Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 07, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

UNIQUE CONTINUATION FOR SOLUTIONS OF $p(x)$-LAPLACIAN EQUATIONS

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Abstract. We study the unique continuation property for solutions to the quasilinear elliptic equation

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + V(x)|u|^{p(x)-2}u = 0 \quad \text{in } \Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $1 < p(x) < N$ for x in Ω.

1. Introduction and preliminary results

In the recent years increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics and calculus of variations. For information on modelling physical phenomena by equations involving $p(x)$-growth condition we refer to [1, 36, 41]. The understanding of such physical models has been facilitated by the development of variable Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real-valued function. Variable exponent Lebesgue spaces appeared for the first time in literature as early as 1931 in an article by Orlicz [32]. The spaces $L^{p(x)}$ are special cases of Orlicz spaces L^{φ} originated by Nakano [31] and developed by Musielak and Orlicz [29, 30], where $f \in L^{\varphi}$ if and only if $\int \varphi(x, |f(x)|)dx < \infty$ for a suitable φ. Variable exponent Lebesque spaces on the real line have been independently developed by Russian researchers. In that context we refer to the studies of Tsenov [40], Sharapudinov [38] and Zhikov [44, 45].

This article is motivated by the phenomena that can be modelled with the equation

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary and $1 < p(x)$, $p(x) \in C(\overline{\Omega})$. Our goal to show strong unique continuation nontrivial for weak solutions for (1.1) in the generalized Sobolev space $W^{1,p(x)}(\Omega)$ for some particular nonlinearities of the type $f(x,u)$. Problems of type (1.1) have been intensively studied in the past decades. We refer to [2, 11, 12, 24, 25, 26, 27, 34, 35, 43], for some interesting results. We point out the presence in (1.1) of the $p(x)$-Laplace

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2000 *Mathematics Subject Classification.* 35D05, 35J60, 58E05.
Key words and phrases. $p(x)$-Laplace operator; unique continuation.

Submitted September 8, 2011. Published January 12, 2012.

operator. This is a natural extension of the p-Laplace operator, with p a positive constant. However, such generalizations are not trivial since the $p(x)$-Laplace operator possesses a more complicated structure than p-Laplace operator, for example it is inhomogeneous.

We recall some definitions and properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. Roughly speaking, anisotropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For $p \in C_+(\overline{\Omega})$, we introduce *the variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \big\{u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\big\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\big\{\mu > 0;\ \int_\Omega \big|\frac{u(x)}{\mu}\big|^{p(x)}\,dx \le 1\big\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [22]. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous, [22, Theorem 2.8].

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise that is, $1/p(x) + 1/p'(x) = 1$, [22, Corollary 2.7]. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality

$$\big|\int_\Omega uv\,dx\big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{1.2}$$

is valid.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold

$$|u|_{p(\cdot)} < 1\ (= 1;\ > 1) \ \Leftrightarrow\ \rho_{p(\cdot)}(u) < 1\ (= 1;\ > 1) \tag{1.3}$$

$$|u|_{p(\cdot)} > 1 \ \Rightarrow\ |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+} \tag{1.4}$$

$$|u|_{p(\cdot)} < 1 \ \Rightarrow\ |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-} \tag{1.5}$$

$$|u_n - u|_{p(\cdot)} \to 0 \ \Leftrightarrow\ \rho_{p(\cdot)}(u_n - u) \to 0, \tag{1.6}$$

since $p^+ < \infty$. For a proof of these facts see [22]. Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [8].

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = Np(x)/(N-p(x))$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$ [22, Theorem 3.9 and 3.3] (see also [10, Theorem 1.3 and 1.1]).

The bounded variable exponent p is said to be Log-Hölder continuous if there is a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}$$

for all $x, y \in \mathbb{R}^N$, such that $|x-y| \leq 1/2$. A bounded exponent p is Log-Hölder continuous in Ω if and only if there exists a constant $C > 0$ such that

$$|B|^{p_B^- - p_B^+} \leq C$$

for every ball $B \subset \Omega$ [7, Lemma 4.1.6, page 101]. As a result of the condition Log-Hölder continuous we have

$$r^{-(p_B^+ - p_B^-)} \leq C, \tag{1.7}$$

$$C^{-1} r^{-p(y)} \leq r^{p(x)} \leq C r^{-p(y)} \tag{1.8}$$

for all $x, y \in \ B := B(x_0, r) \subset \Omega$ and the constant C depends only on the constant Log-Hölder continuous. Under the Log-Hölder condition smooth function are dense in variable exponent Sobolev space [7, Proposition 11.2.3, page 346].

