MULTIPLE POSITIVE SOLUTIONS FOR A THIRD-ORDER THREE-POINT BVP WITH SIGN-CHANGING GREEN'S FUNCTION

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Abstract. This article concerns the third-order three-point boundary-value problem

\[ u'''(t) = f(t, u(t)), \quad t \in [0, 1], \]
\[ u'(0) = u(1) = u''(\eta) = 0. \]

Although the corresponding Green’s function is sign-changing, we still obtain the existence of at least \(2m-1\) positive solutions for arbitrary positive integer \(m\) under suitable conditions on \(f\).

1. Introduction

Third-order differential equations arise from a variety of areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [5].

Recently, the existence of single or multiple positive solutions to some third-order three-point boundary-value problems (BVPs for short) has received much attention from many authors. For example, in 1998, by using the Leggett-Williams fixed point theorem, Anderson [2] proved the existence of at least three positive solutions to the problem

\[ -x'''(t) + f(x(t)) = 0, \quad t \in [0, 1], \]
\[ x(0) = x'(t_2) = x''(1) = 0, \]

where \(t_2 \in [\frac{1}{2}, 1]\). In 2003, Anderson [1] obtained some existence results of positive solutions for the problem

\[ x'''(t) = f(t, x(t)), \quad t_1 \leq t \leq t_3, \]
\[ x(t_1) = x'(t_2) = 0, \quad \gamma x(t_3) + \delta x''(t_3) = 0. \]

The main tools used were the Guo-Krasnosel’skii and Leggett-Williams fixed point theorems. In 2005, Sun [13] studied the existence of single and multiple positive solutions for the problem

\[ x'''(t) = f(t, x(t)), \quad t_1 \leq t \leq t_3, \]
\[ x(t_1) = x'(t_2) = 0, \quad \gamma x(t_3) + \delta x''(t_3) = 0. \]
solutions for the singular BVP
\[ u'''(t) - \lambda a(t)F(t, u(t)) = 0, \quad t \in (0, 1), \]
\[ u(0) = u'(\eta) = u''(1) = u'''(1) = 0, \]
where \( \eta \in \left[\frac{1}{2}, 1\right] \), \( \lambda \) was a positive parameter and \( a(t) \) was a nonnegative continuous function defined on \((0, 1)\). His main tool was the Guo-Krasnosel’skii fixed point theorem. In 2008, by using the Guo-Krasnosel’skii fixed point theorem, Guo, Sun and Zhao [6] obtained the existence of at least one positive solution for the problem
\[ u'''(t) + h(t)f(u(t)) = 0, \quad t \in (0, 1), \]
\[ u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta), \]
where \( 0 < \eta < 1 \) and \( 1 < \alpha < 1/\eta \). For more results concerning the existence of positive solutions to third-order three-point BVPs, one can refer to [3, 4, 9, 10, 12, 14].

It is necessary to point out that all the above-mentioned works are achieved when the corresponding Green’s functions are positive, which is a very important condition. A natural question is that whether we can obtain the existence of positive solutions to some third-order three-point BVPs when the corresponding Green’s functions are sign-changing. It is worth mentioning that Palamides and Smyrlis [8] discussed the existence of at least one positive solution to the singular third-order three-point BVP with an indefinitely signed Green’s function
\[ u'''(t) = a(t)f(t, u(t)), \quad t \in (0, 1), \]
\[ u(0) = u(1) = u''(\eta) = 0, \quad \eta \in \left(\frac{17}{24}, 1\right). \]

Their technique was a combination of the Guo-Krasnosel’skii fixed point theorem and properties of the corresponding vector field. The following equality
\[ \max_{t \in [0, 1]} \int_0^1 G(t, s)a(s)f(s, u(s))ds = \int_0^1 \max_{t \in [0, 1]} G(t, s)a(s)f(s, u(s))ds \tag{1.1} \]
played an important role in the process of their proof. Unfortunately, the equality (1.1) is not right. For a counterexample, one can refer to our paper [11].

