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EXISTENCE OF SCALE INVARIANT SOLUTIONS TO HORIZONTAL FLOW WITH A FUJITA TYPE DIFFUSION COEFFICIENT

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Abstract. In this article, we study a boundary-initial value problem on the half-line for the diffusion equation with a Fujita type diffusion coefficient that carries a parameter α. The equation models the flow of water in soil within an approximation where gravitational effects are not taken into account and, when $\alpha = 1$, an explicit self-similar solution $\psi(x/\sqrt{t})$ can be found. We prove that if $\alpha > 1$ then the above problem, with uniform boundary conditions, posses self-similar solutions. This is the first step towards a multiscale (renormalization group) asymptotic analysis of solutions to more general equations than the ones studied here.

1. Introduction

The aim of this note is to study the following boundary-initial value problem (BIVP) for the unknown $\theta = \theta(x, t)$

$$
\begin{gathered}
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\Big(\Big[\frac{C\,(\theta-\theta_r)^{\alpha-1}}{(\theta_s-\theta)^2}\Big]\frac{\partial \theta}{\partial x}\Big), \quad 0 < x < \infty,\ t > 0, \\
\theta(0,t) = \theta_0,\ \forall t \ge 0; \quad \theta(x,0) = \theta_i,\ \theta_r < \theta_0,\ \theta_i < \theta_s,
\end{gathered}
\tag{1.1}
$$

with $C > 0$. When $\alpha = 1$, the above problem describes the flow of water in soil within an approximation where gravitational effects are not taken into account, which is the case, for instance, of horizontal flow or vertical flow at early times, see [1]. Here, we consider α as a parameter satisfying $\alpha \ge 1$. Sticking to the hydrology's nomenclature, $\theta(x, t)$ is the soil's *water content* at height x, measured from the soil's surface downwards, and at time t. θ_r and θ_s are the *residual* and the *saturated* values of the soil's water content, respectively, and we assume that $0 < \theta_r < \theta_s < 1$. The quantity between square brackets in (1.1) will be denoted by $D = D(\theta)$ and it is called the *hydraulic diffusion coefficient*. Observe that $D(\theta_r) = 0$ if $\alpha > 1$ and that $D(\theta)$ is convex around θ_r if $\alpha > 2$.

The above BIVP is a natural extension of the $\alpha = 1$ case, which is within Fujita's class defined in [1], and it has been studied by several authors [2, 3, 4]. Assuming that the soil is uniformly wet at the begining (θ_i is assumed to be constant) and

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that the water content at the soil's surface is kept constant and equal to θ_0 at later times, the $\alpha = 1$ case can be explicitly solved under the additional condition $\theta(x,t) \equiv \varphi(x/\sqrt{t})$. Solutions of this form are said to be *self-similar* or *scale invariant* because $\theta(x,t) = \theta(Lx, L^2 t)$ for any $L > 0$. Existence and uniqueness of self-similar solutions is an important issue within the context of asymptotic analysis as $t \to \infty$ of solutions to partial differential equations, see [5]. Nonlinearities could be added to the right hand side of equation (1.1) and, if so, one would like to know under which conditions they will be "irrelevant", "marginal" or "relevant" in the Renormalization Group (RG) sense, see [6, 7]. The RG method is based upon a multiscale analysis that provides the right asymptotic behavior of solutions to differential equations and an essential step towards its rigorous study is to prove the existence of the RG fixed points (the self-similar solutions), see [8, 9].

In this article, we take $\alpha \geq 1$ and we prove that (1.1) has a scale invariant solution for specific choices of $\theta_0, \theta_i \in (0,1)$. We will show that

Theorem 1.1. *Let $\alpha \geq 1$ and $\theta_0, \theta_i \in (\theta_r, \theta_s)$. If $\theta_0 \leq \theta_i$ then (1.1) has a unique classical solution $\theta(x,t)$ of the form $\theta(x,t) = \psi(x/\sqrt{t})$, where $\psi : [0,\infty) \to (\theta_r, \theta_s)$ is a $C^2([0,\infty))$ function satisfying $\psi(0) = \theta_0$ and $\psi(\infty) = \theta_i$. Furthermore, given $\theta_0 \in (\theta_r, \theta_s)$, there exists $\varepsilon > 0$ such that (1.1) has a self-similar solution for any choice of $\theta_i \in (\theta_0 - \varepsilon, \theta_s)$.*