Concerning to the Unique Continuation in his paper on Schrödinger semigroup [39], B.Simon formulated the following conjecture:

> Let Ω be a bounded subset $\mathbb{R}^N$ and V a function defined in Ω whose extension with values outside Ω belong to the Stummel-Kato $\mathrm{S}(\mathbb{R}^N)$. Then the Schrödinger operator $H := -\Delta + V$ has the unique continuation property.

That is, $u \in H^1(\Omega)$ is a solutions of equations $Hu = 0$ which vanishes of infinite order (For definitions see section 3.) at one point $x_0 \in \Omega$, then u must be identically zero in Ω. A positive answer to Simon 's conjeture was given by Fabes,Garofalo and Lin for radial potential V.

At the same time Chanilo and Sawyer in [5] proved the unique continuation property for solutions of the inequality $|\Delta u| \leq |V||u|$, assuming V in the Morrey spaces $L^{r,N-2r}(\mathbb{R}^N)$ for $r > \frac{N-1}{2}$. Jarison and Kening proved the continuation unique for Schrödinger operator [20].The same work is done Gossez and Figueiredo, but for linear elliptic operator in the case $V \in L^{\frac{N}{2}}(\Omega)$, $N > 2$, [14]. Also, Loulit extended this property to $N = 2$ by introducing Orlicz's space [23]. In this paper we extended to Variable Exponent Space a result of Zamboni [42] to the solution of a quasilinear elliptic equation

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + V(x)|u|^{p(x)-2}u = 0 \quad \text{in } \Omega, \tag{1.9}$$

where $1 < p(x) < N$, $V \in L^{\frac{N}{p(x)}}(\Omega)$.

2. Fefferman's type inequality

For every $u \in W_0^{1,p(\cdot)}(\Omega)$ the norm Poincaré inequality

$$|u|_{L^{p(\cdot)}(\Omega)} \leq c\,\mathrm{diam}(\Omega)|\nabla u|_{L^{p(\cdot)}}$$

$c = C(N, \Omega, c\,log(p))$ holds (we refer to [19] for notation and proofs). Nevertheless, the modular inequality

$$\int_\Omega |u|^{p(x)} dx \leq C \int_\Omega |\nabla u|^{p(x)} dx, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega) \tag{2.1}$$

not always holds (see [12, Thm. 3.1]). It is known that (2.1) holds if, for instance: i) $N > 1$, and the function $f(t) := p(x_o + tw)$ is monotone [12, Thm.3.4] with $x_o + tw$ with an appropriate setting in Ω; ii) if there exists a function $\xi \geq 0$ such that $\nabla p \cdot \nabla \xi \geq 0$, $\|\nabla \xi\| \neq 0$ [3, Thm. 1]; iii) If there exists $a : \Omega \to \mathbb{R}^N$ bounded such that $div\, a(x) \geq a_0 > 0$ for all $x \in \bar{\Omega}$ and $a(x) \cdot \nabla p(x) = 0$ for all $x \in \Omega$, [28, Thm. 1]. To the best of our knowledge necessary and sufficient conditions in order to ensure that

$$\inf_{u \in W^{1,p(\cdot)}(\Omega)/\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}}{\int_\Omega |u|^{p(x)}} > 0$$

has not been obtained yet, except in the case $N = 1$, [12, Thm. 3.2]. The following definition is in order.

Definition 2.1. We say that $p(\cdot)$ belongs to the Modular Poincaré Inequality Class, $MPIC(\Omega)$, if there exists necessary conditions to ensure that

$$\int_\Omega |u|^{p(x)} \leq C \int_\Omega |\nabla u|^{p(x)}, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega)$$

$C = C(N, \Omega, c_{log}(p)) > 0$ holds.

