GLOBAL CLASSICAL SOLUTIONS FOR REACTION-DIFFUSION SYSTEMS WITH A TRIANGULAR MATRIX OF DIFFUSION COEFFICIENTS

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ABSTRACT. The goal of this article is to study the existence of classical solutions global in time for reaction-diffusion systems with strong coupling in the diffusion and with exponential growth (or without any growth) conditions on the nonlinear reactive terms. This extends some similar results in the case of a diagonal diffusion-operator associated with nonlinearities preserving the positivity and the total mass of the solutions or for which the total mass is a priori bounded.

1. INTRODUCTION

In this study, we are interested in the existence of classical global solutions to the reaction-diffusion system

\[ \frac{\partial u}{\partial t} - a \Delta u = \lambda - f(u,v) - \mu u \quad \text{in } (0, +\infty) \times \Omega, \]

\[ \frac{\partial v}{\partial t} - c \Delta u - d \Delta v = f(u,v) - \mu v \quad \text{in } (0, +\infty) \times \Omega, \]

where \( \Omega \) is an open bounded domain of class \( C^1 \) in \( \mathbb{R}^n \) with boundary \( \partial \Omega \), the constants \( a, c, d, \lambda, \mu \) are such that

(A1) \( a > 0, c > 0, d > a, 2\sqrt{ad} > c, \lambda \geq 0 \) and \( \mu > 0 \),

and the function \( f \) is a nonnegative and continuously differentiable on \( [0, +\infty) \) such that

(A2) \( f(0, \eta) = 0 \) and \( f(\xi, \eta) \geq 0 \), with \( f(\xi, \frac{c}{d-a}(\frac{\lambda}{\mu} - \xi)) = 0 \) when \( a + c \geq d \),

(A3) \( f(\xi, \eta) \leq C\varphi(\xi)\eta^r e^{\alpha \eta} \) for some constants \( C > 0 \) and \( \alpha > 0 \) when \( a + c < d \),

where \( r \) is a positive constant such that \( r \geq 1 \) and \( \varphi \) is any nonnegative continuously differentiable function on \( [0, +\infty) \) such that \( \varphi(0) = 0 \).

We assume that the solutions of (1.1)-(1.2) also satisfy: the boundary conditions

\[ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } (0, +\infty) \times \partial \Omega, \]

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where $\frac{\partial}{\partial \nu}$ is the outward normal derivative to $\partial \Omega$; and the initial conditions
\begin{equation}
  u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \quad \text{in } \Omega, \tag{1.4}
\end{equation}

where $u_0$, $v_0$ are nonnegative and bounded functions satisfying the following restrictions:
\begin{align}
  \|u_0\|_\infty &\leq \frac{\lambda}{\mu} < \frac{8ad}{n(a - d)^2}, \quad \text{when } a + c < d, \\
  \|u_0\|_\infty &\leq \frac{\lambda}{\mu}, \quad \text{when } a + c \geq d, \\
  v_0 &\geq \frac{c}{d - a} \left(\frac{\lambda}{\mu} - u_0\right). \tag{1.5} \tag{1.6}
\end{align}

Problem (1.1)–(1.4) may be viewed as a diffusive epidemic model where $u$ and $v$ represent the nondimensional population densities of susceptibles and infectives, respectively. This problem can be represent a model describing the spread of an infection disease (such as AIDS for instance) within a population assumed to be divided into the susceptible and infective classes as precised (for further motivation see for instance [6, 7, 10] and the references therein).

When $\lambda = \mu = 0$, Kouachi and Youkana [18] generalized the method of Haraux and Youkana [12] with the reaction term $f(\xi, \eta)$ requiring the condition
\begin{equation}
  \lim_{\eta \to +\infty} \frac{\ln(1 + f(\xi, \eta))}{\eta} < \alpha^*, \quad \text{for any } \xi \geq 0,
\end{equation}

with
\begin{equation}
  \alpha^* = \frac{2ad}{n(a - d)^2\|u_0\|_\infty \min\{1, \frac{a - d}{c}\}},
\end{equation}

where $a$, $c$ and $d$ satisfy $a > 0$, $c > 0$, $d > 0$ and $a > d$. This condition reflects the weak exponential growth of the function $f$.

