,\end{gathered} \tag{1.3}$$

For more details we refer the reader to [9].

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Problems of the form $\Delta_p u = \pm\lambda|u|^{p-2}u + f(x,u)$ with Dirichlet boundary conditions has been extensively studied, see for example [1, 6, 7, 10, 13]. Recently, this problem with nonlinear boundary conditions has been considered in [3, 4, 12].

In this paper we employ Pohozaev's fibering method in order to show that if $\lambda < \lambda_1$, then (1.1) admits a nonnegative solution. In the case $\lambda = \lambda_1$, the fibering method is no longer applicable, so we introduce the term $\varepsilon d(\cdot)|u|^{s-2}u$ in the equation, where $\varepsilon > 0$ and $d(\cdot) \in L^\infty(\Omega), d(\cdot) > 0$ a.e., and examine the behavior of the solutions u_ε as $\varepsilon \to 0$. It turns out that $\|u_\varepsilon\|_{W^{1,p}(\Omega)} \to +\infty$ and the energy of the solutions diverges to $-\infty$.

2. Main results

Our reference space is $W^{1,p}(\Omega)$ equipped with the norm $\|u\|^p = \int_\Omega[|\nabla u|^p + a(x)|u|^p]dx$, which is equivalent to its usual one. In what follows, $\sigma(\cdot)$ is the surface measure on the boundary of Ω.

The energy functional associated with (1.1) is

$$\Phi_\lambda(u) := \frac{1}{p}\int_\Omega[|\nabla u|^p dx + a(x)|u|^p]dx - \frac{1}{q}\int_\Omega b(x)|u|^q dx - \frac{\lambda}{p}\int_{\partial\Omega} c(x)|u|^p d\sigma(x). \tag{2.1}$$

Following [9], let $\lambda_1 \in \mathbb{R}$ be the first positive eigenvalue of (1.3), given by (1.2).

Theorem 2.1. *Suppose that* $1 < q < p < N$ *and* $\lambda < \lambda_1$. *Then* (1.1) *admits a nonnegative solution.*

Proof. We employ the fibering method introduced in [11], see also [2] and [8], in order to prove the existence of a negative energy solution of (1.1). Writing $u = rv$, $r > 0$ and $v \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}\Phi_\lambda(rv) = \frac{r^p}{p}\int_\Omega |\nabla v|^p dx + \frac{r^p}{p}\int_\Omega a(x)|v|^p dx - \frac{r^q}{q}\int_\Omega b(x)|v|^q dx \\ - \frac{\lambda r^p}{p}\int_{\partial\Omega} c(x)|v|^p d\sigma(x).\end{aligned} \tag{2.2}$$

For $u \neq 0$ to be a critical point, it should hold $\frac{\partial \Phi_\lambda(rv)}{\partial r} = 0$, from which we obtain

$$\begin{aligned}&r^{p-q}\int_\Omega |\nabla v|^p dx + r^{p-q}\int_\Omega a(x)|v|^p dx - \lambda r^{p-q}\int_{\partial\Omega} c(x)|v|^p d\sigma(x) \\ &= \int_\Omega b(x)|v|^q dx,\end{aligned} \tag{2.3}$$

ensuring the existence of a unique $r = r(v) > 0$ satisfying (2.3). By the implicit function theorem [14, Thm. 4.B], the function $v \to r(v)$ is continuously differentiable for $v \neq 0$. Notice that

$$r(kv)kv = r(v)v \quad \text{for } k > 0. \tag{2.4}$$

In view of (2.2) and (2.3),

$$\Phi_\lambda(r(v)v) = (\frac{1}{p} - \frac{1}{q})r(v)^q \int_\Omega b(x)|v|^q dx < 0.$$

Consider the functional

$$H(u) := \int_\Omega |\nabla u|^p dx + \int_\Omega a(x)|u|^p dx - \lambda\int_{\partial\Omega} c(x)|u|^p d\sigma(x).$$