Fefferman [13] proved the inequality

$$\int_{\mathbb{R}^N} |u(x)|^p |f(x)|\, dx \leq C \int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{2.2}$$

in the case $p = 2$, assuming f in the Morrey's space $L^{r,N-2r}(\mathbb{R}^N)$, with $1 < r \leq \frac{N}{2}$. Later in [37] Schechter showed the same result taking f in the Stummel-Kato class $S(\mathbb{R}^N)$. Chiarenza and Frasca [6] generalized Fefferman's result proving (2.2) under the assumption $f \in L^{r,N-pr}(\mathbb{R}^N)$, with $1 < r < \frac{N}{p}$ and $1 < p < N$. Zamboni [42] generalized Schecter's result proving (2.2) under the assumption $f \in \tilde{M}_p(\mathbb{R}^N)$, with $1 < p < N$. We stress out that is not possible to compare the assumptions $f \in L^{r,N-pr}(\mathbb{R}^N)$ the Morrey class and $f \in S(\mathbb{R}^N)$, the Stumel-Kato class. The theory for a variable exponent spaces is a growing area but Modular Fefferman type inequalities are more scarce than Poincaré inequalities in variable exponent setting. In the following theorem we provide a basic Fefferman's type result, for variable exponent spaces.

Theorem 2.2. *Let p be a Log-Hölder continuous exponent with $1 < p(x) < N$, and $p \in MPIC(\Omega)$. Let $V \in L^1_{\mathrm{loc}}(\Omega)$ with $0 < \varepsilon < V(x)$ a.e.. Then there exist a positive constant $C = C(N, \Omega, c_{log}(p))$ such that*

$$\int_\Omega V(x)|u(x)|^{p(x)}\, dx \leq C \int_\Omega |\nabla u(x)|^{p(x)}\, dx$$

for any $u \in W_0^{1,p(x)}(\Omega)$.

Proof. Let $u \in W_0^{1,p(x)}(\Omega)$ supported in $B(x_0,r)$. Given that $V \in L^1_{\rm loc}(\Omega)$ the function

$$w(x) := \Big(\int_{x_1^0}^{x_1} V(\xi_1, x_2, \dots, x_n) d\xi_1, \dots, \int_{x_N^0}^{x_N} V(x_1, \dots, x_{N-1}, \xi_N) d\xi_N \Big),$$

where $x_0 = (x_1^0, \dots, x_N^0)$ and $x = (x_1, \dots, x_N) \in B(x_0, r)$, is well defined. Notice that $\int_{x_i^0}^{x_i} V(x_1, \dots, \xi_i, \dots, x_n) d\xi_i \in \mathcal{C}[x_i^0, x_i]$ for $i = 1, \dots, N$ [4, Lemme VIII.2]. So that $\operatorname{div} w(x) = NV(x)$. Moreover

$$|V(x)|_{L^1(B(x_0,r))} \geq \int_{x_1^0}^{x_1} \cdots \int_{x_N^0}^{x_N} V(\xi)\, d\xi_n \cdots d\xi_1$$

where $\xi = (\xi_1, \dots, \xi_N)$. Therefore, $|w(x)| \leq \sqrt{N} |V(x)|_{L^1(B(x_0,r))}$.

A direct calculation leads to

$$\begin{aligned}\operatorname{div}(|u|^{p(x)} w(x)) &= |u(x)|^{p(x)} \operatorname{div}\, w(x) + p(x) |u|^{p(x)-2} u \nabla u \cdot w(x) \\ &\quad + |u|^{p(x)} \log u \nabla p(x) \cdot w(x).\end{aligned}$$

Now the Divergence Theorem implies $\int_{B(x_0,r)} \operatorname{div}\,(|u|^{p(x)} w(x)) = 0$, and so

$$\begin{aligned}\int_{B(x_0,r)} |u(x)|^{p(x)} \operatorname{div} w(x) dx &\leq p^+ \int_{B(x_0,r)} |u(x)|^{p(x)-1} |\nabla u(x)| |w(x)| dx \\ &\quad + \int_{B(x_0,r)} |u(x)|^{p(x)} \log |u(x)| |\nabla p(x)| |w(x)| dx.\end{aligned}$$