To prove Theorem 1.1, we restate (1.1) in terms of a boundary value problem as follows. Define

$$\varphi = \frac{1}{1-\sigma} - 1,$$

where $\varphi = \varphi(\zeta)$, with ζ equals the similarity variable $x/\sqrt{t}$, and $\sigma = \sigma(x,t) \equiv (\theta(x,t) - \theta_r)/\Delta\theta$, with $\Delta\theta \equiv \theta_s - \theta_r$. Then, $0 < \sigma < 1$ and $0 < \varphi < \infty$ so that (1.1) is rewritten as the boundary value problem (BVP)

$$\begin{gathered} \varphi'' + \Big(\frac{\alpha-1}{\varphi(\varphi+1)}\Big)(\varphi')^2 + \frac{\zeta}{2K_1}\Big(\frac{(\varphi+1)^{\alpha-3}}{\varphi^{\alpha-1}}\Big)\varphi' = 0, \\ \varphi(0) = \varphi_0, \quad \varphi(\infty) = \varphi_i, \end{gathered} \tag{1.2}$$

where $K_1 \equiv C(\Delta\theta)^{\alpha-3}$ and φ_0 (φ_i) corresponds to θ_0 (θ_i) through the relation

$$\frac{\theta_k - \theta_r}{\theta_s - \theta_k} = \varphi_k, \quad k = 0, i.$$

It is straightforward to see that Theorem 1.1 is a corollary of following result.

Theorem 1.2. *Consider the Boundary Value Problem (1.2) with $\alpha \geq 1$, and φ_0 and φ_i in $(0,\infty)$.*

(1) *If $\varphi_0 \leq \varphi_i$ then (1.2) has a unique classical $C^2([0,\infty))$ solution $\varphi : [0,\infty) \to [\varphi_0, \varphi_i]$;*
(2) *if φ_0 is fixed then there exists $\varepsilon > 0$ such that (1.2), with $\varphi_i \in (\varphi_0 - \varepsilon, \varphi_0)$, has a classical $C^2([0,\infty))$ solution $\varphi : [0,\infty) \to [\varphi_0 - \varepsilon, \varphi_0]$;*
(3) *if $\alpha = 2$ then $\varepsilon = \varphi_0$ in the above statements.*

Finally, instead of studying (1.2) directly, we replace it by the initial value problem

$$\begin{gathered} \varphi'' + \Big(\frac{\alpha-1}{\varphi(\varphi+1)}\Big)(\varphi')^2 + \frac{t}{2K_1}\Big(\frac{(\varphi+1)^{\alpha-3}}{\varphi^{\alpha-1}}\Big)\varphi' = 0, \\ \varphi(0) = \varphi_0 > 0, \quad \varphi'(0) = \varphi'_0 \in \mathbb{R}, \end{gathered} \tag{1.3}$$

where t is the independent variable. Let $\varphi(t) : [0,\omega) \to [0,\infty)$ be the solution to (1.3) on its maximal interval $[0,\omega)$ and let $\varphi_0' \equiv x$. Observe that both $\varphi(t)$ and ω are functions of x and, when appropriate, we express this dependence as $\varphi(t,x)$ and $\omega(x)$, respectively. It is straightforward to see that Theorem 1.1 is a corollary of the following theorem that will be proved in next section.

Theorem 1.3. *Let $\varphi(t) : [0,\omega) \to [0,\infty)$ be the unique classical solution to* (1.3) *on its maximal interval $[0,\omega)$, where $\varphi_0 > 0$ and $\varphi_0' = x \in \mathbb{R}$.*

(1) *Suppose $\alpha \geq 1$. There exists a negative number $\bar{x}$ such that if $x \in (\bar{x},\infty)$ then $\omega(x) = \infty$, i.e., $\varphi(t)$ is a global solution. The limit $\lim_{t\to\infty}\varphi(t,x)$ exists and defines a continuous function $f(x)$ on $(\bar{x},\infty)$ for which $f(0) = \varphi_0$. Furthermore, $f(x)$ is a homeomorphism between $[0,\infty)$ and $[\varphi_0,\infty)$.*
(2) *Suppose $1 < \alpha \leq 2$. There exists a negative number $\underline{x}$ such that if $x \in (-\infty,\underline{x})$ then $\omega(x) < \infty$, i.e., $\varphi(t)$ ceases to be a classical solution at finite time. In particular, for all $x < \underline{x}$, $\lim_{t\to\omega^-}\varphi(t,x) = 0$.*
(3) *If $\alpha = 2$ then there exists a negative number λ such that the function $f(x)$ maps the interval $(\lambda,0]$ onto the interval $(0,\varphi_0]$.*

2. Proof of Theorem 1.3

In this section we consider (1.3) with α and φ_0' in $\mathbb{R}$. We will show below that the derivative $\varphi'(t)$ keeps the sign of $\varphi'(0) = \varphi_0'$ for all positive times, i.e., the product $\varphi_0' \cdot \varphi'(t)$ is non negative for all $t \geq 0$ and it is zero if and only if $\varphi_0' = 0$.

