Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 119, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NON-EXISTENCE OF GLOBAL SOLUTIONS TO GENERALIZED DISSIPATIVE KLEIN-GORDON EQUATIONS WITH POSITIVE ENERGY

MAXIM OLEGOVICH KORPUSOV

Abstract. In this article the initial-boundary-value problem for generalized dissipative high-order equation of Klein-Gordon type is considered. We continue our study of nonlinear hyperbolic equations and systems with arbitrary positive energy. The modified concavity method by Levine is used for proving blow-up of solutions.

1. Introduction

We consider the initial-boundary-value problem

$$u_{tt} + \mu u_t + \Delta h_1(x, \Delta u) - \operatorname{div}(h_2(x, |\nabla u|)\nabla u) + \operatorname{div}(h_3(x, |\nabla u|)\nabla u) = 0, \quad (1.1)$$

$$u|_{\partial\Omega} = \frac{\partial u}{\partial n_x}\big|_{\partial\Omega} = 0, \quad u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad \mu \geq 0, \quad (1.2)$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega \in \mathbb{C}^{4,\delta}$ for $\delta \in (0,1]$.

Finite time blow-up of solutions of generalized Klein-Gordon equation have been studied by many authors; see for example [2, 3, 6, 4, 24, 5, 16]. In these references, the authors considere problems either for negative energy or for weaker conditions than a condition of negative initial energy (see [16, 23]). Other authors have assumed a condition of positive energy under other two conditions on the initial functions. However, the mentioned authors have not studied the compatibility of these conditions, which is come times hard to understand. Finally these conditions for any fixed u_0 and sufficiently large u_1 are not compatible. These authors have used the classic concavity Levine's method. In this paper we use a modified method, developed in [1], that provide two conditions for which their compatibility is easily checked.

Let us remember that there are five well-known methods for studying a blow-up phenomena. The first method is the concavity method developed by Levine [13, 14, 19, 20, 22, 15]. The second method is the test functions developed by Pokhozhaev, Mitidieri and Zhang [17, 18, 7, 25]. And the third method based on different criterion of comparison and was developed by Samarskii, Galaktionov,

2000 *Mathematics Subject Classification.* 35B44, 35L05, 35L25, 35L67.
Key words and phrases. Blow-up; wave equation; shocks and singularities .

Submitted February 14, 2012. Published July 19, 2012.
Supported by grant N 11-01-12018-ofi-m-2011 from the Russian Foundation for basic research.

Kurdyumov, Mikhailov [8, 21]. The forth method based of positive averageswas developed by Keller and Glassey [12, 10]. The fifth method, for nonlinear damping, was developed by by Georgiev and Todorova [9].

2. Main differential inequality

Consider the important differential inequality

$$\Phi\Phi'' - \alpha(\Phi')^2 + \gamma\Phi'\Phi + \beta\Phi \geq 0, \quad \alpha > 1,\ \beta \geq 0,\ \gamma \geq 0, \tag{2.1}$$

where $\Phi(t) \in \mathbb{C}^{(2)}([0,T])$, $\Phi(t) \geq 0$, $\Phi(0) > 0$. Dividing both sides (2.1) by $\Phi^{1+\alpha}$, we obtain

$$\Big(\frac{\Phi'}{\Phi^\alpha}\Big)' + \gamma\frac{\Phi'}{\Phi^\alpha} + \beta\Phi^{-\alpha} \geq 0.$$

Therefore,

$$\frac{1}{1-\alpha}(\Phi^{1-\alpha})'' + \frac{\gamma}{1-\alpha}(\Phi^{1-\alpha})' + \beta\Phi^{-\alpha} \geq 0. \tag{2.2}$$

By definition, put$Z(t) = \Phi^{1-\alpha}(t)$. Then, from (2.2) we obtain

$$Z'' + \gamma Z' - \beta(\alpha-1)Z^{\alpha_1} \leq 0, \quad \alpha_1 = \frac{\alpha}{\alpha-1}. \tag{2.3}$$

Also by definition, put $Y(t) = e^{\gamma t}Z(t)$; hence from (2.3) we obtain

$$Y'' - \gamma Y' - \beta(\alpha-1)e^{-\delta t}Y^{\alpha_1} \leq 0, \quad \delta = \frac{\gamma}{\alpha-1}. \tag{2.4}$$