Motivated greatly by the above-mentioned works, in this paper we study the following third-order three-point BVP
\[ u'''(t) = f(t, u(t)), \quad t \in [0, 1], \]
\[ u'(0) = u(1) = u''(\eta) = 0, \tag{1.2} \]
where \( f \in C([0, 1] \times [0, +\infty], [0, +\infty]) \) and \( \eta \in \left(\frac{1}{2}, 1\right) \). Although the corresponding Green’s function is sign-changing, we still obtain the existence of at least \( 2m - 1 \) positive solutions for arbitrary positive integer \( m \) under suitable conditions on \( f \).

In the remainder of this section, we state some fundamental concepts and the Leggett-Williams fixed point theorem [7].

Let \( E \) be a real Banach space with cone \( P \). A map \( \sigma : P \to (-\infty, +\infty) \) is said to be a concave functional if
\[ \sigma(tx + (1 - t)y) \geq t\sigma(x) + (1 - t)\sigma(y) \]
for all \( x, y \in P \) and \( t \in [0, 1] \). Let \( a \) and \( b \) be two numbers with \( 0 < a < b \) and \( \sigma \) be a nonnegative continuous concave functional on \( P \). We define the following convex
After a simple computation, we obtain the following expression of Green’s function

\[ P_a = \{ x \in P : \| x \| < a \}, \]

\[ P(\sigma, a, b) = \{ x \in P : a \leq \sigma(x), \| x \| \leq b \}. \]

**Theorem 1.1** (Leggett-Williams fixed point theorem). Let \( A : \overline{P}_c \rightarrow \overline{P}_c \) be completely continuous and \( \sigma \) be a nonnegative continuous concave functional on \( P \) such that \( \sigma(x) \leq \| x \| \) for all \( x \in \overline{P}_c \). Suppose that there exist \( 0 < d < a < b < c \) such that

(i) \( \{ x \in P(\sigma, a, b) : \sigma(x) > a \} \neq \emptyset \) and \( \sigma(Ax) > a \) for \( x \in P(\sigma, a, b) \);

(ii) \( \| Ax \| < d \) for \( \| x \| \leq d \);

(iii) \( \sigma(Ax) > a \) for \( x \in P(\sigma, a, c) \) with \( \| Ax \| > b \).

Then \( A \) has at least three fixed points \( x_1, x_2, x_3 \) in \( \overline{P}_c \) satisfying

\[ \| x_1 \| < d, \ a < \sigma(x_2), \ \| x_3 \| > d, \ \sigma(x_3) < a. \]

2. Preliminaries

In this article, we assume that Banach space \( E = C[0,1] \) is equipped with the norm \( \| u \| = \max_{t \in [0,1]} | u(t) | \).

For any \( y \in E \), we consider the BVP

\begin{align*}
    u''(t) &= y(t), \quad t \in [0,1], \\
    u'(0) &= u(1) = u''(\eta) = 0. \tag{2.1}
\end{align*}

After a simple computation, we obtain the following expression of Green’s function \( G(t,s) \) of the BVP (2.1): for \( s \geq \eta \),

\[ G(t,s) = \begin{cases} 
    -\frac{(1-s)^2}{2}, & 0 \leq t \leq s \leq 1, \\
    \frac{t^2-2st+2s-1}{2}, & 0 \leq s \leq t \leq 1
\end{cases} \]

and for \( s < \eta \),

\[ G(t,s) = \begin{cases} 
    -\frac{t^2-s^2+2s}{2}, & 0 \leq t \leq s \leq 1, \\
    -st + s, & 0 \leq s \leq t \leq 1
\end{cases} \]

Obviously, \( G(t,s) \geq 0 \) for \( 0 \leq s < \eta \), and \( G(t,s) \leq 0 \) for \( \eta \leq s \leq 1 \). Moreover, for \( s \geq \eta \),

\[ \max \{ G(t,s) : t \in [0,1] \} = G(1,s) = 0 \]

and for \( s < \eta \),

\[ \max \{ G(t,s) : t \in [0,1] \} = G(0,s) = -\frac{s^2}{2} + s. \]

To obtain the existence of positive solutions for (1.2), we need to construct a suitable cone in \( E \). Let \( u \) be a solution of (1.2). Then it is easy to verify that \( u(t) \geq 0 \) for \( t \in [0,1] \) provided that \( u'(1) \leq 0 \). In fact, since \( f \) is nonnegative, we know that \( u''(t) \geq 0 \) for \( t \in [0,1] \), which together with \( u''(\eta) = 0 \) implies that

\[ u''(t) \leq 0 \text{ for } t \in [0,\eta] \quad \text{and} \quad u''(t) \geq 0 \text{ for } t \in [\eta,1]. \tag{2.2} \]