Set

$$I_1 := p^+ \int_{B(x_0,r)} |u(x)|^{p(x)-1} |\nabla u(x)| |w(x)| dx$$

and

$$I_2 := \int_{B(x_0,r)} |u(x)|^{p(x)} \log |u(x)| |\nabla p(x)| |w(x)| dx.$$

Now we estimate I_2 by distinguishing the case when $|u(x)| \leq 1$ and $|u(x)| > 1$. Notice that the relations

$$\sup_{0 \leq t \leq 1} t^\eta |\log t| < \infty, \tag{2.3}$$

$$\sup_{t > 1} t^{-\eta} \log t < \infty \tag{2.4}$$

hold for $\eta > 0$. Let $\Omega_1 =: \{x \in B_r : |u(x)| \leq 1\}$ and $\Omega_2 =: \{x \in B_r : |u(x)| > 1\}$, then by (2.3) and (2.4) we have

$$I_2 \leq C_1 \int_{\Omega_1} |w(x)| |u(x)|^{p(x) - \eta_1} dx + C_2 \int_{\Omega_2} |w(x)| |u(x)|^{p(x) + \eta_2} dx.$$

We can choose $k \in \mathbb{N}$ such that $p(x) - 1/k \geq p^-$. Since $u \in L^{p^-}(B(x_0,r))$ and in Ω_1, $|u(x)| \leq 1$ we have

$$|u(x)|^{p(x) - 1/n} \leq |u(x)|^{p^-},$$

for $n > k$. The Lebesgue Dominated Convergence Theorem implies

$$\lim_{n \to \infty} \int_{\Omega_1} |u(x)|^{p(x) - 1/n} dx = \int_{\Omega_1} |u(x)|^{p(x)} dx.$$

For Ω_2 we can choose k' such that $p(x) + 1/k' \leq (p(x))^* = Np(x)/(N - p(x))$. So

$$|u(x)|^{p(x)+1/n} \leq |u(x)|^{(p(x))^*},$$

for $n > k'$, and $x \in \Omega_2$. Since $u \in L^{(p(x))^*}(B(x_0, r))$ [7, Thm. 8.3.1] we may use the Lebesgue Theorem again to obtain

$$\lim_{n\to\infty} \int_{\Omega_2} |u(x)|^{p(x)+1/n} dx = \int_{\Omega_2} |u(x)|^{p(x)} dx.$$

Given that $p \in MPI(\Omega)$, we have

$$I_2 \leq C \int_{B(x_0,r)} |u|^{p(x)} dx \leq C \int_{B(x_0,r)} |\nabla u|^{p(x)} dx.$$

Now we estimate I_1 by using the modular Young's inequality [19, Theorem 3.2.21],

$$I_1 \leq p^+ C_1 \int_{B(x_0,r)} |w(x)|^{p(x)/(p(x)-1)} |u(x)|^{p(x)} + p^+ C_2 \int_{B(x_0,r)} |\nabla u(x)|^{p(x)}.$$

Again, since $p \in MPI(\Omega)$ we obtain

$$I_1 \leq C \int_{B(x_0,r)} |\nabla u|^{p(x)} dx.$$

Finally, recalling that div $w(x) = NV(x)$ we obtain

$$N \int_{B(x_o,r)} V(x)|u(x)|^{p(x)} \leq C \int_{B(x_0,r)} |\nabla u(x)|^{p(x)} dx,$$

which leads to the claim of the theorem. $\square$

3. Unique Continuation

Consider the equation

$$Hu := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + V(x)|u|^{p(x)-2} u = 0, \quad x \in \Omega, \tag{3.1}$$

$u \in W^{1.p(x)}_{\rm loc}(\Omega)$, $1 < p(x) < N$, $V \in L^{\frac{N}{p(x)}}(\Omega)$. A weak solution of (3.1) is a function $u \in W^{1.p(x)}_{\rm loc}(\Omega)$ such that

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} V(x)|u|^{p(x)-2} u \cdot \varphi dx = 0, \tag{3.2}$$

for all $\varphi \in W^{1,p(x)}_0(\Omega)$.

Note that $L^{\frac{N}{p(x)}}(\Omega)$ implies $V \in L^1(\Omega)$ by [19, Theorem 3.3.1]. The main interest of this section is to prove some unique continuation results for solution of (3.1) according to the following definitions.