It is easily shown that the following chain of equalities holds:

$$Y' = \big(\Phi^{1-\alpha}e^{\gamma t}\big)' = \Phi^{-\alpha}(\alpha-1)e^{\gamma t}\big[-\Phi'(t) + \frac{\gamma}{\alpha-1}\Phi(t)\big]. \tag{2.5}$$

Take the initial condition

$$\Phi'(0) > \frac{\gamma}{\alpha-1}\Phi(0); \tag{2.6}$$

then there exists $t_0 > 0$ such that

$$\Phi'(t) > \frac{\gamma}{\alpha-1}\Phi(t) \quad \text{for } t \in [0,t_0). \tag{2.7}$$

Combining (2.7) and (2.5), we obtain

$$Y'(t) < 0 \quad \text{for} \quad t \in [0,t_0).$$

Since $-\gamma Y'(t) \geq 0$, for $t \in [0,t_0)$, it follows from (2.4) that

$$Y'' - \beta(\alpha-1)e^{-\delta t}Y^{\alpha_1} \leq 0, \quad \delta = \frac{\gamma}{\alpha-1} \quad \text{for} \quad t \in [0,t_0). \tag{2.8}$$

Now multiplying both sides (2.8) by Y', we obtain

$$Y'Y'' - \beta(\alpha-1)e^{-\delta t}Y^{\alpha_1}Y' \geq 0, \quad \delta = \frac{\gamma}{\alpha-1} \quad \text{for } t \in [0,t_0). \tag{2.9}$$

Let us remark that

$$e^{-\delta t}Y^{\alpha_1}Y' = \frac{d}{dt}[e^{-\delta t}Y^{1+\alpha_1}] + \delta e^{-\delta t}Y^{1+\alpha_1} - \alpha_1 e^{-\delta t}Y^{\alpha_1}Y',$$

Thus we have

$$e^{-\delta t}Y^{\alpha_1}Y' = \frac{1}{1+\alpha_1}\frac{d}{dt}[e^{-\delta t}Y^{1+\alpha_1}] + \frac{1}{1+\alpha_1}\delta e^{-\delta t}Y^{1+\alpha_1}. \tag{2.10}$$

Combining (2.10) with (2.9), we obtain

$$Y'Y'' - \frac{\beta(\alpha-1)}{1+\alpha_1}\frac{d}{dt}[e^{-\delta t}Y^{1+\alpha_1}] - \frac{\beta(\alpha-1)\delta}{1+\alpha_1}e^{-\delta t}Y^{1+\alpha_1} \geq 0 \quad \text{for } t \in [0,t_0),$$

clearly, from this inequality we obtain

$$Y'Y'' - \frac{\beta(\alpha-1)}{1+\alpha_1}\frac{d}{dt}[e^{-\delta t}Y^{1+\alpha_1}] \geq 0 \quad \text{for } t \in [0,t_0). \tag{2.11}$$

Integrating the above expression,

$$(Y')^2 \geq A^2 + \frac{2\beta(\alpha-1)^2}{2\alpha-1}e^{-\delta t}Y^{1+\alpha_1} \geq A^2, \tag{2.12}$$

where

$$A^2 \equiv (Y'(0))^2 - \frac{2\beta(\alpha-1)^2}{2\alpha-1}Y^{1+\alpha_1}(0). \tag{2.13}$$

We assume the condition

$$A^2 > 0.$$

The reader will have no difficulty in showing that this condition is equivalent to the condition

$$A^2 = (\alpha-1)^2\Phi^{-2\alpha}(0)[\big(\Phi'(0) - \frac{\gamma}{\alpha-1}\Phi(0)\big)^2 - \frac{2\beta}{2\alpha-1}\Phi(0)] > 0. \tag{2.14}$$

Therefore, the condition $A^2 > 0$ and the following condition are equivalent.

$$\Big(\Phi'(0) - \frac{\gamma}{\alpha-1}\Phi(0)\Big)^2 > \frac{2\beta}{2\alpha-1}\Phi(0). \tag{2.15}$$

Thus, combining (2.12) and (2.14), we obtain

$$Y'(t) \leq -A < 0 \Rightarrow \Phi'(t_0) > \frac{\gamma}{\alpha-1}\Phi(t_0).$$

But now we have that $Y'(t_0) < 0$. Therefore, using this algorithm of "continue in time", we obtain

$$Y'(t) < 0 \quad \text{for all } t \in [0,T].$$

This implies that

$$|Y'| \geq A > 0 \Rightarrow Y'(t) \leq -A \Rightarrow Y(t) \leq Y(0) - At \Rightarrow$$
$$\Rightarrow \Phi^{1-\alpha}(t) \leq e^{-\gamma t}[\Phi^{1-\alpha}(0) - At] \Rightarrow \Phi(t) \geq \frac{e^{\gamma t/(\alpha-1)}}{[\Phi^{1-\alpha}(0) - At]^{1/(\alpha-1)}}.$$

The result is the following theorem.