By the way we chose λ, for $u \in W^{1,p}(\Omega)$, $H(u) \geq 0$ (equality holds exactly when $u = 0$). Define $V = \{v \in W^{1,p}(\Omega) : H(v) = 1\}$. Evidently, $(H'(v), v) \neq 0$ for $v \in V$. In view of [2, Lemma 3.4], any conditional critical point of $\widehat{\Phi}_\lambda(v) = \Phi_\lambda(r(v)v)$ subject to $H(v) = 1$, provides a critical point $r(v)v$ of Φ_λ. Notice that V is bounded. To see this, let $\varepsilon > 0$ be such that $\lambda + \varepsilon < \lambda_1$. Then, for $v \in V$, by the definition of λ_1,

$$\lambda + \varepsilon < \frac{\int_\Omega |\nabla v|^p + \int_\Omega a|v|^p}{\int_{\partial\Omega} c(x)|v|^p d\sigma(x)},$$

which implies that

$$1 = \int_\Omega |\nabla v|^p + \int_\Omega a|v|^p - \lambda \int_{\partial\Omega} c(x)|v|^p d\sigma(x) > \varepsilon \int_{\partial\Omega} c(x)|v|^p d\sigma(x).$$

Thus $\int_{\partial\Omega} c(x)|v|^p d\sigma(x)$, $v \in V$, is bounded. Consequently, V is a bounded set. Because of the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, (2.3) guarantees that $r(V)$ is bounded. Consequently, $I = \{\Phi_{\lambda,\mu}(r(v)v) : v \in V\}$ is a bounded interval in $\mathbb{R}$ with endpoints a and b, $a < b \leq 0$. We are now going to show that $a \in I$. To this end, let $\{v_n\}_{n\in\mathbf{N}}$ be a sequence in V, with $\Phi_\lambda(r(v_n)v_n) \to a$. Without loss of generality, we may assume that $v_n \to v$ weakly in $W^{1,p}(\Omega)$. We may also assume that $r(v_n) \to r \in \mathbb{R}$. Thus $r(v_n)v_n \to rv$ weakly in $W^{1,p}(\Omega)$. Since $\Phi_\lambda(\cdot)$ is weakly lower semicontinuous,

$$\Phi_\lambda(rv) \leq \liminf_{n\to\infty} \Phi_\lambda(r(v_n)v_n) = a, \tag{2.5}$$

ensuring that $rv \neq 0$. Because of the compactness of the Sobolev and trace embeddings, $r(v_n)v_n \to rv$ strongly in $L^q(\Omega)$, $L^p(\partial\Omega)$, respectively. Taking into account the lower semicontinuity of the norm in (2.3), we have

$$r^{p-q} H(v) \leq \int_\Omega b(x)|v|^q dx. \tag{2.6}$$

Combining (2.3) and (2.6), we get $r \leq r(v)$. Our purpose is to prove equality. Let us assume the contrary; that is $r < r(v)$. We define $F(y) = \Phi_\lambda(yv)$, $y \geq 0$. For $y \in [r, r(v)]$, we have

$$F'(y) = y^{q-1}\Big(y^{p-q} H(v) - \int_\Omega b(x)|v|^q dx\Big), \tag{2.7}$$

which is negative everywhere, but at $y = r(v)$. Thus $F(y)$ decreases strictly in the considered interval, giving

$$\Phi_\lambda(r(v)v) < \Phi_\lambda(rv) \leq a, \tag{2.8}$$

because of (2.5). Notice that for suitable $k \geq 1$, $kv \in V$. Then, combining (2.4) and (2.8), we obtain

$$\Phi_\lambda(r(kv)kv) = \Phi_\lambda(r(v)v) < \Phi_\lambda(rv) \leq a,$$

which is a contradiction. So, $r = r(v)$, and necessarily $\Phi_\lambda(r(kv)kv) = a$. This means that kv is a conditional critical point of $\widehat{\Phi}_\lambda(\cdot)$ subject to $H(v) = 1$, and, consequently, $r(kv)kv = r(v)v$ is a critical point of $\Phi_\lambda(\cdot)$. Since for a minimizer w, $|w|$ is also a minimizer, we may assume $v \geq 0$, and $r(v)v$ is a nontrivial nonnegative solution of (1.1). □