Definition 3.1. A function $u \in L^{p(x)}_{\rm loc}(\Omega)$ vanishes of infinite order in the $p(x)$-mean at a point $x_0 \in \Omega$ if , for each $k \in \mathbb{N}$

$$\lim_{R\to 0} \frac{1}{R^k} \int_{|x-x_0|<R} |u|^{p(x)} dx = 0. \tag{3.3}$$

Definition 3.2. The operator H has the unique continuation property in Ω if the only solution to $Hu = 0$ such that u vanishes of infinity order in the $p(x)$-mean at a point $x_0 \in \Omega$ is u must be identically zero in Ω.

Lemma 3.3 ([42]). *Assume $w \in L^1loc\Omega$, $w \geq 0$ almost everywhere in Ω, $w \not\equiv 0$. If there exists C such that*

$$\int_{B(x_0,2r)} w(x)\,dx \leq C \int_{B(x_0,r)} w(x)\,dx, \quad \forall r > 0$$

Then $w(x)$ has no zero of infinity order in Ω.

Recall that $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is a bounded open set. We want to prove estimates independent of p^+ for bounded solutions. For this purpose we assume throughout this section that $1 < p^- \leq p^+ < \infty$ and p is Lipschitz continuous. In particular, p is Log-Hölder continuous. The new feature in the estimate is the choice of a test function which include the variable exponent. This has both advantages and disadvantages: we need to assume that p is differentiable almost everywhere, but, on the other hand, we avoid terms involving p^+, which would be impossible to control later, see[19].

In this section we prove the unique continuation property for the operator Hu, defined in 3.1 extending in some sense the results obtained by Zamboni [42] to variable exponent spaces. To prove this property we need the following Lemma.

Lemma 3.4. *(Caccioppoli estimate) Let $p : \Omega \to (1, N)$ be an exponent with $1 < p^- \leq p^+ < \infty$ and such that $p \in MPI(\Omega)$ is Lipschitz continuous. Let u be a non negative solution of (3.1) in Ω and $\eta : \Omega \to [0,1]$ be a Lipschitz function with compact support in Ω satisfying $\eta \log \frac{1}{\eta} \leq a|\nabla \eta|$ a.e. in $\{\eta > 0\}$ for some constant $a > 0$. Then*

$$\int_\Omega |\nabla \log u|^{p(x)} \eta^{p(x)}\,dx \leq C \int_\Omega |\eta|^{p(x)}\,dx$$

for non-negative Lipschitz function $\eta \in C_0^\infty$.

Proof. Let $x_0 \in \Omega$, Let $B(x_0,h)$ be a ball such that $B(x_0,2h)$ is contained in Ω. Consider any ball $B(x_0,r)$ with $r < h$. Let $\eta \in C_0^\infty$ with compact support in $B(x_0,2r)$ such that $\eta \log \frac{1}{\eta} \leq a|\nabla \eta|$ a.e. in $\{x \in B_{2r} : \eta > 0\}$ for some constant $a > 0$, and $\eta = 1$ in B_r and $|\nabla \eta| \leq \frac{C}{r}$. Then using

$$\varphi(x) = |u(x)|^{1-p(x)} \eta^{p(x)}$$

as test function in (3.2) we obtain

$$\begin{aligned}
0 = &\int_{B_{2r}} (1 - p(x)) \eta^{p(x)} |\nabla u|^{p(x)} |u|^{-p(x)}\,dx \\
&- \int_{B_{2r}} \eta^{p(x)} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla p(x) |u|^{1-p(x)} \log u \\
&+ \int_{B_{2r}} p(x) \eta^{p(x)-1} \nabla u \cdot \nabla \eta |\nabla u|^{p(x)-2} |u|^{1-p(x)}\,dx \\
&+ \int_{B_{2r}} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla p(x) |u|^{1-p(x)} \eta^{p(x)} \log \eta\,dx \\
&+ \int_{B_{2r}} V |u|^{p(x)-2} u \eta^{p(x)} |u|^{1-p(x)}\,dx;
\end{aligned}$$

therefore,

$$(p^- - 1) \int_{B_{2r}} \eta^{p(x)} |\nabla \log u|^{p(x)}\,dx \leq |I_1| + |I_2| + |I_3| + |I_4|,$$