Theorem 2.1. *Suppose* $\Phi(t) \in \mathbb{C}^{(2)}([0,T])$*, satisfies inequality* (2.1) *and*

$$\Phi'(0) > \frac{\gamma}{\alpha-1}\Phi(0), \tag{2.16}$$

$$\big(\Phi'(0) - \frac{\gamma}{\alpha-1}\Phi(0)\big)^2 > \frac{2\beta}{2\alpha-1}\Phi(0) \tag{2.17}$$

where $\Phi(t) \geq 0$, $\Phi(0) > 0$, *then the time* $T > 0$ *can not be arbitrarily large, but the following inequality holds*

$$T \leq T_\infty \leq \Phi^{1-\alpha}(0)A^{-1},$$
$$A^2 \equiv (\alpha-1)^2\Phi^{-2\alpha}(0)\big[\big(\Phi'(0) - \frac{\gamma}{\alpha-1}\Phi(0)\big)^2 - \frac{2\beta}{2\alpha-1}\Phi(0)\big],$$

where $\limsup_{t\uparrow T}\Phi(t) = +\infty$.

3. Conditions

We begin with conditions on the functions $h_1(x,s)$, $h_2(x,s)$, and $h_3(x,s)$.

Conditions on $h_1(x,s)$:

(H1.1) $h_1(x,s) : \Omega \times \mathbb{R}^1 \to \mathbb{R}^1$ is a Caratheodory function;

(H1.2) for almost all $x \in \Omega$ the function $h_1(x,s) \in \mathbb{C}^{(1)}(\mathbb{R}^1)$ and a "growth conditions" take place

$$|h_1(x,s)| \le c_1 + c_2|s|^{p_1-1}, \quad |h'_{1s}(x,s)| \le c_1 + c_2|s|^{p_1-2} \quad \text{for } p_1 \ge 2; \tag{3.1}$$

(H1.3) for any $v(x) \in \mathbb{W}_0^{2,p_1}(\Omega)$ there exist the inequalities

$$0 \le \int_\Omega h_1(x, \Delta v(x))\Delta v(x)\,dx \le \theta_1 \int_\Omega H_1(x, \Delta v(x))\,dx, \tag{3.2}$$

where $\theta_1 > 0$ and $H_1(x,s) = \int_0^s d\sigma\, h_1(x,\sigma)$;

conditions on $h_2(x,s)$:

(H2.1) $h_2(x,s) : \Omega \times \mathbb{R}_+^1 \to \mathbb{R}_+^1$ is a Caratheodory function;

(H2.2) for almost all $x \in \Omega$ the function $h_2(x,s) \in \mathbb{C}^{(1)}(\mathbb{R}_+^1)$ and we suppose the following inequalities

$$0 \le h_2(x,s) \le c_3 + c_4 s^{p_2-2}, \quad |h'_{2s}(x,s)s| \le c_3 + c_4 s^{p_2-2} \quad \text{for } p_2 \ge 2; \tag{3.3}$$

(H2.3) for any $v(x) \in \mathbb{W}_0^{1,p_2}(\Omega)$ an inequality holds

$$0 \le \int_\Omega h_2(x, |\nabla v|)|\nabla v|^2\,dx \le \theta_2 \int_\Omega dx\, H_2(x, |\nabla v|) \quad \text{for } \theta_2 > 0, \tag{3.4}$$

where $H_2(x,s) = \int_0^s d\sigma\, h_2(x,\sigma)\sigma$.

Conditions on $h_3(x,s)$:

(H3.1) $h_3(x,s) : \Omega \times \mathbb{R}_+^1 \to \mathbb{R}_+^1$ is a Caratheodory function;

(H3.2) for almost all $x \in \Omega$ the function $h_3(x,s) \in \mathbb{C}^{(1)}(\mathbb{R}_+^1)$ and

$$0 \le h_3(x,s) \le c_5 + c_6 s^{p_3-2}, \quad |h'_{3s}(x,s)s| \le c_5 + c_6 s^{p_3-2}, \quad p_3 > 2; \tag{3.5}$$

(H3.3) for all $v(x) \in \mathbb{W}_0^{1,p_3}(\Omega)$ we assume that

$$\int_\Omega h_3(x, |\nabla v|)|\nabla v|^2\,dx \ge \theta_3 \int_\Omega dx\, H_3(x, |\nabla v|) \quad \text{for } \theta_3 > 2, \tag{3.6}$$

where $H_3(x,s) = \int_0^s d\sigma\, h_3(x,\sigma)\sigma$.