Lemma 2.1. *Let $\varphi(t) : [0,\omega) \to \mathbb{R}$ be the solution to* (1.3) *with $\varphi_0 > 0$ and $\alpha, \varphi_0' \in \mathbb{R}$, where $[0,\omega)$ is the solution's maximal interval of existence. Then the product $\varphi_0' \cdot \varphi'(t)$ is non negative for all $t \in [0,\omega)$ and it is zero if and only if $\varphi_0' = 0$.*

Proof. For any $\alpha, \varphi_0' \in \mathbb{R}$, the IVP (1.3) has a unique local positive solution as long as $\varphi_0 > 0$, see [10], and it will keep itself positive as long as it exists as a classical solution. Of course, if $\varphi_0' = 0$ then, by uniqueness, $\varphi(t) = \varphi_0$ for all t. If $\varphi_0' \neq 0$ then $\varphi'(t) \neq 0$ for t close to 0 and, of course, φ_0' and $\varphi'(t)$ will have the same sign. Let $[0,\omega') \subset [0,\omega)$ be the maximal interval on which φ_0' and $\varphi'(t)$ will have the same sign. We will prove below that $\omega' = \omega$. Suppose, by contradiction, that $\omega' < \omega$. Then, by continuity, $\varphi'(\omega') = 0$. Divide (1.3) through by $\varphi'(t)$, $t < \omega'$, and integrate out to get that

$$\varphi'(t) = \varphi_0'\Big(\frac{\varphi(t)+1}{\varphi(t)}\Big)^{\alpha-1}\Big(\frac{\varphi_0}{\varphi_0+1}\Big)^{\alpha-1} h(t), \tag{2.1}$$

where

$$h(t) = \exp\Big(-\int_0^t \frac{t(\varphi(t)+1)^{\alpha-3}}{2K(\varphi(t))^{\alpha-1}}\,\mathrm{d}t\Big). \tag{2.2}$$

We conclude from (2.1) and (2.2) that $\lim_{t\to\omega'^-}\varphi'(t)$ exists and is different from 0, a contradiction. □

2.1. Proof of Part I when $\varphi_0' > 0$.

Lemma 2.2. *Let $\varphi(t) : [0,\omega) \to \mathbb{R}$ be the solution to* (1.3) *with $\alpha \geq 1$ and $\varphi_0' > 0$, where $[0,\omega)$ is the solution's maximal interval of existence. Then $\varphi'(t)$ is a positive, monotonically decreasing, function on $[0,\omega)$.*

Proof. It follows from Lemma 2.1 that $\varphi'(t) > 0$ for all $t \in [0, \omega)$ if $\varphi'_0 > 0$. That $\varphi'(t)$ is monotonically decreasing comes from $\alpha \geq 1$ and from the ODE, rewritten as

$$\varphi''(t) = -\Big(\frac{\alpha - 1}{\varphi(t)(\varphi(t)+1)}\Big)(\varphi'(t))^2 - \frac{t}{2K_1}\Big(\frac{(\varphi(t)+1)^{\alpha-3}}{\varphi^{\alpha-1}(t)}\Big)\varphi'(t). \tag{2.3}$$

□

Lemma 2.3. *Under the hypothesis of Lemma 2.2, the solution to* (1.3) *is well defined for all* $t \geq 0$.

Proof. Since $\alpha \geq 1$ and $\varphi'_0 > 0$, it follows from Lemma 2.2 that $\varphi'(t)$ is a positive, monotonically decreasing, function on $[0, \omega)$. In particular, $\varphi'(t) \leq \varphi'_0$ for all $t \in [0, \omega)$ which implies that $\varphi(t) \leq \varphi_0 + t\,\varphi'_0$. But since $\varphi_0 \leq \varphi(t)$ for all t, it follows that $\varphi(t)$ is bounded above and below for all $t \in [0, \omega)$ an it follows from the theorem of existence of solutions to differential equations, see [10], that the solution can be extended up to $\omega = \infty$. □

For the rest of this article, we replace φ'_0 by x in (1.3). To make the x dependence explicit we sometimes write $\varphi(t, x)$, $\varphi'(t, x)$ and $h(t, x)$ instead of $\varphi(t)$, $\varphi'(t)$ and $h(t)$, respectively. If $x \geq 0$ and $\alpha \geq 1$ then it follows from lemmas (2.2) and (2.3) that $\varphi(t, x)$ is a monotonically increasing function of t for all $t \geq 0$, implying that the limit

$$\lim_{t \to \infty} \varphi(t, x) \equiv f(x) \tag{2.4}$$

is well defined although it could be infinity. $f(x)$ is the boundary value of $\varphi(t, x)$ at $t = \infty$. The next two lemmas show that if $x \geq 0$ and $\alpha \geq 1$ then $f(x)$ is a continuous homeomorphism between $[0, \infty)$ and $[\varphi_0, \infty)$. It follows from this result that (1.2) has a unique classical solution if $\varphi_0 < \varphi_i$.