where

$$
\begin{aligned}
I_1 &:= -\int_{B_{2r}} \eta^{p(x)}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla p(x)|u|^{1-p(x)}\log u\,dx,\\
I_2 &:= \int_{B_{2r}} p(x)\eta^{p(x)-1}\nabla u\nabla\eta|\nabla u|^{p(x)-2}|u|^{1-p(x)}\,dx,\\
I_3 &:= \int_{B_{2r}} |\nabla u|^{p(x)-2}\nabla u\nabla p(x)|u|^{1-p(x)}\eta^{p(x)}\log\eta\,dx,\\
I_4 &:= \int_{B_{2r}} V|u|^{p(x)-2}u\eta^{p(x)}|u|^{1-p(x)}\,dx\,.
\end{aligned}
$$

Now we estimate I_1, I_2, I_3 and I_4. We have

$$
\begin{aligned}
|I_1| &\le \int_{B_{2r}} \eta^{p(x)}|\nabla p(x)||\nabla u|^{p(x)-1}|u|^{1-p(x)}\log u\,dx\\
&\le \int_{B_{2r}} \eta^{p(x)}|\nabla p(x)||\nabla u|^{p(x)-1}|u|^{1-p(x)}|u|^{\pm\eta}\,dx\,,
\end{aligned}
$$

where $\eta > 0$ and

$$
\pm\eta = \begin{cases} -\eta, & \text{if } |u|\le 1,\\ \eta, & \text{if } |u|>1. \end{cases}
$$

Using the Lebesgue Dominated Convergence Theorem as in the proof of Theorem 2.2 and Young's inequality we obtain

$$
\begin{aligned}
I_1 &\le \int_{B_{2r}} \eta^{p(x)}|\nabla p(x)||\nabla u|^{p(x)-1}|u|^{1-p(x)}\,dx\\
&\le \varepsilon C_p\int_{B_{2r}} \eta^{p(x)}|\nabla\log u|^{p(x)}dx + \varepsilon C_p\int_{B_{2r}} \big(\frac{1}{\varepsilon}\big)^{p(x)-1}\eta^{p(x)}dx\,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|I_2| &\le p^+\Big|\int_{B_{2r}} \eta^{p(x)-1}\nabla u\cdot\nabla\eta|\nabla u|^{p(x)-2}|u|^{1-p(x)}\,dx\Big|\\
&\le p^+\int_{B_{2r}} \eta^{p(x)-1}|\nabla u||\nabla\eta||\nabla u|^{p(x)-2}|u|^{1-p(x)}\,dx\\
&= p^+\int_{B_{2r}} \eta^{p(x)-1}|\nabla\eta||\nabla u|^{p(x)-1}|u|^{1-p(x)}\,dx\\
&= p^+\int_{B_{2r}} |\nabla\eta|\eta^{p(x)-1}|\nabla\log u|^{p(x)-1}\,dx\\
&\le p^+\int_{B_{2r}} \Big(\frac{1}{\varepsilon}\Big)^{p(x)-1}|\nabla\eta|^{p(x)}dx + p^+\varepsilon\int_{B_{2r}} |\eta|^{p(x)}|\nabla\log u|^{p(x)}\,dx\,.
\end{aligned}
$$