We define

$$p^* = \begin{cases} Np/(N-p), & \text{for } N > p; \\ +\infty, & \text{for } N \le p. \end{cases}$$

It can easily be checked that from the conditions on the functions $h_1(x,s)$, $h_2(x,s)$, and $h_3(x,s)$ we have

$$\begin{gathered} \Delta h_1(x, \Delta v) : \mathbb{W}_0^{2,p_1}(\Omega) \to \mathbb{W}^{-2,p'_1}(\Omega), \quad p'_1 = p_1/(p_1-1), \\ \operatorname{div}(h_2(x, |\nabla v|)\nabla v) : \mathbb{W}_0^{1,p_2}(\Omega) \to \mathbb{W}^{-1,p'_2}(\Omega), \quad p_2 = p_2/(p_2-1), \\ \operatorname{div}(h_3(x, |\nabla v|)\nabla v) : \mathbb{W}_0^{1,p_3}(\Omega) \to \mathbb{W}^{-1,p'_3}(\Omega), \quad p_3 = p_3/(p_3-1), \end{gathered}$$

and this operators are continuous in the corresponding topologies.

Definition 3.1. A strong generalized solution of (1.1), (1.2) is a function $u(x)(t)$ in the class

$$u(x)(t) \in \mathbb{C}^{(2)}([0,T];\mathbb{W}_0^{2,p_1}(\Omega)), \quad T > 0,$$

for some large $s > 0$, if the following condition hold:

$$\begin{aligned}
&\int_\Omega u''(x)(t)w(x)\,dx + \mu\int_\Omega u'(x)(t)w(x)\,dx + \int_\Omega h_1(x,\Delta u)\Delta w(x)\,dx \\
&+ \int_\Omega h_2(x,|\nabla u|)(\nabla u,\nabla w)\,dx - \int_\Omega h_3(x,|\nabla u|)(\nabla u,\nabla w)\,dx = 0, \quad t \in [0,T]
\end{aligned} \tag{3.7}$$

for all $w(x) \in \mathbb{W}_0^{2,p_1}(\Omega)$; and

$$u(x)(0) = u_0(x) \in \mathbb{W}_0^{2,p_1}(\Omega), \quad u'(x)(0) = u_1(x) \in \mathbb{L}^2(\Omega). \tag{3.8}$$

4. Blow-up of solutions

Assume that there is a weak solution $u(x)(t)$ in the class $\mathbb{C}^{(2)}([0,T];\mathbb{W}_0^{2,p_1}(\Omega))$ for some $T > 0$. Let us put $w = u(x)(t)$ in equation (3.7), then we obtain the first energy equality

$$\begin{aligned}
&\frac{1}{2}\frac{d^2\Phi}{dt^2} + \frac{\mu}{2}\frac{d\Phi}{dt} - J + \int_\Omega h_1(x,\Delta u)\Delta u\,dx + \int_\Omega h_2(x,|\nabla u|)|\nabla u|^2\,dx \\
&= \int_\Omega h_3(x,|\nabla u|)|\nabla u|^2\,dx,
\end{aligned} \tag{4.1}$$

where we denote

$$\Phi(t) \equiv \int_\Omega |u|^2\,dx, \quad J(t) \equiv \int_\Omega |u'|^2\,dx.$$

Let us put $w = u'(x)(t)$ in (3.7), we obtain the second energy equality

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}J + \int_\Omega H_1(x,\Delta u)\,dx + \int_\Omega H_2(x,|\nabla u|)\,dx\Big) + \mu J \\
&= \frac{d}{dt}\int_\Omega H_3(x,|\nabla u|)\,dx.
\end{aligned} \tag{4.2}$$

Furthermore, integrating (4.2) over time, we obtain the inequality

$$\frac{1}{2}J + \int_\Omega H_1(x,\Delta u)\,dx + \int_\Omega H_2(x,|\nabla u|)\,dx - E(0) \le \int_\Omega H_3(x,|\nabla u|)\,dx, \tag{4.3}$$

where

$$\begin{aligned}
E(0) \equiv \frac{1}{2}\int_\Omega |u_1|^2\,dx &+ \int_\Omega H_1(x,\Delta u_0)\,dx \\
&+ \int_\Omega H_2(x,|\nabla u_0|)\,dx - \int_\Omega H_3(x,|\nabla u_0|)\,dx.
\end{aligned} \tag{4.4}$$