Lemma 2.4. *If $x \geq 0$ and $\alpha \geq 1$ then $f(x)$, defined by* (2.4)*, is a monotonically increasing function that satisfies $f(x) \to \infty$ as $x \to \infty$.*

Proof. We first show that $f(x) \to \infty$ as $x \to \infty$. It follows from Lemma 2.2 that, under the above hypothesis on α and x, $\varphi(t, x)$ is an increasing function of t. In particular, $\varphi(t, x) \geq \varphi_0$ for all $t \geq 0$ which, together with (2.1) and with $\alpha \geq 1$, leads to

$$\varphi'(t, x) \geq x\Big(\frac{\varphi_0}{\varphi_0 + 1}\Big)^{\alpha-1} \mathrm{e}^{-[\frac{(1+\varphi_0)^{\alpha-3}}{K\varphi_0^{\alpha-1}}]\frac{t^2}{4}}.$$

Integration on both sides of the above inequality gives the result.

To prove that f is increasing in the interval $[0, \infty)$, we take $\bar{x} > x \geq 0$ and define $\delta(t) \equiv \bar{\varphi}(t) - \varphi(t)$ where $\varphi(t) = \varphi(t, x)$ and $\bar{\varphi}(t) = \varphi(t, \bar{x})$ are the solutions to (1.3) with initial derivatives x and $\bar{x}$, respectively. In the sequel we will prove that $\delta'(t) > 0$ for all $t \geq 0$. It then follows that our result is proven because $0 < \bar{x} - x = \delta(0) \leq \delta(t) \leq f(\bar{x}) - f(x)$, where the last inequality is obtained by taking the limit of $\delta(t)$ as $t \to \infty$ and using its monotonicity.

Let $[0, \bar{t})$ be the maximal interval where $\delta'(t) > 0$. Of course, $[0, \bar{t})$ is not empty because $\delta(t)$ is continuous and $\delta'(0) = \bar{x} - x > 0$. We will show that $\bar{t} = \infty$. Suppose by contradiction that $\bar{t} < \infty$; i.e., $\delta'(\bar{t}) = 0$. It follows from (1.3) that

$$\delta''(t) = -\frac{(\alpha-1)\bar{\varphi}'^2}{\bar{\varphi}^2 + \bar{\varphi}} - \frac{t}{2K}\Big[\frac{(\bar{\varphi}+1)}{\bar{\varphi}}\Big]^{\alpha-1}\frac{1}{(\bar{\varphi}+1)^2}\bar{\varphi}'$$

$$+\frac{(\alpha-1)\varphi'^2}{\varphi^2+\varphi}+\frac{t}{2K}\Big[\frac{(\varphi+1)}{\varphi}\Big]^{\alpha-1}\frac{1}{(\varphi+1)^2}\varphi'.$$

Now, since $\delta'(t)>0$ for $t\in[0,\bar t)$, then $\delta(t)$ is increasing and $\delta(t)=\bar\varphi(t)-\varphi(t)\geq 0$ for $t\in[0,\bar t)$. It then follows from the above identity that

$$\delta''(t)\geq-\frac{\alpha-1}{\varphi^2+\varphi}(\bar\varphi'^2-\varphi'^2)-\frac{t}{2K}\,[\frac{(\bar\varphi+1)}{\bar\varphi}]^{\alpha-1}\frac{1}{(\bar\varphi+1)^2}(\bar\varphi'-\varphi').$$

For $t\in[0,\bar t)$, we divide the last inequality by $\bar\varphi'(t)-\varphi'(t)=\delta'(t)$ and integrate out from 0 to $\bar t$ to obtain

$$0=\delta'(\bar t)\geq(\bar x-x)\,\exp(U),$$
$$U=\Big[-\int_0^{\bar t}\Big(\frac{\alpha-1}{\varphi^2+\varphi}(\bar\varphi'+\varphi')+\frac{t}{2K}\,[\frac{(\bar\varphi+1)}{\bar\varphi}]^{\alpha-1}\frac{1}{(\bar\varphi+1)^2}\Big)\,\mathrm{d}t\Big],$$

which is a contradiction because the right hand side of the last inequality is positive. $\square$

Lemma 2.5. *If $x\geq 0$ and $\alpha\geq 1$ then $f(x)$, defined by* (2.4)*, is a continuous function.*

Proof. The continuity of $f(x)$ comes from the following claim which we will prove below: If $x\geq 0$ and $\alpha\geq 1$ then there exist positive constants C and t_0 such that

$$\varphi'(t,x)\leq Cx/t^2 \tag{2.5}$$

for all $t>t_0$. It follows from the above upper bound that $\varphi(t,x)$ converges, as $t\to\infty$, to $f(x)$, the convergence being uniform on compact sets because:

$$|\varphi(t',x)-\varphi(t,x)|=\Big|\int_t^{t'}\varphi'(t,x)\,\mathrm{d}t\Big|\leq Cx\int_t^{t'}\frac{1}{t^2}\,\mathrm{d}t\leq\frac{Cx}{t}$$

for all $t'\geq t>t_0$.