Kanel and Kirane [15] proved the existence of a classical global solutions for a coupled reaction-diffusion system without any conditions on the growth of the function $f$ under the following conditions:
\begin{itemize}
  \item $a > d$ and $c \geq d - a > 0,$
  \item $f(\xi, \eta) = F(\xi)G(\eta)$.
\end{itemize}

Later they improved their results in [16] where they extended the result of Herrero et al [14] to the case of a bounded domain under the following assumptions:
\begin{itemize}
  \item $a < d$ and $0 \leq c < d - a,$
  \item $f(\xi, \eta) \leq C\varphi(\xi)e^{\alpha \eta}$, for some $C > 0$, $\alpha > 0$ and any nonnegative continuous and locally Lipschitzian function $\varphi$ on $\mathbb{R}$ such that $\varphi(0) = 0$.
\end{itemize}

In the case, where $\lambda \geq 0$ and $\mu > 0$, Abdelmalek and Youkana [11] proved the global existence of nonnegative classical solutions for the nonlinearities of weakly exponential growth under the following assumptions:
\begin{itemize}
  \item $a > 0$, $c > 0$, $d - a \geq c$, and $2\sqrt{ad} > c,$
  \item $\|u_0\|_\infty \leq \lambda/\mu$.
\end{itemize}

We note that solving problem (1.1)–(1.4) is quite difficult. As a consequence of the blow-up examples found in [23], we can prove that there is blow-up of the solutions in finite time for such triangular systems even though the initial data are regular, the solutions are positive and the nonlinear terms are negative, a structure that ensured the global existence in the diagonal case.
The aim of this article is to prove the existence of global classical solutions to (1.1)–(1.4) without any restrictions on the growth of the function $f$ when $a + c \geq d$ and with possibility of exponential growth for this function when $a + c < d$. For this purpose, we demonstrate that for any initial conditions satisfying (1.5) and (1.6), the problem (1.1)–(1.4) is equivalent to a problem for which the global existence follows from a similar Lyapunov functionals appeared in [12, 11, 18, 24] under the assumptions (A1)–(A3).

2. Existence of local and positive solutions

The study of existence and uniqueness of local solutions $(u, v)$ of (1.1)–(1.4) follows from the basic existence theory for parabolic semilinear equations (see, e.g., [3, 9, 13, 22]). As a consequence, for any initial data in $C(\bar{\Omega})$ or $L^\infty(\Omega)$ there exists $T^* \in (0, +\infty]$ such that (1.1)–(1.4) has a unique classical solution on $[0, T^*) \times \Omega$.

Furthermore, if $T^* < +\infty$, then

$$
\lim_{t \uparrow T^*} (\|u(t)\|_\infty + \|v(t)\|_\infty) = +\infty.
$$

Therefore, if there exists a positive constant $C$ such that

$$
\|u(t)\|_\infty + \|v(t)\|_\infty \leq C \quad \forall t \in [0, T^*),
$$

then $T^* = +\infty$.

Since the initial conditions (1.5) and (1.6) are satisfied under assumptions (A1) and (A3), the next lemma says that the classical solution of (1.1)–(1.4) on $[0, T^*) \times \Omega$ remains nonnegative on $[0, T^*) \times \Omega$.