By (4.3) we obtain the inequality

$$\begin{aligned}
&\frac{\theta_3}{2}J + \theta_3\int_\Omega H_1(x,\Delta u)\,dx + \theta_3\int_\Omega H_2(x,|\nabla u|)\,dx - \theta_3 E(0) \\
&\le \theta_3\int_\Omega H_3(x,|\nabla u|)\,dx,
\end{aligned}$$

combining this with the condition (H3.3), we obtain

$$\frac{\theta_3}{2}J + \theta_3\int_\Omega H_1(x,\Delta u)\,dx + \theta_3\int_\Omega H_2(x,|\nabla u|)\,dx - \theta_3 E(0)$$

$$\leq \int_\Omega h_3(x, |\nabla u|)|\nabla u|^2\,dx.$$

Using the above inequality and (4.1), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d^2\Phi}{dt^2} + \frac{\mu}{2}\frac{d\Phi}{dt} - J + \int_\Omega h_1(x, \Delta u)\Delta u\,dx + \int_\Omega h_2(x, |\nabla u|)|\nabla u|^2\,dx \\ &\geq \frac{\theta_3}{2} J + \theta_3 \int_\Omega H_1(x, \Delta u)\,dx + \theta_3 \int_\Omega H_2(x, |\nabla u|)\,dx - \theta_3 E(0). \end{aligned} \tag{4.5}$$

Now taking into account the conditions (H3.1) and (H3.2), from (4.5) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d^2\Phi}{dt^2} + \frac{\mu}{2}\frac{d\Phi}{dt} - J + \theta_1 \int_\Omega H_1(x, \Delta u)\,dx + \theta_2 \int_\Omega H_2(x, |\nabla u|)\,dx \\ &\geq \frac{\theta_3}{2} J + \theta_3 \int_\Omega H_1(x, \Delta u)\,dx + \theta_3 \int_\Omega H_2(x, |\nabla u|)\,dx - \theta_3 E(0). \end{aligned} \tag{4.6}$$

Under the conditions $\theta_3 \geq \theta_1$, $\theta_3 \geq \theta_2$ using the inequalities

$$\int_\Omega H_1(x, \Delta u)\,dx \geq 0, \quad \int_\Omega H_2(x, |\nabla u|)\,dx \geq 0,$$

from (4.6), we obtain

$$\frac{1}{2}\frac{d^2\Phi}{dt^2} + \frac{\mu}{2}\frac{d\Phi}{dt} + \theta_3 E(0) \geq \big(1 + \frac{\theta_3}{2}\big)J(t). \tag{4.7}$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, it is easily shown the differential inequality

$$(\Phi')^2 \leq 4J\Phi. \tag{4.8}$$

Combining (4.7) and (4.8), we obtain the important differential inequality

$$\Phi\Phi'' - \frac{1}{2}\big(1 + \frac{\theta_3}{2}\big)(\Phi')^2 + \mu\Phi\Phi' + 2\theta_3 E(0)\Phi \geq 0. \tag{4.9}$$

Comparing this differential inequality with (2.1), we obtain that

$$\alpha = \frac{1}{2}\big(1 + \frac{\theta_3}{2}\big) > 1 \quad \text{for } \theta_3 > 2,$$

$$\beta = 2\theta_3 E(0), \quad \gamma = \mu, \quad \frac{2\beta}{2\alpha - 1} = 8E(0), \quad \frac{\gamma}{\alpha - 1} = \frac{4\mu}{\theta_3 - 2}.$$

We assume the following conditions

$$\Phi'(0) > \frac{4\mu}{\theta_3 - 2}\Phi(0) > 0, \tag{4.10}$$

$$\Big(\Phi'(0) - \frac{4\mu}{\theta_3 - 2}\Phi(0)\Big)^2 > 8E(0)\Phi(0), \tag{4.11}$$

$$\begin{aligned} E(0) \equiv{}& \frac{1}{2}\int_\Omega |u_1|^2\,dx + \int_\Omega H_1(x, \Delta u_0)\,dx \\ &+ \int_\Omega H_2(x, |\nabla u_0|)\,dx - \int_\Omega H_3(x, |\nabla u_0|)\,dx > 0, \end{aligned} \tag{4.12}$$