In view of (2.2) and \( u'(0) = 0 \), we have

\[ u'(t) \leq 0 \text{ for } t \in [0,\eta] \quad \text{and} \quad u'(t) \leq u'(1) \text{ for } t \in [\eta,1]. \tag{2.3} \]
If \( u'(1) \leq 0 \), then it follows from (2.3) that \( u'(t) \leq 0 \) for \( t \in [0,1] \), which together with \( u(1) = 0 \) implies that \( u(t) \geq 0 \) for \( t \in [0,1] \). Therefore, we define a cone in \( E \) as follows:
\[
\hat{P} = \{ y \in E : y(t) \text{ is nonnegative and decreasing on } [0,1] \}.
\]

**Lemma 2.1** \([11]\). Let \( y \in \hat{P} \) and \( u(t) = \int_0^1 G(t,s)y(s)ds, \ t \in [0,1] \). Then \( u \in \hat{P} \) and \( u \) is the unique solution of (2.1). Moreover, \( u \) satisfies
\[
\min_{t \in [1-\theta,\theta]} u(t) \geq \theta^*\|u\|,
\]
where \( \theta \in (\frac{1}{2}, \eta) \) and \( \theta^* = (\eta - \theta)/\eta \).

**3. Main results**

In the remainder of this paper, we assume that \( f : [0,1] \times [0, +\infty) \to [0, +\infty) \) is continuous and satisfies the following two conditions:

- (D1) For each \( x \in [0, +\infty) \), the mapping \( t \mapsto f(t,x) \) is decreasing;
- (D2) For each \( t \in [0,1] \), the mapping \( x \mapsto f(t,x) \) is increasing.

Let
\[
P = \{ u \in \hat{P} : \min_{t \in [1-\theta,\theta]} u(t) \geq \theta^*\|u\| \}.
\]
Then it is easy to check that \( P \) is a cone in \( E \). Now, we define an operator \( A \) on \( P \) by
\[
(Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds, \ t \in [0,1].
\]
Obviously, if \( u \) is a fixed point of \( A \) in \( P \), then \( u \) is a nonnegative solution of (1.2).

For convenience, we denote
\[
H_1 = \int_0^\eta \left(-\frac{s^2}{2} + s\right)ds, \quad H_2 = \min_{t \in [1-\theta,\theta]} \int_{1-\theta}^\theta G(t,s)ds.
\]

**Theorem 3.1.** Assume that there exist numbers \( d, a \) and \( c \) with \( 0 < d < a < a^{\frac{a}{\theta^*}} \leq c \) such that
\[
\begin{align*}
    f(t,u) &< \frac{d}{H_1}, \quad t \in [0,\eta], \ u \in [0,d], \quad (3.1) \\
    f(t,u) &> \frac{a}{H_2}, \quad t \in [1-\theta,\theta], \ u \in [a,\frac{\eta}{\theta}], \quad (3.2) \\
    f(t,u) &< \frac{c}{H_1}, \quad t \in [0,\eta], \ u \in [0,c]. \quad (3.3)
\end{align*}
\]

Then (1.2) has at least three positive solutions \( u, v \) and \( w \) satisfying
\[
\|u\| < d, \quad a < \min_{t \in [1-\theta,\theta]} v(t), \quad d < \|w\|, \quad \min_{t \in [1-\theta,\theta]} w(t) < a.
\]