For I_3 we have

$$
\begin{aligned}
|I_3| &= \Big|\int_{B_{2r}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla p(x)|u|^{1-p(x)}\eta^{p(x)}|\log\eta|\,dx\Big|\\
&\le \int_{B_{2r}} |\nabla u|^{p(x)-2}|\nabla u||\nabla p(x)||u|^{1-p(x)}\eta^{p(x)}|\log\eta|\,dx\\
&\le L\int_{B_{2r}} |\nabla u|^{p(x)-1}|u|^{1-p(x)}\eta^{p(x)-1}\eta|\log\eta|\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= L\int_{B_{2r}} \eta^{p(x)-1}|\nabla \log u|^{p(x)-1}\eta \log\frac{1}{\eta}\,dx\\
&\le aL\int_{B_{2r}} |\nabla\eta|\eta^{p(x)-1}|\nabla \log u|^{p(x)-1}\,dx\\
&\le aL\int_{B_{2r}} \Big(\frac{1}{\varepsilon}\Big)^{p(x)-1}|\nabla\eta|^{p(x)}dx + aL\varepsilon\int_{B_{2r}}|\eta|^{p(x)}|\nabla \log u|^{p(x)}\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &\le \int_{B_{2r}} V|u|^{p(x)-2}u\eta^{p(x)}|u|^{1-p(x)}\,dx\\
&\le \int_{B_{2r}} V|u|^{p(x)-2}|u|\eta^{p(x)}|u|^{1-p(x)}\,dx\\
&= \int_{B_{2r}} V\eta^{p(x)}\,dx;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
&(p^- -1)\int_{B_{2r}}\eta^{p(x)}|\nabla \log u|^{p(x)}\,dx\\
&\le (p^+ + aL)\varepsilon\int_{B_{2r}}\eta^{p(x)}|\nabla \log u|^{p(x)}\,dx + \int_{B_{2r}} V\eta^{p(x)}\,dx\\
&\quad + (p^+ + aL)\int_{B_{2r}}\big(\frac{1}{\varepsilon}\big)^{p(x)-1}|\nabla\eta|^{p(x)}\,dx
\end{aligned}
$$

Let $0<\epsilon\le 1$ such that $\epsilon<\min\big\{1,\frac{p^- -1}{2(p^+ + aL)}\big\}$. Since $(\frac{1}{\varepsilon})^{p(x)-1}\le\frac{1}{\varepsilon})^{p^+-1}$, we obtain

$$
\int_{B_{2r}}\eta^{p(x)}|\nabla \log u|^{p(x)}\,dx \le C\int_{B_{2r}}|\nabla\eta|^{p(x)}\,dx + \int_{B_{2r}} V\eta^{p(x)}\,dx
$$

and by Theorem 2.2, we have

$$
\begin{aligned}
\int_{B_{2r}}\eta^{p(x)}|\nabla \log u|^{p(x)}\,dx &\le C\int_{B_{2r}}|\nabla\eta|^{p(x)}\,dx + C\int_{B_{2r}}|\nabla\eta|^{p(x)}\,dx\\
&\le C\left(p^+, a, L, \Omega\right)\int_{B_{2r}}|\nabla\eta|^{p(x)}\,dx\\
&= C\int_{B_{2r}}|\nabla\eta|^{p(x)}\,dx
\end{aligned}
$$

Since $C>0$, this completes the proof. ☐

Theorem 3.5. *Let $p:\Omega\to(1,N)$ be an exponent with $1<p^-\le p^+<\infty$ and such that $p\in MPI(\Omega)$ is Lipschitz continuous. Let $u\in W^{1,p(x)}(\Omega)$, $u\ge 0$, be a solution of* (3.1)*, then u has no zero of infinite order in Ω, for all $V\in L^{\frac{N}{p(x)}}(\Omega)$.*

Proof. Let $\varphi(x)$ as in the proof of Lemma 3.4 then, we have

$$
\int_{B_{2r}}\eta^{p(x)}|\nabla \log u|^{p(x)}\,dx \le C\int_{B_{2r}}|\nabla\eta|^{p(x)}\,dx.
$$

And, since $p(x)$ is Log-Hölder, $r^{-p(x)}\le Cr^{-p(x_0)}$ for all $x_0\in B_{2r}$, by (1.7), we have

$$
\int_{B_{2r}}|\nabla\eta|^{p(x)}\,dx \le \int_{B_{2r}}\big(\frac{C}{r}\big)^{p(x)}\,dx
$$

$$\begin{aligned}
&\le \frac{C}{r^{p(x_0)}}\int_{B_{2r}} dx\\
&\le Cr^{-p(x_0)}|B_{2r}|\\
&\le Cr^{N-p(x_0)};
\end{aligned}$$

therefore,

$$\int_{B_{2r}} \eta^{p(x)}|\nabla \log u|^{p(x)}\,dx \le Cr^{N-p(x_0)}$$

and hence

$$\int_{B_r} |\nabla \log u|^{p(x)}\,dx \le Cr^{N-p(x_0)}$$

since $\eta = 1$ in B_r. Now by the Poincaré inequality [7, Proposition 8.2.8],

$$\int_{B_r} \Big(\frac{|v - v_{B_r}|}{r}\Big)^{p(x)}\,dx \le C\int_{B_r} |\nabla v|^{p(x)}\,dx + C|B_r|$$