Under conditions (4.10)–(4.12), the time $T > 0$ of existence of $u(x)(t)$ is bounded from above

$$T \leq \Phi^{(2-\theta_3)/4}(0)A^{-1},$$

$$A^2 \equiv \big(\frac{\theta_3 - 2}{4}\big)^2 \Phi^{-1-\theta_3/2}(0)\Big[\Big(\Phi'(0) - \frac{4\mu}{\theta_3 - 2}\Phi(0)\Big)^2 - 8E(0)\Phi(0)\Big],$$

at the same time

$$\Phi(t) \geq \frac{e^{4\mu t/(\theta_3-2)}}{[\Phi^{(2-\theta_3)/4}(0) - At]^{4/(\theta_3-2)}}, \tag{4.13}$$

where

$$\Phi'(0) = 2\int_\Omega u_1(x)u_0(x)\,dx, \quad \Phi(0) = \int_\Omega |u_0|^2\,dx.$$

Therefore, our main result of is the following theorem.

Theorem 4.1. *Assume all conditions on h_1, h_2 and h_3 hold. Under the following conditions*

$$\Phi'(0) > \frac{4\mu}{\theta_3-2}\Phi(0) + (8E(0)\Phi(0))^{1/2} > 0, \quad E(0) > 0, \tag{4.14}$$

$$\theta_3 \geq \theta_1, \quad \theta_3 \geq \theta_2, \tag{4.15}$$

there exists the estimation from above for the time of solution existence T,

$$T_1 \leq T_\infty = \Phi^{(2-\theta_3)/4}(0)A^{-1};$$

i. e., we have that $\limsup_{t\uparrow T_1}\Phi(t) = +\infty$*, where*

$$\Phi(0) = \int_\Omega |u_0|^2\,dx, \quad \Phi'(0) = 2\int_\Omega u_1u_0\,dx,$$

$$\begin{aligned} E(0) \equiv \frac{1}{2}\int_\Omega |u_1|^2\,dx + \int_\Omega H_1(x,\Delta u_0)\,dx \\ + \int_\Omega H_2(x,|\nabla u_0|)\,dx - \int_\Omega H_3(x,|\nabla u_0|)\,dx, \end{aligned}$$

$$A^2 \equiv \big(\frac{\theta_3-2}{4}\big)^2\Phi^{-1-\theta_3/2}(0)\big[\big(\Phi'(0) - \frac{4\mu}{\theta_3-2}\Phi(0)\big)^2 - 8E(0)\Phi(0)\big].$$

Remark 4.2. Now we shall see that all conditions (4.14) are compatible for enough small $\mu \geq 0$. Indeed, first we shall choose $u_0 \in \mathbb{W}_0^{2,p_1}(\Omega)$ enough large to satisfy the inequality

$$\begin{aligned} &\int_\Omega H_3(x,|\nabla u_0|)\,dx \\ &> \int_\Omega H_1(x,\Delta u_0)\,dx + \int_\Omega H_2(x,|\nabla u_0|)\,dx + \frac{2}{\theta_3-2}\int_\Omega |u_0|^2\,dx. \end{aligned} \tag{4.16}$$

Secondly we fix u_0 and choose $u_1 = \lambda u_0$ for $\lambda > 2\mu/(\theta_3-2)$. In this case we have

$$\Phi'(0) - \frac{4\mu}{\theta_3-2}\Phi(0) = 2\Big(\lambda - \frac{2\mu}{\theta_3-2}\Big)\Phi(0) > 0. \tag{4.17}$$

Finally we choose $\lambda > 2\mu/(\theta_3-2)$ enough large to satisfy the inequality

$$\begin{aligned} E(0) = \frac{\lambda^2}{2}\int_\Omega |u_0|^2\,dx + \int_\Omega H_1(x,\Delta u_0)\,dx \\ + \int_\Omega H_2(x,|\nabla u_0|)\,dx - \int_\Omega H_3(x,|\nabla u_0|)\,dx > 0. \end{aligned}$$