To obtain the upper bound (2.5) we first prove that $\varphi'(t,x)\to 0$ as $t\to\infty$. It follows from (2.1) that

$$\varphi'(t,x)\leq xh(t,x)$$

if $\alpha\geq 1$ and $x\geq 0$ because the product $((\varphi+1)/\varphi)^{\alpha-1}((\varphi_0+1)/\varphi_0)^{\alpha-1}$ will be at most 1. Then, it is sufficient to prove that $h(t,x)\to 0$ as $t\to\infty$. But, since $h(t,x)\leq 1$ for all $t,x\geq 0$, the above inequality implies that $\varphi(t,x)\leq xt+\varphi_0$ for $t\in[0,\infty)$ which gives rise to the following lower bound for the integrand in (2.2):

$$\frac{t}{2K}\frac{(\varphi+1)^{\alpha-3}}{(\varphi)^{\alpha-1}}=\frac{t}{2K}(\frac{\varphi+1}{\varphi})^{\alpha-1}\frac{1}{(\varphi+1)^2}\geq\frac{t}{2K}\frac{1}{(xt+\varphi_0+1)^2} \tag{2.6}$$

and implying that $h(t,x)\to 0$ as $t\to\infty$.

Now, since $\varphi'(t,x)\to 0$ as $t\to\infty$ and since $\varphi'(t,x)$ is monotonically decreasing, see the proof of Lemma 2.3, it follows that given $r>0$ and small, there exist $s>0$ and $t_0>0$ such that $\varphi(t,x)\leq rt+s$ for all $t\geq t_0$ and, as in (2.6), we get

$$\frac{t}{2K}\frac{(\varphi+1)^{\alpha-3}}{(\varphi)^{\alpha-1}}\geq\frac{1}{4r^2K}\;\frac{1}{t}$$

for $t>t_0$. Then, (2.5) is proven once we choose r such that $1/(4r^2K)\geq 2$. $\square$

2.2. **Proof of Part I when $\varphi_0' < 0$.** In this section and in Section 2.3 we consider (1.3) with $\varphi_0' < 0$. Let $\varphi(t) : [0, \omega) \to \mathbb{R}$ be its solution on the maximal interval $[0, \omega)$. It follows from Lemma 2.1 that $\varphi'(t) < 0$ for all $t \in [0, \omega)$ and therefore that $\varphi(t)$ is a monotonically decreasing function on $[0, \omega)$. It also follows from (2.3) that if $\alpha \geq 1$ then $\varphi''(t) < 0$, i.e. $\varphi(t)$ is concave, at least for small values of t. For larger values of t there will be a competition between the two parcels on the right hand side of (2.3), one keeping itself positive while the other one keeping negative, and it is a matter to decide who is going to win as t gets larger. The question whether $\varphi(t)$ changes concavity or not as t increases is related to how negatively large or small is φ_0' and we deal with this problem in sections 2.3 and 2.4. In this section we show that there exists a negative real number $\bar{x}$ such that $\varphi(t, x)$ is well defined for all t and for $x > \bar{x}$. In particular, $\varphi(t, x)$ changes concavity at some point t_0 and, similarly to Lemma 2.5, we prove that the limit (2.4) is a well defined continuous function on $(\overline{x}, 0)$.

Lemma 2.6. *Let $\varphi(t) : [0, \omega) \to \mathbb{R}$ be the solution to (1.3) with $\alpha \geq 1$ and x satisfying*

$$-\frac{2}{\sqrt{\pi}}\frac{1}{\alpha}\Big[\frac{(\varphi_0+1)^{\alpha-3}}{4K(\varphi_0)^{\alpha-3}}\Big]^{1/2} < x \leq 0, \tag{2.7}$$

where $[0, \omega)$ is the solution's maximal interval of existence. Then $\omega = \infty$.

Proof. We will show that if Condition (2.7) is satified then the limit $\lim_{t\to\omega^-} \varphi(t)$ is positive. This is enough to conclude that $\omega = \infty$ because it follows from this limit and from (2.1) that the limit $\lim_{t\to\omega^-} \varphi'(t)$ exists and is also positive, implying that the solution can always be extended to the right of ω, for any positive ω.

If $\alpha \geq 1$ and $x < 0$ then it follows from Lemma 2.1 that $\varphi'(t) < 0$ for all $t \in [0, \omega)$ implying, together with (2.1), (2.2) and that $\varphi(t) \leq \varphi_0$ for all $t \in [0, \omega)$, that

$$\varphi^{\alpha-1}(t)\varphi'(t) \geq x\varphi_0^{\alpha-1}\ \exp\Big(-\frac{t^2(\varphi_0+1)^{\alpha-3}}{4K(\varphi_0)^{\alpha-1}}\Big)$$

for any $t \in [0, \omega)$. Define

$$\gamma^2 \equiv \frac{(\varphi_0+1)^{\alpha-3}}{4K(\varphi_0)^{\alpha-1}}$$

and integrate the above inequality from 0 to t to obtain

$$\varphi^\alpha(t) \geq \varphi_0^\alpha + \alpha x\varphi_0^{\alpha-1}\int_0^t \mathrm{e}^{-\gamma^2 t^2}\mathrm{d}t = \varphi_0^\alpha + \frac{\alpha x\varphi_0^{\alpha-1}}{\gamma}\int_0^{\gamma t}\mathrm{e}^{-u^2}\mathrm{d}u \equiv (\varphi_{m1}(t))^\alpha.$$