In attempting to obtain the existence of a solution to problem (1.1) for $\lambda = \lambda_1$, following a similar procedure, we encounter an unsurpassable difficulty, due to the fact that (2.3) does no longer guarantee the existence of a suitable $r(v)$. In order to study this situation, we add an additional term in (1.1), with the problem taking the following form

$$
\begin{gathered}
\Delta_p u = a(x)|u|^{p-2}u - b(x)|u|^{q-2}u + \varepsilon d(x)|u|^{s-2}u \quad \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda_1 c(x)|u|^{p-2}u \quad \text{on } \partial\Omega,
\end{gathered} \tag{2.9}
$$

where, $\varepsilon > 0$, $q < s < p^*$, and $d(\cdot) \in L^\infty(\Omega)$ with $d(\cdot) > 0$ a.e. in Ω. The energy functional is

$$
F_{\lambda_1,\varepsilon}(u) := \Phi_{\lambda_1}(u) + \frac{\varepsilon}{s} D(u), \tag{2.10}
$$

where

$$
D(u) := \int_\Omega d(x)|u|^s dx.
$$

Theorem 2.2. *Suppose that* $1 < q < s < p^*$, $\varepsilon > 0$ *and* $d(\cdot) \in L^\infty(\Omega)$ *with* $d(\cdot) > 0$ *a.e. in* Ω*.Then problem* (2.9) *admits a nonnegative solution* u_ε *for every* $\varepsilon > 0$*. Furthermore,* $F_{\lambda_1,\varepsilon}(u_\varepsilon) \to -\infty$ *and* $\|u_\varepsilon\| \to +\infty$ *as* $\varepsilon \to 0$*.*

Proof. Following a similar reasoning, we obtain the counterpart of (2.3),

$$
\begin{aligned}
& r^{p-q}\big[\int_\Omega |\nabla v|^p dx + \int_\Omega a(x)|v|^p dx - \lambda_1 \int_{\partial\Omega} c(x)|v|^p d\sigma(x)\big] \\
& + \varepsilon r^{s-q}\int_\Omega d(x)|v|^s dx \\
& = \int_\Omega b(x)|v|^q dx.
\end{aligned} \tag{2.11}
$$

The function $R(y) = Hy^{p-q} + \varepsilon D y^{s-q} - B$, with $H \geq 0$, D, $B > 0$, has a unique root in $(0, +\infty)$, since it is strictly increasing, $R(0) = -B$ and $R(y) \to +\infty$, for $y \to +\infty$. Thus, for $v \in W^{1,p}(\Omega)$ there exists a unique positive $r_\varepsilon(v)$ satisfying (2.11). The so defined function $v \to r_\varepsilon(v)$ is once more continuously differentiable for $v \neq 0$, by another application of the implicit function theorem. In addition, it is easily checked that (2.4) remains true. We notice also that, due to (2.11), if $v \neq 0$,

$$
F_{\lambda_1,\varepsilon}(r_\varepsilon(v)v) = \big(\frac{1}{p} - \frac{1}{q}\big) r_\varepsilon(v)^p H(v) + \varepsilon\big(\frac{1}{s} - \frac{1}{q}\big) r_\varepsilon(v)^s D(v) < 0. \tag{2.12}
$$

We define next the positive functional (except at $u = 0$),

$$
L(u) := H(u) + D(u). \tag{2.13}
$$

Consider the set

$$
W = \{v \in W^{1,p}(\Omega) : L(v) = 1\}.
$$

Because of our hypothesis on $d(\cdot)$, $(L'(v), v) > D(v) > 0$ for $v \in W$. As usual, the conditional critical points of $\widehat{F}_{\lambda_1,\varepsilon}(v) = F_{\lambda_1,\varepsilon}(r_\varepsilon(v)v)$ subject to $L(v) = 1$ provide critical points $r_\varepsilon(v)v$ of F_{λ_1}. We claim that W is bounded. Indeed, if not, there would exist $v_n \in W, n \in \mathbb{N}$, such that $\|v_n\| \to +\infty$. Let $v_n := t_n u_n$ with $t_n > 0$ and $\|u_n\| = 1$. Since u_n, $n \in \mathbb{N}$, is bounded, by passing to a subsequence if necessary, we may assume that $u_n \to u_0$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(c, \partial\Omega)$ and $L^s(\Omega)$. By (2.13),