Under condition (4.17), inequality (4.14) is equivalent to

$$\Big(\Phi'(0) - \frac{4\mu}{\theta_3-2}\Phi(0)\Big)^2 > 8E(0)\Phi(0), \tag{4.18}$$

by our substitution we obtain from the left and right side of this inequality

$$\begin{aligned}\Big(\Phi'(0)-\frac{4\mu}{\theta_3-2}\Phi(0)\Big)^2&=4\Big(\lambda-\frac{2\mu}{\theta_3-2}\Big)^2\Phi^2(0)\\&=\Big(4\lambda^2-\frac{16\mu\lambda}{\theta_3-2}+\frac{16\mu^2}{(\theta_3-2)^2}\Big)\Phi^2(0),\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}8E(0)\Phi(0)=4\lambda^2\Phi^2(0)+8\Phi(0)\Big[&\int_\Omega H_1(x,\Delta u_0)\,dx\\&+\int_\Omega H_2(x,|\nabla u_0|)\,dx-\int_\Omega H_3(x,|\nabla u_0|)\,dx\Big]\end{aligned}\tag{4.20}$$

Combining (4.18) with (4.19) and (4.20), we obtain that

$$\begin{aligned}\Big(\Phi'(0)-\frac{4\mu}{\theta_3-2}\Phi(0)\Big)^2&=\Big(4\lambda^2-\frac{16\mu\lambda}{\theta_3-2}+\frac{16\mu^2}{(\theta_3-2)^2}\Big)\Phi^2(0)>8E(0)\Phi(0)\\&=4\lambda^2\Phi^2(0)+8\Phi(0)\Big[\int_\Omega H_1(x,\Delta u_0)\,dx\\&\quad+\int_\Omega H_2(x,|\nabla u_0|)\,dx-\int_\Omega H_3(x,|\nabla u_0|)\,dx\Big],\end{aligned}\tag{4.21}$$

Now it is not hard to prove that

$$\begin{aligned}&\int_\Omega H_3(x,|\nabla u_0|)\,dx+\frac{2\mu^2}{(\theta_3-2)^2}\int_\Omega |u_0|^2\,dx\\&>\frac{2\mu\lambda}{\theta_3-2}\int_\Omega |u_0|^2\,dx+\int_\Omega H_1(x,\Delta u_0)\,dx+\int_\Omega H_2(x,|\nabla u_0|)\,dx.\end{aligned}\tag{4.22}$$

Moreover, we choose

$$\lambda=\frac{1}{\mu}\quad\text{for } \mu\in\big(0,(\frac{\theta_3-2}{2})^{1/2}\big),$$

and if $\mu=0$, then $\lambda>0$ and large enough. We see that all foregoing conditions are satisfied for small enough $\mu\geq 0$. Now combining this large enough λ and (4.22), we obtain the inequality

$$\begin{aligned}&\int_\Omega H_3(x,|\nabla u_0|)\,dx+\frac{2\mu^2}{(\theta_3-2)^2}\int_\Omega |u_0|^2\,dx\\&>\frac{2}{\theta_3-2}\int_\Omega |u_0|^2\,dx+\int_\Omega H_1(x,\Delta u_0)\,dx+\int_\Omega H_2(x,|\nabla u_0|)\,dx.\end{aligned}$$

Obviously, this inequality holds by (4.16). Therefore, we have to prove (4.16) for some functions on $h_1(x,s)$, $h_2(x,s)$, and $h_3(x,s)$. At the same time we check the condition (4.15). Suppose

$$h_1(x,s)=|s|^{p_1-2}s,\quad h_2(x,s)=s^{p_2-2},\quad h_3(x,s)=s^{p_3-2},$$

where $p_3>p_1>2$, $p_3>p_2\geq 2$. Then

$$H_1(x,s)=\frac{|s|^{p_1}}{p_1},\quad H_2(x,s)=\frac{|s|^{p_2}}{p_2},\quad H_3(x,s)=\frac{|s|^{p_3}}{p_3},$$

and $\theta_3=p_3>\theta_1=p_1>2$, $\theta_3=p_3>\theta_2=p_2$. Therefore, first note that the condition (4.15) holds, and further note that for large enough $u_0(x)\in\mathbb{W}_0^{2,p_1}(\Omega)$ the condition (4.16) also holds.

References

[1] A. B. Al'shin, M. O. Korpusov, A. G. Sveshnikov; *Blow-up in nonlinear Sobolev type equations*, De Gruyter, Series: De Gruyter Series in Nonlinear Analysis and Applications 15, (2011).

[2] Lianjun An, Anthony Peirce; "A Weakly Nonlinear Analysis of Elasto-plastic-Microstructure Models," *SIAM J. Appl. Math.*, **55**, 136–155 (1995).