**Proof.** For \( u \in P \), we define
\[
\sigma(u) = \min_{t \in [1-\theta,\theta]} u(t).
\]
It is easy to check that \( \sigma \) is a nonnegative continuous concave functional on \( P \) with \( \sigma(u) \leq \|u\| \) for \( u \in P \) and that \( A : P \to P \) is completely continuous.
We first assert that if there exists a positive number \( r \) such that \( f(t, u) < \frac{r}{H_t} \) for \( t \in [0, \eta] \) and \( u \in [0, r] \), then \( A : \overline{P}_r \rightarrow P_r \). Indeed, if \( u \in \overline{P}_r \), then

\[
\|Au\| = \max_{t \in [0,1]} \int_0^1 G(t, s)f(s, u(s))ds \\
\leq \int_0^1 \max_{t \in [0,1]} G(t, s)f(s, u(s))ds \\
= \int_0^\eta \max_{t \in [0,1]} G(t, s)f(s, u(s))ds + \int_\eta^1 \max_{t \in [0,1]} G(t, s)f(s, u(s))ds \\
= \int_0^\eta (1 - \frac{s^2}{2} + s)f(s, u(s))ds \\
< \frac{r}{H_t} \int_0^\eta (1 - \frac{s^2}{2} + s)ds = r;
\]

that is, \( Au \in P_r \).

Hence, we have shown that if (3.1) and (3.3) hold, then \( A \) maps \( \overline{P}_d \) into \( P_d \) and \( \overline{P}_r \) into \( P_r \).

Next, we assert that \( \{ u \in P(\sigma, a, \frac{\sigma}{\sigma_\infty}) : \sigma(u) > a \} \neq \emptyset \) and \( \sigma(Au) > a \) for all \( u \in P(\sigma, a, \frac{\sigma}{\sigma_\infty}) \). In fact, the constant function \( \frac{a + \sigma}{\sigma_\infty} \) belongs to \( \{ u \in P(\sigma, a, \frac{\sigma}{\sigma_\infty}) : \sigma(u) > a \} \).

On the one hand, for \( u \in P(\sigma, a, \frac{\sigma}{\sigma_\infty}) \), we have

\[
a \leq \sigma(u) = \min_{t \in [1 - \theta, \theta]} u(t) \leq u(t) \leq \|u\| \leq \frac{a}{\theta^*},
\]

for all \( t \in [1 - \theta, \theta] \).

Also, for any \( u \in P \) and \( t \in [1 - \theta, \theta] \), we have

\[
\int_0^{1-\theta} G(t, s)f(s, u(s))ds + \int_\theta^\eta G(t, s)f(s, u(s))ds + \int_\eta^1 G(t, s)f(s, u(s))ds \\
\geq \int_0^{1-\theta} (1 - t)sf(s, u(s))ds - \int_\eta^1 \frac{(1 - s)^2}{2}f(s, u(s))ds \\
\geq f(\eta, u(\eta)) \int_0^{1-\theta} (1 - t)ds - \int_\eta^1 \frac{(1 - s)^2}{2}ds \\
\geq f(\eta, u(\eta)) \int_0^{1-\theta} (1 - t)ds - \int_\theta^1 \frac{(1 - s)^2}{2}ds \\
= f(\eta, u(\eta)) \left[ \frac{(1 - t)(1 - \theta)^2}{2} - \frac{(1 - \theta)^3}{6} \right] \\
\geq f(\eta, u(\eta)) \left[ \frac{(1 - \theta)(1 - \theta)^2}{2} - \frac{(1 - \theta)^3}{6} \right] \\
= f(\eta, u(\eta)) \frac{(1 - \theta)^3}{3} \geq 0,
\]

which together with (3.2) and (3.4) implies

\[
\sigma(Au) = \min_{t \in [1 - \theta, \theta]} \int_0^1 G(t, s)f(s, u(s))ds
\]
for $u \in P(\sigma, a, \frac{a}{\theta})$.

Finally, we verify that if $u \in P(\sigma, a, c)$ and $\|Au\| > a/\theta^*$, then $\sigma(Au) > a$. To see this, we suppose that $u \in P(\sigma, a, c)$ and $\|Au\| > a/\theta^*$. Then it follows from $Au \in P$ that

$$
\sigma(Au) = \min_{t \in [1-\theta, \theta]} (Au)(t) \geq \theta^* \|Au\| > a.
$$

To sum up, all the hypotheses of the Leggett-Williams fixed point theorem are satisfied. Therefore, $A$ has at least three fixed points; that is, (1.2) has at least three positive solutions $u, v$ and $w$ satisfying

$$
\|u\| < d, \quad a < \min_{t \in [1-\theta, \theta]} v(t), \quad d < \|w\|, \quad \min_{t \in [1-\theta, \theta]} w(t) < a.
$$

\[\square\]