for all $v \in W^{1,p(x)}(B_r)$. We apply this to the function $v := \log u$:

$$\begin{aligned}
\fint_{B_r} \Big(\frac{|\log u - (\log u)_{B_r}|}{r}\Big)^{p(x)} &\le C\fint_{B_r} |\nabla \log u|^{p(x)}\,dx + C\\
&\le Cr^{-p(x_0)}
\end{aligned}$$

by Log-Hölder continuity of $p(x)$, we have

$$\frac{1}{r^{p(x_0)}}\fint_{B_r} |\log u - (\log u)_{B_r}|^{p(x)}\,dx \le \fint_{B_r} \Big(\frac{|\log u - (\log u)_{B_r}|}{r}\Big)^{p(x)}\,dx \le Cr^{-p(x_0)};$$

thus

$$\fint_{B_r} |\log u - (\log u)_{B_r}|^{p(x)}\,dx \le Cr^{-p(x_0)}r^{p(x_0)} = C,$$

and since

$$\fint_{B_r} |\log u - (\log u)_{B_r}|\,dx \le \fint_{B_r} |\log u - (\log u)_{B_r}|^{p(x)} + 1\,dx \le C,$$

it follows that $\log u \in BMO(B_r)$ uniformly, see [15]. The measure theoretic John-Nirenberg [21] implies that there exist positive constants α and C depending on the BMO-norm such that

$$\fint_{B_r} e^{\alpha|f - f_{B_r}|}\,dx \le C,$$

where $f := \log u$. Using this we can conclude that

$$\begin{aligned}
\fint_{B_r} e^{\alpha f}\,dx \fint_{B_r} e^{-\alpha f}\,dx &= \fint_{B_r} e^{\alpha(f - f_{B_r})}\,dx \fint_{B_r} e^{-\alpha(f - f_{B_r})}\,dx\\
&\le \Big(\fint_{B_r} e^{\alpha|f - f_{B_r}|}\,dx\Big)^2 \le C
\end{aligned}$$

which implies

$$\int_{B_r} e^{\alpha f}\,dx \int_{B_r} e^{-\alpha f}\,dx \le C|B_r|^2.$$

So

$$\int_{B_r} |u|^{\alpha}\,dx \int_{B_r} |u|^{-\alpha}\,dx \le C|B_r|^2;$$

that is, $|u|^\alpha$ belongs to the Muckenhoupt class A_2 for $\alpha > 0$, see [15]. Now it is well known that A_2 implies the doubling property for $|u|^\alpha$, that is the assumption of Lemma(3.3). So the conclusion follows for $|u|^\alpha$ and hence also for u. $\square$

Acknowledgements. The authors want to thank Peter Hästö for the careful reading of a draft of this article, and for his suggestions. Johnny Cuadro was supported by a CONACYT México's Ph. D. Scholarship.

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ADDENDUM POSTED ON OCTOBER 14, 2012

The authors want to correct the following misprints:
Page 3, line 4: the inclusion is just continuous.
Page 6, Definition 3.1 must say:
Definition 3.1 Assume $w \in L^1_{\rm loc}(\Omega)$, $w \geq 0$ almost everywhere in Ω. We say that w has a zero of infinite order at $x_0 \in \Omega$ if

$$\lim_{\sigma \to 0} \frac{\int_{B(x_0,\sigma)} w(x)\,dx}{|B(x_0,\sigma)|^k} = 0, \quad \forall k > 0.$$

Page 6, Definition 3.2 must say:

Definition 3.2 The operator H has the strong unique continuation property in Ω if the only solution to $Hu = 0$ such that u vanishes of infinity order at a point $x_0 \in \Omega$ is $u \equiv 0$ in Ω.
Page 7, in Lemma 3.3 must say: $w \in L^1_{\rm loc}(\Omega)$.
Page 7, in Lemma 3.4: The constant C is missing.
Page 9, Theorem 3.5 should include: "$w \not\equiv 0$ a.e."
Page 9, In Theorem 3.5: The constant C is missing.

End of addendum.

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