**Lemma 2.1.** Assume (A1), (A3). Then for any initial conditions $u_0$ and $v_0$ satisfying (1.5) and (1.6), the classical solution $(u, v)$ of problem (1.1)–(1.4) on $[0, T^*) \times \Omega$ satisfies

$$
0 \leq u \leq \frac{\lambda}{\mu}, \quad v \geq \frac{c}{d-a}(\frac{\lambda}{\mu} - u).
$$

**Proof.** In system (1.1)–(1.2) the change of variables

$$
w = v - \frac{c}{d-a}(\frac{\lambda}{\mu} - u),
$$

$$
F(u, w) = f(u, w + \frac{c}{d-a}(\frac{\lambda}{\mu} - u))
$$

leads to the system

$$
\frac{\partial u}{\partial t} - a \Delta u = \lambda - F(u, w) - \mu u \quad \text{in } (0, +\infty) \times \Omega,
$$

$$
\frac{\partial w}{\partial t} - d \Delta w = \frac{d-a-c}{d-a}F(u, w) - \mu w \quad \text{in } (0, +\infty) \times \Omega,
$$

with boundary conditions

$$
\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad \text{on } (0, +\infty) \times \partial \Omega,
$$

and initial conditions

$$
u(0, x) = u_0(x), \quad w(0, x) = w_0(x) \quad \text{in } \Omega,
$$

$$
(2.1)
$$

$$
(2.2)
$$

$$
(2.3)
$$

$$
(2.4)
$$
If we assume (A1) and (A3), then a simple application of comparison theorem
[26, Theorem 10.1] to system (2.1)-(2.2) implies that for any initial conditions
$u_0$ and $v_0$ satisfying (1.5) and (1.6), we have
\[ 0 \leq u(t,x) \leq \frac{\lambda}{\mu}, \quad v(t,x) \geq \frac{c}{d-a}(\frac{\lambda}{\mu} - u(t,x)) \quad \forall (t,x) \in [0,T^*) \times \Omega. \]

\[ \square \]

3. Existence of global solutions

It is clear that to prove the existence of global solutions for problem (1.1)-(1.4)
we need to prove it for problem (2.1)-(2.4).

At first, when $a + c \geq d$, the local classical solutions of (1.1)-(1.4) may be
extended as a classical and uniformly bounded solutions on $[0, +\infty) \times \Omega$ for any
nonnegative initial data $u_0$ and $v_0$ satisfying (1.5) and (1.6) without any restrictions
on the growth of the function $f$ under the assumptions (A1) and (A2). Indeed, since
$u, w$ and $f$ are nonnegative, then from (A1) and (A2), we have by the comparison
theorem that
\[ 0 \leq u(t,x) \leq \frac{\lambda}{\mu}, \quad 0 \leq w(t,x) \leq \|w_0\|_{\infty} \quad \forall (t,x) \in [0,T^*) \times \Omega. \]

Now, the main result for the case $a + c < d$ is stated in the following theorem.

**Theorem 3.1.** Under assumptions (A1)-(A3) and restrictions (1.5) and (1.6),
the solutions of problem (1.1)-(1.4) are global and uniformly bounded on $[0, +\infty) \times \Omega$.

Since $0 \leq u \leq \frac{\lambda}{\mu}$, then the problem of global existence reduces to establish
the uniform boundedness of $w$ on $[0,T^*)$. By $L^p$-regularity theory for parabolic
operator (see, e.g., [19, 25]) it follows that it is sufficient to derive a uniform estimate
of $\|\rho F(u,w) - \mu w\|_p$ on $[0,T^*)$ for some $p > \frac{n}{2}$ where $\rho = \frac{d-a-c}{d-a}$. The proof of
Theorem 3.1 is based on the following key proposition.

**Proposition 3.1.** Suppose (A1)-(A3), (1.5) and (1.6). For every classical solution
$(u,w)$ of (2.1)-(2.4) on $[0, T^*) \times \Omega$, consider the function
\[ L(t) = \int_\Omega \left[ \delta u + (M - u)^{-\gamma}(w + 1)^{\beta p e^{\alpha p w}} \right](t,x)dx, \]
where $\alpha, \beta, \gamma, \delta, p$ and $M$ are positive constants such that
\[ \frac{\lambda}{\mu} < M < \frac{2\gamma}{\alpha n}, \quad \gamma < \tau = \frac{4ad}{(a-d)^2}, \quad \beta = \max\{r, \frac{\tau(1+\gamma)}{p(\tau-\gamma)}\}. \quad (3.1) \]

Then, there exists $\delta > 0$, $\sigma > 0$ and $p > \frac{n}{2}$ such that
\[ \frac{d}{dt} L(t) \leq -\mu L(t) + \sigma \quad \forall t \in [0,T^*). \quad (3.2) \]

Before proving this proposition we need the following lemmas.