**Theorem 3.2.** Let $m$ be an arbitrary positive integer. Assume that there exist numbers $d_i$ ($1 \leq i \leq m$) and $a_j$ ($1 \leq j \leq m-1$) with $0 < d_1 < a_1 < \frac{a_2}{\theta^*} < d_2 < a_2 < \frac{a_3}{\theta^*} < \cdots < d_{m-1} < a_{m-1} < \frac{a_m}{\theta^*} < d_m$ such that

$$
(3.5) \quad f(t, u) < \frac{d_i}{H_1}, \quad t \in [0, \eta], \quad u \in [0, d_i], \quad 1 \leq i \leq m,
$$

$$
(3.6) \quad f(t, u) > \frac{a_j}{H_2}, \quad t \in [1-\theta, \theta], \quad u \in [a_j, \frac{a_j}{\theta^*}], \quad 1 \leq j \leq m-1.
$$

Then (1.2) has at least $2m - 1$ positive solutions in $P_{d_m}$.

\[\text{Proof.}\] We use induction on $m$. First, for $m = 1$, we know from (3.5) that $A : P_{d_1} \to P_{d_1}$. Then it follows from Schauder fixed point theorem that (1.2) has at least one positive solution in $P_{d_1}$.

Next, we assume that this conclusion holds for $m = k$. To show that this conclusion also holds for $m = k + 1$, we suppose that there exist numbers $d_i$ ($1 \leq i \leq k + 1$) and $a_j$ ($1 \leq j \leq k$) with $0 < d_1 < a_1 < \frac{a_2}{\theta^*} < d_2 < a_2 < \frac{a_3}{\theta^*} < \cdots < d_k < a_k < \frac{a_{k+1}}{\theta^*} < d_{k+1}$ such that

$$
(3.7) \quad f(t, u) < \frac{d_i}{H_1}, \quad t \in [0, \eta], \quad u \in [0, d_i], \quad 1 \leq i \leq k + 1,
$$

$$
(3.8) \quad f(t, u) > \frac{a_j}{H_2}, \quad t \in [1-\theta, \theta], \quad u \in [a_j, \frac{a_j}{\theta^*}], \quad 1 \leq j \leq k.
$$

By assumption, (1.2) has at least $2k - 1$ positive solutions $u_i$ ($i = 1, 2, \ldots, 2k - 1$) in $P_{d_k}$. At the same time, it follows from Theorem 3.1 (3.7) and (3.8) that (1.2) has at least three positive solutions $u, v$ and $w$ in $P_{d_{k+1}}$ such that

$$
\|u\| < d_k, \quad a_k < \min_{t \in [1-\theta, \theta]} v(t), \quad d_k < \|w\|, \quad \min_{t \in [1-\theta, \theta]} w(t) < a_k.
$$

Obviously, $v$ and $w$ are different from $u_i$ ($i = 1, 2, \ldots, 2k - 1$). Therefore, (1.2) has at least $2k + 1$ positive solutions in $P_{d_{k+1}}$, which shows that this conclusion also holds for $m = k + 1$. \[\square\]
Example 3.3. We consider the BVP

\[ u'''(t) = f(t, u(t)), \quad t \in [0, 1], \]

\[ u'(0) = u(1) = u''(\frac{2}{3}) = 0, \quad (3.9) \]

where

\[ f(t, u) = \begin{cases} 
(1-t)(u+1)^2, & (t, u) \in [0, 1] \times [0, 1], \\
(1-t)(122(u-1) + 4), & (t, u) \in [0, 1] \times [1, 2], \\
14(1-t)(u+1)^2, & (t, u) \in [0, 1] \times [2, 20], \\
6174(1-t), & (t, u) \in [0, 1] \times [20, +\infty). 
\end{cases} \]

Let \( \theta = 3/5 \). Then \( \theta^* = 1/10 \). A simple calculation shows that \( H_1 = 14/81 \) and \( H_2 = 1/25 \). If we choose \( d = 1, a = 2, c = 1068 \), then all the conditions of Theorem 3.1 are satisfied. Therefore, it follows from Theorem 3.1 that (3.9) has at least three positive solutions.

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