It follows from the above inequality that $\varphi(t)$ is positive whenever $\varphi_{m1}(t)$ is positive; i.e., whenever x is such that

$$\frac{-1}{\alpha x}\Big[\frac{(\varphi_0+1)^{\alpha-3}}{4K(\varphi_0)^{\alpha-3}}\Big]^{1/2} > \int_0^{\gamma t}\mathrm{e}^{-u^2}\mathrm{d}u,$$

which is fulfilled if Condition (2.7) is satisfied because

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty \mathrm{e}^{-u^2}\mathrm{d}u > \int_0^{\gamma t}\mathrm{e}^{-u^2}\mathrm{d}u$$

for all positive t. □

Define

$$\overline{x} \equiv -\frac{2}{\sqrt{\pi}}\frac{1}{\alpha}\Big[\frac{(\varphi_0+1)^{\alpha-3}}{4K(\varphi_0)^{\alpha-3}}\Big]^{1/2}.$$

It follows from Lemma 2.6 that if $\alpha \geq 1$ then $f(x)$, given by the limit (2.4), is well defined on the interval $(\overline{x}, 0)$. In next lemma we prove that $f(x)$ is continuous on $(\overline{x}, 0)$ and this result implies that $f(x)$ is onto the interval $f(\overline{x}, 0)$.

Lemma 2.7. *If $\alpha \geq 1$ and $x \in (\bar{x}, 0)$ then $f(x)$, defined by (2.4), is a continuous function.*

Proof. Let $I \equiv [x_1, x_2] \subset (\bar{x}, 0)$. The proof of this lemma is similar to the proof of Lemma 2.5 and it is enough to show that $|\varphi'(t)|$ is bounded by an integrable function of t, uniformly in $x \in I$. To do so, we observe that it follows from the proof of Lemma 2.6 that there exists $m > 0$ such that $m \leq f(x) \leq \varphi_0$ for any $x \in I$. Therefore, from (2.1), we obtain

$$|\varphi'(t)| \leq |x| \Big(\frac{m+1}{m}\Big)^{\alpha-1} \Big(\frac{\varphi_0}{\varphi_0 + 1}\Big)^{\alpha-1} \exp\Big(-\frac{t^2(\varphi_0+1)^{\alpha-3}}{4K(\varphi_0)^{\alpha-1}}\Big)$$

which completes the proof. $\square$

2.3. **Proof of Part II.** In Lemma 2.8 below we prove that if $\alpha \geq 2$ and if $\varphi(t, x)$ changes concavity at some point t_0 then $\varphi(t, x)$ exists for all t. We use this result to prove, in Lemma 2.9, that classical solutions will not be defined on $[0, \infty)$.

Lemma 2.8. *Let $\alpha \geq 2$ and suppose that there exists $t_0 > 0$ such that $\varphi''(t_0) = 0$. Then, t_0 is unique and the solution $\varphi(t)$ to (1.3) is well defined for all $t \geq 0$.*

Proof. Since $\{t : \varphi''(t) = 0\}$ is nonempty, $t_0 \equiv \inf\{t : \varphi''(t) = 0\}$ is well defined. We first prove that if $\alpha \geq 2$ then $\{t : \varphi''(t) = 0\} = \{t_0\}$. From (1.3), $\varphi''(t)$ can be written as

$$\varphi''(t) = g(t)\Big(\frac{-\varphi'(t)}{\varphi^2(t) + \varphi(t)}\Big), \tag{2.8}$$

where $g(t)$ is given by

$$g(t) \equiv (\alpha - 1)\,\varphi'(t) + \frac{t}{2\,K}\Big(\frac{\varphi(t)+1}{\varphi(t)}\Big)^{\alpha-2}. \tag{2.9}$$

It follows from (2.8) that both $\varphi''(t)$ and $g(t)$ have the same sign because $\varphi(t)$ and $-\varphi'(t)$ are both positive for $t \in [0, \omega)$. From (2.9), we obtain

$$g'(t) = (\alpha - 1)\,\varphi''(t) + \frac{1}{2\,K}\Big(\frac{\varphi(t)+1}{\varphi(t)}\Big)^{\alpha-2} + \frac{(\alpha-2)}{2\,K}\Big(\frac{\varphi(t)+1}{\varphi(t)}\Big)^{\alpha-3}\Big(\frac{-\varphi'(t)}{\varphi^2(t)}\Big). \tag{2.10}$$

It follows from (2.10) that $g'(t_0) > 0$ if $\alpha \geq 2$. We claim that $g'(t) > 0$ for all $t \geq t_0$. If not, there would exist $t_2 > t_0$ such that $g'(t_2) = 0$. Of course, $g(t_2) > 0$ because $g(t)$ is increasing up to t_2 and because, from (2.8), $g(t_0) = 0$. Then we conclude, from (2.10), that $\varphi''(t_2) < 0$. Therefore, again from (2.8), we conclude that $g(t_2) < 0$, a contradiction. In particular, $g(t)$ is an increasing function for $t > t_0$.