**Lemma 3.1.** Let $(u,w)$ be a solution of (2.1)-(2.4) on $[0, T^*) \times \Omega$, then under
assumption (A3), we have
\[ \int_\Omega F(u(t,x),w(t,x))dx \leq \lambda|\Omega| - \frac{d}{dt} \int_\Omega u(t,x)dx. \quad (3.3) \]

**Proof.** Since $u$ is a nonnegative function, it suffices to integrate both sides of (2.1)
on $\Omega$, to obtain (3.3), which competes the proof. \[ \square \]
Lemma 3.2. Let $\phi$ and $\psi$ be two nonnegative continuous functions on $[0, +\infty)$ with $\phi(\eta)$ goes to $+\infty$ as $\eta \to +\infty$. Then there exists a positive constant $A$ such that

$$
(1 - \phi(\eta))\psi(\eta) \leq A \quad \forall \eta \geq 0.
$$

(3.4)

Proof. Since $\phi(\eta)$ goes to $+\infty$ as $\eta \to +\infty$, there exists $\eta_0 > 0$ such that for all $\eta > \eta_0$, we obtain

$$(1 - \phi(\eta))\psi(\eta) \leq 0.
$$

On the other hand, if $\eta$ is in the compact interval $[0, \eta_0]$, then the continuous function $\eta \mapsto (1 - \phi(\eta))\psi(\eta)$ is bounded. So that (3.4) immediately follows. □

Proof of Proposition 3.1. Differentiating $L$ with respect to $t$, one obtains

$$
\frac{d}{dt}L(t) = \delta \frac{d}{dt} \int_\Omega u(t,x)dx + I + J,
$$

(3.5)

where

$$
I = \int_\Omega \left( a\gamma(M - u)^{-\gamma-1}(w + 1)^{\beta p}e^{\alpha pw} \Delta u \\
+ dp(M - u)^{-\gamma}[\alpha(w + 1)^{\beta p} + \beta(w + 1)^{\beta p-1}]e^{\alpha pw} \Delta w \right)dx,
$$

and

$$
J = \int_\Omega \mu \left( \gamma(M - u)^{-1}\left( \frac{\Delta}{\mu} - u \right) - p[\alpha + \beta(w + 1)^{-1}]w \right) \\
\times (M - u)^{-\gamma}(w + 1)^{\beta p}e^{\alpha pw} dx \\
+ \int_\Omega \left( \mu p(M - u)[\alpha + \beta(w + 1)^{-1}] - \gamma \right) \\
\times F(u,w)(M - u)^{-\gamma-1}(w + 1)^{\beta p}e^{\alpha pw} dx.
$$

Using Green’s formula in the first integral and taking into account (2.3), we obtain

$$
I \leq \int_\Omega Q(\nabla u, \nabla w)(M - u)^{-\gamma-2}(w + 1)^{\beta p}e^{\alpha pw} dx,
$$

where

$$
Q(\nabla u, \nabla w) = a\gamma(1 + \gamma)|\nabla u|^2 + \gamma p(a + b)(M - u)[\alpha + \beta(w + 1)^{-1}]\nabla u \nabla w \\
+ dp(M - u)^{2}[\alpha^2 p + 2\alpha \beta p(w + 1)^{-1} + \beta(p - 1)(w + 1)^{-2}] |\nabla w|^2,
$$

is a quadratic form with respect to $\nabla u$ and $\nabla w$.