[3] Guowang Chen, Fang Da; "Blow-up of solution of Cauchy problem for three-dimensional damped nonlinear hyperbolic equation," *Nonlinear Analysis: Theory, Methods and Applications.*, **71**, No. 1-2, 358–372 (2009).

[4] Guo-wang Chen, Chang-shun Hou; "Initial value problem for a class of fourth-order nonlinear wave equations," *Applied Mathematics and Mechanics*, **30**, No. 3, 391–401 (2009).

[5] Guowang Chen, Zhijian Yang; "Existence and non-existence of global solutions for a class of non-linear wave equations," *Mathematical Methods in the Applied Sciences*, **23**, No. 7, 615–631 (2000).

[6] Xiangying Chena, Guowang Chen; "Asymptotic behavior and blow-up of solutions to a nonlinear evolution equation of fourth order," *Nonlinear Analysis: Theory, Methods and Applications*, **68**, No. 4, 892–904 (2008).

[7] V. A. Galaktionov; S. I. Pohozaev; "Blow-up, critical exponents and asymptotic spectra for nonlinear hyperbolic equations" *Preprint Univ. Bath. Math.*, **00/10**, (2000).

[8] V. A. Galaktionov, S. I. Pohozaev; "Third-order nonlinear dispersive equations: Shocks, rarefaction, and blowup waves" *CMMP*, **48**, No. 10, 1784–1810 (2008).

[9] V. Georgiev, G. Todorova; "Existence of a solution of the wave equation with nonlinear damping and source terms" *J. Diff. Equations*, **109**, 295–308 (1994).

[10] R. T. Glassey; "Blow-up theorems for nonlinear wave equations" *Math. Z.*, **132**, 183–203 (1973).

[11] V. K. Kalantarov, O. A. Ladyzhenskaya; "The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types" *JMS*, **10**, No. 1, 53–70 (1975).

[12] J. B. Keller; "On solutions of nonlinear wave equations" *Comm. Pure Appl. Math*, **10**, 523–530 (1957).

[13] H. A. Levine; "Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$," *Transactions of the American mathematical society*, 1–21 (1974).

[14] H. A. Levine, P. Pucci, J. Serrin; "Some remarks on the global nonexistence problem for nonautonomous abstract evolution equations" *Contemporary Math.*,**208**, 253–263 (1997).

[15] H. A. Levine, G. Todorova; "Blow up of solutions of the Cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy in four space-time dimensions" *Proc. Amer. Math. Soc.*, **129**, 793–805 (2003).

[16] Yacheng Liu, Runzhang Xu; "A class of forth order wave equations with dissipative and nonlinear strain terms," *Journal of Differential Equations*, **244**, 200–228 (2008).

[17] E. L. Mitidieri, S. I. Pohozaev; "A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities", *Tr. Mat. Inst. Steklova*, **234**, (2001).

[18] S. I. Pohozaev; "Critical nonlinearities in partial differential equations" *Milan J. Math.*, **77**, No. 1, 127–150 (2009).

[19] P. Pucci, J. and Serrin; "Some new results on global nonexistence for abstract evolution equation with positive initial energy" *Topological Methods in Nonlinear Analys, Journal of J. Schauder Center for Nonlinear Studies.*, **10**, 241–247 (1997).

[20] P. Pucci, J. Serrin; "Global nonexistence for abstract evolution equations with positive initial energy" *J. Diff. Equations.*, **150**, 203–214 (1998).

[21] A. A. Samarskij, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mihailov; "On blow up of nonlinear quasilinear equations of parabolic type" *Nauka*, **480**, (1987).

[22] B. Straughan; "Further global nonexistence theorems for abstract nonlinear wave equations" *Proc. Amer. Math. Soc.*, **2**, 381–390 (1975).

[23] Yanjin Wang; "Nonexistence of global solutions of a class of coupled nonlinear Klein-Gordon equations with nonnegative potentials and arbitrary initial energy," *IMA Journal of Applied Mathematics*, **24**, No. 3, 392–415 (2009).

[24] Zhijian Yang; "Global existence, asymptotic behavior and blowup of solutions for a class of nonlinear wave equations with dissipative term," *Journal of Differential Equations*, **187**, No. 2, 520–540 (2003).

[25] Q. S. Zhang; "Blow-up results for nonlinear parabolic equations on manifolds" *Duke Math. J.*, **97**, 515–539 (1999).

Maxim Olegovich Korpusov

Chair of Mathematics, Faculty of Physics, M. V. Lomonosov Moscow State University, Moscow, Russia

E-mail address: korpusov@gmail.com