In the sequel, we will prove that $\varphi(t)$ remains bounded away from zero if it changes concavity at some point t_0. Since $\varphi(t)$ is also bounded above by φ_0, the solution will exist for all $t \geq 0$, see [10]. Fix $t_i > t_0$ and consider $t \in [t_i, \omega)$. Since $\varphi(t)$ is decreases and $g(t)$ increases as t increases above t_0, we get

$$\frac{g(t)}{1+\varphi(t)} \geq \frac{g(t_i)}{1+\varphi(t_i)} \equiv K_i > 0,$$

which implies from (2.8) that

$$K_i \Big(\frac{-\varphi'(t)}{\varphi(t)}\Big) \le \varphi''(t)$$

for all $t \ge t_i$. Integrating both sides of the above inequality from t_i to t, we get that

$$K_i \ln \Big(\frac{\varphi(t_i)}{\varphi(t)}\Big) \le \varphi'(t) - \varphi'(t_i) \le -\varphi'(t_i),$$

where we have used that $\varphi'(t) < 0$ for all $t \in [0,\omega)$ to get the last inequality. After exponentiating both sides of the above inequality we get

$$\varphi(t) \ge \varphi(t_i)\, \exp \frac{\varphi'(t_i)}{K_i} > 0$$

for all $t > t_i$ and we are done. □

Lemma 2.9. *Let $\varphi(t) : [0,\omega) \to \mathbb{R}$ be the solution to* (1.3) *with $1 < \alpha \le 2$ and*

$$x < -\sqrt{\frac{\varphi_0}{2K(\alpha-1)}}, \tag{2.11}$$

where $[0,\omega)$ is the solution's maximal interval of existence. Then $\omega < \infty$.

Proof. Let $L(t) \equiv \varphi_0 + t\,x$ and observe that $L(-\varphi_0/x) = 0$. It follows from the arguments given at the beginning of this section that $\varphi(t) < L(t)$ holds, at least for small positive values of t, because $\varphi''(t) < 0$ if $t \ge 0$ and small and if $\alpha > 1$. We claim that $\omega < -\varphi_0/x$. If not, there would exist a positive time t_1, with $t_1 < -\varphi_0/x$, such that $\varphi(t_1) = L(t_1)$. It then follows that $\varphi(t)$ changes concavity at some time $t_0 < t_1$, i.e., $\varphi''(t_0) = 0$. From (2.3) we read that

$$\varphi'(t_0) = -\frac{t_0}{2K(\alpha-1)}\Big(\frac{\varphi(t_0)+1}{\varphi(t_0)}\Big)^{\alpha-2}.$$

Since $x > \varphi'(t_0)$ and $t_0 < -\varphi_0/x$, we take $\alpha \le 2$ to obtain, from the above inequality, that

$$x > \frac{\varphi_0}{x}\frac{1}{2K(\alpha-1)},$$

which is in contradiction with (2.11). □

2.4. **Proof of Part III.** For $\alpha = 2$, we show that the set of solutions, parametrized by $\varphi_0' < 0$, is organized increasingly as φ_0' varies from $-\infty$ to some negative number λ and we use this result to prove that $f(x)$ maps the interval $(\lambda, 0]$ onto the interval $(0, \varphi_0]$. As in the proof of Lemma 2.4, we take $0 > \bar{x} > x$ and write $\delta(t) = \bar{\varphi}(t) - \varphi(t)$, where $\varphi(t) = \varphi(t,x)$ and $\bar{\varphi}(t) = \varphi(t,\bar{x})$ are the solutions to (1.3) with initial derivatives x and $\bar{x}$, respectively.

Lemma 2.10. *Let $\alpha = 2$ and suppose that $\omega(\bar{x}) < \infty$. Then $\omega(x) < \omega(\bar{x})$ for all $x < \bar{x}$.*

Proof. Observe that $\delta(0) = 0$ and $\delta'(0) > 0$. Then $\delta'(t) > 0$ for small values of t. We will show below that $\delta'(t) > 0$ for all values of $t \in (0,\omega_m)$, where $\omega_m = \min\{\omega(x), \omega(\bar{x})\}$; i.e., $\delta(t)$ is an increasing and strictly positive function on $(0,\omega_m)$. It follows from this result that $\omega(x) < \omega(\bar{x})$. In fact, if it happened that $\omega(x) \ge \omega(\bar{x})$ then $\delta(t) = 0$ at some point $t < \omega_m$ but this is not possible because $\delta(t) > 0$ for all $t \in (0, \omega_m)$. So, suppose, by contradiction, that there exists a first point $\tilde{t}$ such that $\delta'(\tilde{t}) = 0$, i.e., $\bar{\varphi}'(\tilde{t}) = \varphi'(\tilde{t})$. It then follows from (1.3) with $\alpha = 2$ that