$$
\begin{aligned}
& t_n^p\big[\int_\Omega |\nabla u_n|^p dx + \int_\Omega a(x)|u_n|^p dx \\
& - \lambda_1 \int_{\partial\Omega} c(x)|u_n|^p d\sigma(x)\big] + t_n^s \int_\Omega d(x)|u_n|^s dx = 1,
\end{aligned}
$$

and so

$$0 \le \int_\Omega |\nabla u_n|^p dx + \int_\Omega a(x)|u_n|^p dx - \lambda_1 \int_{\partial\Omega} c(x)|u_n|^p d\sigma(x) \le \frac{1}{t_n^p} \to 0 \tag{2.14}$$

and

$$0 < \int_\Omega d(x)|u_n|^s dx \le \frac{1}{t_n^s} \to 0. \tag{2.15}$$

By (2.15), $u_0 = 0$. On the other hand, since $\|u_n\| = 1$, (2.14) yields

$$\lambda_1 \int_{\partial\Omega} c(x)|u_0|^p d\sigma(x) = 1$$

and so $u_0 \ne 0$, a contradiction, thereby proving the claim. We can now continue as in the previous case. Namely, we notice that by the way it was defined, $r_\varepsilon(v)$ is bounded on W (we use now the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$). Thus $I' = \{F_{\lambda_1,\varepsilon}(r_\varepsilon(v)v) : v \in W\}$ is a bounded interval with endpoints a' and b', $a' < b' \le 0$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence of W with $F_{\lambda_1,\varepsilon}(r_\varepsilon(v_n)v_n) \to a'$. We may assume that $v_n \to v_\varepsilon$ weakly in $W^{1,p}(\Omega)$, and $r_\varepsilon(v_n) \to r_\varepsilon \in \mathbb{R}$. Thus $r_\varepsilon(v_n)v_n \to r_\varepsilon v_\varepsilon$ weakly in $W^{1,p}(\Omega)$, and consequently, at least for a subsequence, strongly in $L^s(\Omega)$. Since $\Phi_{\lambda_1}(\cdot)$ is weakly lower semicontinuous, so is $F_{\lambda_1,\varepsilon}(\cdot)$, and the obvious counterpart of (2.5) ensures that $r_\varepsilon v_\varepsilon \ne 0$. Combining (2.11) with the lower semicontinuity of the involved norms, the compactness of the Sobolev and trace embeddings $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$, and $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, respectively, we obtain

$$r_\varepsilon^{p-q} H(v_\varepsilon) + r_\varepsilon^{s-q} \varepsilon D(v_\varepsilon) \le \int_\Omega b(x)|v_\varepsilon|^q dx = B(v_\varepsilon). \tag{2.16}$$

Evidently, (2.11) and (2.16) ensure that $r_\varepsilon \le r_\varepsilon(v_\varepsilon)$. We are going to prove equality. Assuming the contrary, the function $G(y) = F_{\lambda_1,\varepsilon}(yv_\varepsilon)$, $y > 0$, has its derivative

$$G'(y) = y^{q-1}\big(y^{p-q}H(v_\varepsilon) + y^{s-q}\varepsilon D(v_\varepsilon) - B(v_\varepsilon)\big)$$

which is negative in $[r_\varepsilon,\, r_\varepsilon(v_\varepsilon)]$ except at $y = r_\varepsilon(v_\varepsilon)$, where it is zero. Thus $G(y)$ decreases strictly in the above interval, meaning

$$F_{\lambda_1,\varepsilon}(r_\varepsilon(v_\varepsilon)v_\varepsilon) < F_{\lambda_1,\varepsilon}(r_\varepsilon v_\varepsilon) \le a', \tag{2.17}$$