The discriminant of $Q$ is given by

$$
D = \gamma p(d - a)^2(M - u)^2 \left( \beta[\beta p(\gamma - \tau) + \tau(1 + \gamma)](w + 1)^{-2} \\
+ \alpha^2 p(\gamma - \tau)[\alpha + 2\beta(w + 1)^{-1}] \right).
$$

From conditions (3.1), we have $D \leq 0$, then we obtain $Q(\nabla u, \nabla v) \geq 0$ and consequently

$$
I \leq 0.
$$

(3.6)
Concerning the term $J$, since $0 \leq u \leq \frac{\lambda}{\mu} < M$, we observe that

$$J \leq \int_{\Omega} \mu (\gamma - p[\alpha + \beta(w + 1)^{-1}]w)(M - u)^{-\gamma}(w + 1)^{\beta} e^{\alpha w} dx$$

$$+(M - \frac{\lambda}{\mu})^{-\gamma-1} \int_{\Omega} (pM[\alpha + \beta(w + 1)^{-1}] - \gamma) F(u, w)(w + 1)^{\beta} e^{\alpha w} dx, $$

or

$$J \leq -\mu L(t) + \lambda \delta |\Omega|$$

$$+ \mu (M - \frac{\lambda}{\mu})^{-\gamma} \int_{\Omega} (\gamma + 1 - p[\alpha + \beta(w + 1)^{-1}]w)(w + 1)^{\beta} e^{\alpha w} dx$$

$$+(M - \frac{\lambda}{\mu})^{-\gamma-1} \int_{\Omega} |\beta pM - (\gamma - \alpha pM)(w + 1)| F(u, w)(w + 1)^{\beta-1} e^{\alpha w} dx.$$ From (3.1), we obtain $\frac{n}{2} < \gamma$ and $\frac{n}{2} < \gamma$. Then we can choose $p$ such that $\frac{n}{2} < p < \frac{n}{\alpha M}$. Using Lemma 3.2, we get $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$J \leq -\mu L(t) + (2 \lambda \delta + \mu \delta_1 (M - \frac{\lambda}{\mu})^{-\gamma}) |\Omega| + \delta_2 (M - \frac{\lambda}{\mu})^{-\gamma-1} \int_{\Omega} F(u, w) dx.$$ Let $\delta = \delta_2 (M - \frac{\lambda}{\mu})^{-\gamma-1}$ and using Lemma 3.1 we obtain

$$J \leq -\mu L(t) + (2 \lambda \delta + \mu \delta_1 (M - \frac{\lambda}{\mu})^{-\gamma}) |\Omega| - \delta \frac{d}{dt} \int_{\Omega} u(t, x) dx. \quad (3.7)$$

From (3.6) and (3.7), we conclude that

$$\frac{d}{dt} L(t) \leq -\mu L(t) + \sigma,$$

where $\sigma = (2 \lambda \delta + \mu \delta_1 (M - \frac{\lambda}{\mu})^{-\gamma}) |\Omega|$. This concludes the proof. \qed

Proof of Theorem 3.1. Let $p$ be the same as in Proposition 3.1. Since $M^{-\gamma} \leq (M - u)^{-\gamma}$, from (3.1) and (A3) it follows that

$$\|w\|^p_p = \int_{\Omega} |w|^p dx \leq M^\gamma L(t),$$

$$\|F(u, w)\|^p_p = \int_{\Omega} |F(u, w)|^p dx \leq M^\gamma K^p L(t),$$

where

$$K = \max_{0 \leq \xi \leq \frac{\mu}{\lambda}} \varphi(\xi).$$

By (3.2), it is seen that there exists a positive constant $B$ such that

$$L(t) \leq B \quad \forall t \in [0, T^*),$$

and consequently

$$\|w\|^p_p \leq BM^\gamma,$$

$$\|F(u, w)\|^p_p \leq BM^\gamma K^p.$$ Hence $\rho F(u(t, .), w(t, .)) - \mu w(t, .)$ is uniformly bounded in $L^p(\Omega)$ for all $t \in [0, T^*)$ with $p > \frac{2}{n}$. Using the regularity results for solutions of parabolic equations in [19, 20], we conclude that the solutions of the problem (1.1)–(1.4) are uniformly bounded on $[0, +\infty) \times \Omega.$ \qed
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