$$[\bar{\varphi}(\tilde{t})^2 + \bar{\varphi}(\tilde{t})]\bar{\varphi}''(\tilde{t}) = [\varphi(\tilde{t})^2 + \varphi(\tilde{t})]\varphi''(\tilde{t}),$$

and we conclude that $\bar{\varphi}''(\tilde{t})$ and $\varphi''(\tilde{t})$ are both positive or negative. But, by hypothesis, $\bar{\omega} < \infty$ which implies from Lemma 2.8 that $\bar{\varphi}''(t) < 0$ for all $t \in [0, \bar{\omega})$. Therefore, $\bar{\varphi}''(\tilde{t})$ and $\bar{\varphi}''(\tilde{t})$ are both negative. From the above identity we get

$$0 < \frac{\bar{\varphi}''(\tilde{t})}{\varphi''(\tilde{t})} = \frac{\varphi(\tilde{t})^2 + \varphi(\tilde{t})}{\bar{\varphi}(\tilde{t})^2 + \bar{\varphi}(\tilde{t})} < 1,$$

leading that $\bar{\varphi}''(\tilde{t}) - \varphi''(\tilde{t}) = \delta''(\tilde{t}) > 0$. On the other hand, since $\delta(t)$ is increasing in the interval $[0, \tilde{t}]$ and since $\delta'(\tilde{t}) = 0$ then $\delta''(\tilde{t}) \leq 0$, a contradiction with $\delta''(\tilde{t}) > 0$. ◻

Once φ_0 is fixed and $1 < \alpha \leq 2$, it follows from Lemma 2.9 that $\lambda \equiv \sup\{x \in (-\infty, 0) : \omega(x) < \infty\}$ is well defined and it follows from Lemma 2.6 that $\lambda < 0$. Next result states that the set $\{x \in (-\infty, 0) : \omega(x) < \infty\}$ is an interval.

Corollary 2.11. *If $\alpha = 2$, then $(-\infty, \lambda] = \{x \in (-\infty, 0) : \omega(x) < \infty\}$.*

Proof. It is sufficient to show that $(-\infty, \lambda] \subset \{x \in (-\infty, 0) : \omega(x) < \infty\}$. We first observe that $\lambda \in \{x \in (-\infty, 0) : \omega(x) < \infty\}$ because the set $\{x \in (-\infty, 0) : \omega(x) = \infty\}$ is open (this is so because of Lemma 2.8 and the smoothly continuous dependence of solutions on the initial conditions, see [10]). Then $\omega(\lambda) < \infty$ and it follows from Lemma 2.10 that $\omega(x) < \omega(\lambda)$ if $x < \lambda$ and we are done. ◻

It follows from the corollary that $f(x)$ is a well defined function on $(\lambda, 0]$. In next lemma we characterize the set $f((\lambda, 0])$.

Lemma 2.12. *If $\alpha = 2$ then $f((\lambda, 0]) = (0, \varphi_0]$.*

Proof. It follows from Lemma 2.7 that $f(x)$ is continuous on $(\lambda, 0]$. Since $f(0) = \varphi_0$ then or $f((\lambda, 0]) = (0, \varphi_0]$ or there exists $a > 0$ such that $f(\lambda, 0] \subset [a, \varphi_0]$. In what follows we discard the second option. Since $\omega(\lambda) < \infty$ then $\varphi(t, \lambda) \to 0$ as $t \to \omega(\lambda)$. In particular, given $a > 0$ there exists $t_a > 0$ such that $\varphi(t_a, \lambda) < a/2$. By the continuous dependence on initial conditions, $\varphi(t_a, x) < a/2$ if x is close enough of λ, with $x > \lambda$. But since $\varphi(t, x)$ is decreasing as a function of t, then $\varphi(t, x) < a/2$ for all $t > t_a$ and then $f(x) \leq a/2 < a$. It follows from here that $f((\lambda, 0]) = (0, \varphi_0]$. ◻

We remark that Lemma 2.8 is proven only for $\alpha \geq 2$ and Lemma 2.9 is proven only for $1 < \alpha \leq 2$, but numerical simulations indicate that both hold for all $\alpha \geq 1$. If so then Lemma 2.12, equivalently the third part of Theorem 1.3, would hold for any $\alpha \geq 1$. Numerical simulations also suggest that the solutions to (1.3) are organized increasingly as a function of x and we have proved that this is the case if $x \geq 0$ and $\alpha \geq 1$ or if $x < \lambda$ and $\alpha = 2$ but it is an open problem to prove it for $x \in (\lambda, 0)$. If so then the result would imply that $f(x) : (\lambda, 0] \to (0, \varphi_0]$ is one-to-one and, together with lemmas (2.7) and (2.12), a homeomorphism. Therefore, based upon numerical experiments, we conjecture that Theorem 1.1 holds also when $\theta_i < \theta_0$, for any $\theta_0, \theta_i \in (\theta_r, \theta_s)$.

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