since $F_{\lambda_1,\varepsilon}(r_\varepsilon v_\varepsilon) \le \liminf_{n\to+\infty} F_{\lambda_1,\varepsilon}(r_\varepsilon(v_n)v_n) = a'$. Next we choose a positive k, such that $kv_\varepsilon \in W$. Since (2.4) holds, we arrive at an obvious contradiction. Thus $r_\varepsilon = r_\varepsilon(v_\varepsilon)$, and $F_{\lambda_1,\varepsilon}(r_\varepsilon(kv_\varepsilon)kv_\varepsilon) = a'$, thus obtaining a conditional critical point of $\widehat{F}_{\lambda_1,\varepsilon}(\cdot)$ subject to $L(v) = 1$, and, consequently, $u_\varepsilon := r_\varepsilon(v_\varepsilon)v_\varepsilon$ is a critical point of $F_{\lambda_1,\varepsilon}(\cdot)$. Once more, we may assume $v_\varepsilon \ge 0$, and so u_ε is a nontrivial nonnegative solution of (2.9).

Next we study the behavior of the solutions $u_\varepsilon = r_\varepsilon(v_\varepsilon)v_\varepsilon$ as $\varepsilon \to 0$. Let $\varphi_1 > 0$ be the eigenfunction of (1.3) corresponding to λ_1, with $L(\varphi_1) = 1$. By (2.11),

$$r_\varepsilon(\varphi_1)^{s-q} = \frac{\int_\Omega b(x)|\varphi_1|^q dx}{\varepsilon \int_\Omega d(x)|\varphi_1|^s dx}, \tag{2.18}$$

which implies that $r_\varepsilon(\varphi_1) \to +\infty$ as $\varepsilon \to 0$. In view of (2.12) and (2.11)

$$F_{\lambda_1,\varepsilon}(r_\varepsilon(\varphi_1)\varphi_1) = \varepsilon\big(\frac{1}{s} - \frac{1}{q}\big) r_\varepsilon(\varphi_1)^s D(\varphi_1) = \big(\frac{1}{s} - \frac{1}{q}\big) r_\varepsilon(\varphi_1)^q \int_\Omega b(x)|\varphi_1|^q dx.$$

Since $F_{\lambda_1,\varepsilon}(r_\varepsilon(v_\varepsilon)v_\varepsilon) \le F_{\lambda_1,\varepsilon}(r_\varepsilon(\varphi_1)\varphi_1)$, we conclude that

$$F_{\lambda_1,\varepsilon}(u_\varepsilon) = F_{\lambda_1,\varepsilon}(r_\varepsilon(v_\varepsilon)v_\varepsilon) \to -\infty$$

as $\varepsilon \to 0$. By (2.12) we also get that $r_\varepsilon(v_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Let $\widehat{v}$ be a weak accumulation point of v_ε; that is, $\widehat{v} = w - \lim_{n\to+\infty} v_{\varepsilon_n}$ where $\varepsilon_n \to 0$ as $n \to +\infty$. Since $L(v_{\varepsilon_n}) = 1$, necessarily

$$0 \le \int_\Omega |\nabla v_{\varepsilon_n}|^p dx + \int_\Omega a(x)|v_{\varepsilon_n}|^p dx - \lambda_1 \int_{\partial\Omega} c(x)|v_{\varepsilon_n}|^p d\sigma(x) \to 0.$$

Consequently, either $\widehat{v} = 0$ or $\widehat{v} = \gamma\varphi_1$ for some $\gamma \neq 0$. We cannot have $\widehat{v} = 0$, because then, since $v_{\varepsilon_n} \in W$, we would get that $\int_\Omega d(x)|\widehat{v}|^s dx = \lim \int_\Omega d(x)|v_{\varepsilon_n}|^s dx = 1$. Therefore, $\widehat{v} = \gamma\varphi_1$ and so $\|u_{\varepsilon_n}\| = r_{\varepsilon_n}(v_{\varepsilon_n})\|v_{\varepsilon_n}\| \to +\infty$ as $n \to +\infty$. □

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